Initial Characterized L-spaces and Characterized L- topological Groups

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Abstract: In this research work, new topological notions are proposed and investigated. The notions are named initial characterized L-spaces and characterized L-topological groups. The properties of such notions are deeply studied. We show that the intitial characterized L-space for an characterized L-spaces exists. By this notion, the notions of characterized L-subspace and characterized product L-space are introduced and studied. On other hand we show that for the notion of characterized L-topological groups the functors ω_L , i_L and i_α are concrete and covariant funvtors. Some sort of relationship were introduced, such as, we show that the ordinary characterized space is characterized topological group if and only if its induced is characterized L-topological group. However, we show that for each characterized L-topological group, the α -level characterized space and the initial characterized space are characterized topological groups.

Keywords: L-filter, topological L-space, the category **L-Top** , operations, isotone and idempotent, characterized L-space, the categories **CRL-Sp** , **SCRL-Sp** and **CR-Sp**, group, the category **Grp,** functor, $\varphi_{1,2}$ L- neighborhood filters, $\varphi_{1,2}$ $\psi_{1,2}$ L-continuous and $\varphi_{1,2}$ $\psi_{1,2}$ -continuous, $\varphi_{1,2}$ $\psi_{1,2}$ L-open and $\varphi_{1,2}$ $\psi_{1,2}$ -open, $\varphi_{1,2}$ $\psi_{1,2}$ L-homeomorphism, $\varphi_{1,2}$ $\psi_{1,2}$ -homeomorphism, $\varphi_{1,2}$ $\psi_{1,2}$ -homomorphism, α -level characterized space, induced characterized space, initial characterized L-space, characterized L-subspace, characterized product L-space, characterized L-topological group, the category **CRL-TopGrp,** characterized topological group, the category **CR-TopGrp**.

1. Introduction

The notion of L-filter has been introduced by Eklund *et al*. [11]. By means of this notion a point-based approach to L- topology related to usual points has been developed. The more general concept for L-filter introduced by G$\ddot{a}$hler in [13] and L-filters are classified by types. Because of the specific type of L-filter however the approach of Eklund is related only to L-topologies which are stratified, that is, all constant L-sets are open. The more specific L-filters considered in the former papers are called now homogeneous. The operation on the ordinary topological space (X, T) has been defined by Kasahara ([19]) as a mapping φ from T into 2^X such that $A \subseteq A^{\varphi}$, for all $A \in T$. In. [4], Abd El-Monsef's *et al.*extended Kasahars's operation to the power set $P(X)$ of a set X . Kandil *et al.* ([18]), extended Kasahars's and Abd El-Monsef's operations by introducing an operation on the class of all L-sets endowed with an L-topology τ as a mapping $\varphi : L^X \to L^X$ such that $\text{int}\,\mu \leq \mu^{\varphi}$ for all $\mu \in L^X$, where μ^{φ} denotes the value of φ at μ . The notions of the L-filters and the operations on the class of all L-sets on X endowed with an L-topology τ are applied in [1,2,3] to introduce a more general theory including all the weaker and stronger forms of the L-topology. By means of these notions the notion of $\varphi_{1,2}$ -interior of L-set, $\varphi_{1,2}$ L-convergence and $\varphi_{1,2}$ L-neighborhood filters are defined and applied to introduced many special classes of separation axioms. The notion of $\varphi_{1,2}$ -interior operator for L-sets is defined as a mapping $\varphi_{1,2}.\text{int} : L^X \to L^X$ which fulfill (I1) to (I5) in [1]. There is a one-to-one correspondence between the class of all $\varphi_{1,2}$ -open L-subsets of X and these operators, that is, the class $\varphi_{1,2}OF(X)$ of all $\varphi_{1,2}$ -

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open L-subsets of X can be characterized by these operators. Then the triple $(X, \varphi_{1,2}.\mathrm{int})$ as will as the triple $(X, \varphi_{1,2}OF(X))$ will be called the characterized L-space of $\varphi_{1,2}$-open L-subsets. The characterized L-spaces are characterized by many of characterizing notions in [1, 2], for example by: $\varphi_{1,2}$ L-neighborhood filters, $\varphi_{1,2}$ L-interior of the L-filters and by the set of $\varphi_{1,2}$-inner points of the L-filters. Moreover, the notions of closeness and compactness in characterized L-spaces are introduced and studied in [3].

This paper is devoted to introduce and study the notions of intial characterized L-spaces and characterized L-topological groups as a generalization of the weaker and stronger forms of the initial topological L-space and L-topological group introduced in [5, 7, 9, 10, 17, 20]. **In section 2**, some definitions and notions related to L-sets, L-topologies, L-filters, operations on L-sets, characterized L-spaces, $\varphi_{1,2}$ L-neighborhood filters, $\varphi_{1,2}\alpha$ L-neighborhood, $\varphi_{1,2}\,\psi_{1,2}$ L-contiuous mappings and $\varphi_{1,2}\,\psi_{1,2}$ L-open mappings are given. The categories of all characterized L-spaces, stratified characterized L- spaces and the ordinary characterized spaces with the $\varphi_{1,2}\,\psi_{1,2}$ L-continuity and $\varphi_{1,2}\,\psi_{1,2}$-continuity as a morphismeis between them are presented. **Section 3**, is devoted to introduce and study the notion of initial characterized L-spaces and some new types of functors between the categories **CRL-Sp, SCRL-Sp** and **CR-Sp** of all characterized L-spaces, stratified characterized L-spaces and characterized spaces. We make the relation of the farness on L-sets and the finer relation between characterized spaces to introduce the notions of α-level and initial characterized spaces for an L-topological space (X, τ) by means of the functors ω and i. For an ordinary topological space (X, T), the induced characterized L-space is also introduced by applied the functor ω. The functors ω and i are extended for any complete distributive lattice L to the functors functors ω_{L} and i_{L} and add new types. We show that the functors ω_{L}, S_L and i_{L} are concrete functors. For an family of a characterized L-spaces, the notions of initial characterized L-space is introduced and studied. At the beginang we show that the intitial characterized L-space for an characterized L-spaces exists. We further notions related to the notion of characterized L-spaces are e.g. those of characterized L-subspace and characterized product L-space are investigated as special cases from the notions of intial characterized L-spaces. By the initial left in **CRL-Sp** we show that the category **CRL-Sp** is topological category in sence of [1,6] and it is also complete and co-complete category, that is, all limits and all co-limits in **CRL-Sp** exist. Finally, we show that the category **SCRL-Sp** is bireflective subcategory of the category **CRL-Sp** and it is also topological. **Section 4**, is devaided to introduce and study the notion of characterized L-topological groups as a generalization of the weaken and stronger forms of the L-topological groups which introduced in [5,7,9,10,17,20]. It will be shown that the characterized L-topological group is an extension with respect to the functor ω_L. As an example we show that the α-level characterized space $(X, (\varphi_{1,2}OF(X))_\alpha)$, $\alpha \in L_1$ and the initial characterized space $(X, i_L(\varphi_{1,2}OF(X)))$ of a characterized L-topological group $(G, \varphi_{1,2}.\mathrm{int}_G)$ are characterized topological groups. Moreover, we show that the ordinary characterized space is characterized topological group if and only if its induced is characterized L-topological group. For a characterized L-topological group $(G, \varphi_{1,2}.\mathrm{int}_G)$, we show that the left translation L_a and the right translation R_a are $\varphi_{1,2}\,\varphi_{1,2}$ L- homeomorphisms for every $a \in G$. Some examples in special choice for the operations φ_1 and φ_2 are given for characterized L-topological groups. Finally we show that for the notion of characterized L-topological groups the functors ω_L, i_L and i_α are concrete and covariant funvtors.

2. Preliminaries

We begin by recalling some facts on the L-filters. Let L be a completely distributive complete lattice with different least and last elements 0 and 1, respectively. Let $L_0 = L \setminus \{0\}$ and $L_1 = L \setminus \{1\}$. Sometimes we will assume more specially that L is complete chain, that is, L is a complete lattice whose partial ordering is a linear one. For a set X, let L^X be the set of all L-subsets of X, that is, of all mappings $f : X \to L$. Assume that an order-reversing involution $\alpha \mapsto \alpha'$ of L is fixed. For each L-set $\mu \in L^X$, let μ' denote the complement of μ and it is defined by: $\mu'(x) = \mu(x)'$ for all $x \in X$. Denote by $\overline{\alpha}$ the constant L-subset of X with value $\alpha \in L$. For all $x \in X$ .and for all $\alpha \in L_0$, the L-subset x_α of X whose value α at x and 0 otherwise is called an L-point in X.

L-filters. The L- filter on a set X ([13]) is a mapping $\mathcal{M} : L^X \to L$ such that the following conditions are fulfilled:

(F1) $\mathcal{M}(\overline{\alpha}) \leq \alpha$ for all $\alpha \in L$ and $\mathcal{M}(\overline{1}) = 1$.

(F2) $\mathcal{M}(\mu \wedge \rho) = \mathcal{M}(\mu) \wedge \mathcal{M}(\rho)$ for all $\mu, \rho \in L^X$.

The L-filter $\mathcal{M}$ is called homogeneous ([13]) if $\mathcal{M}(\overline{\alpha}) = \alpha$ for all $\alpha \in L$. For each $x \in X$, the mapping $\dot{x} : L^X \to L$ defined by $\dot{x}(\mu) = \mu(x)$ for all $\mu \in L^X$ is a homogeneous L-filter on X. For each $\mu \in L^X$, the mapping $\dot{\mu} : L^X \to L$ defined by $\dot{\mu}(\eta) = \bigwedge_{0<\eta(x)} \eta(x)$ for all $\eta \in L^X$ is also homogeneous L-filter on X, called homogenous L- filter at the L-subset $\mu \in L^X$. Let $\mathcal{F}_L X$ and $F_L X$ will be denote the sets of all L-filters and of all homogeneous L- filters on a set X, respectively. If $\mathcal{M}$ and $\mathcal{N}$ are L- filters on a set X, $\mathcal{M}$ is said to be finer than $\mathcal{N}$, denoted by $\mathcal{M} \leq \mathcal{N}$, provided $\mathcal{M}(\mu) \geq \mathcal{N}(\mu)$ holds for all $\mu \in L^X$. Noting that if L is a complete chain then $\mathcal{M}$ is not finer than $\mathcal{N}$, denoted by $\mathcal{M} \not\leq \mathcal{N}$, provided there exists $\mu \in L^X$ such that $\mathcal{M}(\mu) < \mathcal{N}(\mu)$ holds.

For each non-empty set $\mathcal{A}$ of the L- filters on X the supremum $\bigvee_{\mathcal{M} \in \mathcal{A}} \mathcal{M}$ exists ([13]) and given by:

$$(\bigvee_{\mathcal{M} \in \mathcal{A}} \mathcal{M})(\mu) = \bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M}(\mu)$$

for all $\mu \in L^X$. Whereas the infimum $\bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M}$ of $\mathcal{A}$ does not exists in general as an L-filter. If the infimum $\bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M}$ exists, then we have:

$$(\bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M})(\mu) = \bigvee_{\substack{\mu_1 \wedge \ldots \wedge \mu_n \leq \mu, \\ \mathcal{M}_1, \ldots, \mathcal{M}_n \in \mathcal{A}}} (\mathcal{M}_1(\mu_1) \wedge \ldots \wedge \mathcal{M}_n(\mu_n))$$

For all $\mu \in L^X$, where n is a positive integer, $\mu_1, \ldots, \mu_n$ is a collection such that $\mu_1 \wedge \ldots \wedge \mu_n \leq \mu$ and $\mathcal{M}_1, \ldots, \mathcal{M}_n$ are L- filters from $\mathcal{A}$. Let X be a set and $\mu \in L^X$, then the homogeneous L- filter $\dot{\mu}$ at $\mu \in L^X$ is the L-filter on X given by:

$$\dot{\mu} = \bigvee_{0 < \mu(x)} \dot{x}$$

L- filter bases. A family $(\mathcal{B}_\alpha)_{\alpha \in L_0}$ of non-empty subsets of L^X is called a valued L- filter base ([13]) if the following conditions are fulfilled:

(V1) $\mu \in \mathcal{B}_\alpha$ implies $\alpha \leq \sup \mu$.

(V2) For all $\alpha, \beta \in L_0$ with $\alpha \wedge \beta \in L_0$ and all $\mu \in \mathcal{B}_\alpha$ and $\rho \in \mathcal{B}_\beta$ there are $\gamma \geq \alpha \wedge \beta$ and $\eta \geq \mu \wedge \sigma$ such that $\eta \in \mathcal{B}_\gamma$.

Proposition 2.1 [13] Each valued base $(\mathcal{B}_\alpha)_{\alpha \in L_0}$ defines the L-filter $\mathcal{M}$ on X by $\mathcal{M}(\mu) = \bigvee_{\rho \in \mathcal{B}_\alpha, \rho \leq \mu} \alpha$ for all $\mu \in L^X$. Conversely, each L- filter $\mathcal{M}$ can be generated by a valued base, e.g. by $(\alpha\text{-pr}\,\mathcal{M})_{\alpha \in L_0}$ with $\alpha\text{-pr}\,\mathcal{M} = \{\mu \in L^X \mid \alpha \leq \mathcal{M}(\mu)\}$.

$(\alpha\text{-pr}\,\mathcal{M})_{\alpha \in L_0}$ is a family of prefilters on X and is called the large valued base of $\mathcal{M}$. Recall that a prefilter on X ([19]) is a non-empty proper subset $\mathcal{F}$ of L^X such that:

(1) $\mu, \rho \in \mathcal{F}$ Implies $\mu \wedge \rho \in \mathcal{F}$ and (2) from $\mu \in \mathcal{F}$ and $\mu \leq \rho$ it follows $\rho \in \mathcal{F}$.

Proposition 2.2 [13] Let $\mathcal{A}$ be a set of L- filters on a set X. Then the following are equivalent:
(1) The infimum $\bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M}$ of $\mathcal{A}$ with respect to the finer relation for L- filters exists.
(2) $\mathcal{M}_1(\mu_1) \wedge \ldots \wedge \mathcal{M}_n(\mu_n) \leq \sup(\mu_1 \wedge \ldots \wedge \mu_n)$ for all finite subset $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ of $\mathcal{A}$ and $\mu_1, \ldots, \mu_n \in L^X$.
(3) $\alpha \leq \sup(\mu_1 \wedge \ldots \wedge \mu_n)$ holds for all non-empty finite subset $\{\mu_1, \ldots, \mu_n\}$ of $\bigcup_{\mathcal{M} \in \mathcal{A}} \alpha$-pr $\mathcal{M}$ and $\alpha \in L_0$

Topological L-spaces. By an L-topology on a set X ([8, 16]), we mean a subset of $\mu \in L^X$ which is closed with respect to all suprema and all finite infima and contains the constant L-sets $\overline{0}$ and $\overline{1}$. A set X equipped with an L-topology τ on X is called topological L-space. For each topological L-space (X, τ), the elements of τ are called open L-subsets of this space. If τ_1 and τ_2 are L-topologies on a set X, τ_2 is said to be finer than τ_1 and τ_1 is said to be coarser than τ_2 provided $\tau_1 \subseteq \tau_2$ holds. For each L-set $\mu \in L^X$, the strong α-cut and the weak α-cut of μ are ordinary subsets of X defined by:

$$\mathrm{S}_\alpha(\mu) = \{x \in X : \mu(x) > \alpha\} \text{ and } \mathrm{W}_\alpha(\mu) = \{x \in X : \mu(x) \geq \alpha\},$$

respectively. For each complete chain L, the α-level topology and the initial topology ([20]) of an L-topology τ on X are defined as follows:

$$\tau_\alpha = \{\mathrm{S}_\alpha(\mu) \in P(X) : \mu \in \tau\} \text{ and } i(\tau) = \inf\{\tau_\alpha : \alpha \in L_1\},$$

respectively, where inf is the infimum with respect to the finer relation on topologies. On other hand if (X, T) is ordinary topological space, then the induced L-topology on X is defined by Lowen in [20] as follows:

$$\omega(T) = \{\mu \in L^X : \quad \mathrm{S}_\alpha(\mu) \in T \text{ for all } \alpha \in L_1\}.$$

Lowen in [20], show that ω and i are functors in special case of $L = I$. The topological L-space (X, τ) and also τ are said to be stratified provided $\overline{\alpha} \in \tau$ holds for all $\alpha \in L$, that is, all constant L-sets are open ([20]). Denote by **L-Top** and **Top** to the categories of all L-topological spaces and all ordinary topological spaces, respectively.

Operation on L-sets. In the sequel, let a topological L-space (X, τ) be fixed. By the operation ([18]) on a set X we mean a mapping $\varphi : L^X \to L^X$ such that $\operatorname{int} \mu \leq \mu^\varphi$ holds, for all $\mu \in L^X$, where μ^φ denotes the value of φ at μ. The class of all operations on X will be denoted by $O_{(L^X, \tau)}$. By identity operation on $O_{(L^X, \tau)}$, we mean the operation $1_{L^X} : L^X \to L^X$ such that $1_{L^X}(\mu) = \mu$, for all $\mu \in L^X$. Also by constant operation on $O_{(L^X, \tau)}$ we mean the operation $c_{L^X} : L^X \to L^X$ such that $c_{L^X}(\mu) = \overline{1}$, for all $\mu \in L^X$. If $\leq$ is a partially ordered relation on $O_{(L^X, \tau)}$ defined as follows: $\varphi_1 \leq \varphi_2 \Leftrightarrow \mu^{\varphi_1} \leq \mu^{\varphi_2}$ for all $\mu \in L^X$, then obviously, $O_{(L^X, \tau)}$ is a completely distributive lattice. As an application on this partially ordered relation, the operation $\varphi : L^X \to L^X$ will be called:
(i) Isotone if $\mu \leq \rho$ implies $\mu^\varphi \leq \rho^\varphi$, for all $\mu, \rho \in L^X$.
(ii) Weakly finite intersection preserving (wfip, for short) with respect to $\mathcal{A} \subseteq L^X$ if $\rho \wedge \mu^\varphi \leq (\rho \wedge \mu)^\varphi$ holds, for all $\rho \in \mathcal{A}$ and $\mu \in L^X$.
(iii) Idempotent if $\mu^\varphi = (\mu^\varphi)^\varphi$, for all $\mu \in L^X$.

The operations $\varphi, \psi \in O_{(L^X,\tau)}$ are said to be dual if $\mu^{\psi} = co((co\,\mu)^{\varphi}$ or equivalently $\mu^{\varphi} = co((co\,\mu)^{\psi}$ for all $\mu \in L^X$, where $co\,\mu$ denotes the complementation of μ. The dual operation of $\varphi : L^X \to L^X$ will be denoted by $\tilde{\varphi} : L^X \to L^X$. In the classical case of $L = \{0,1\}$, by the operation on a set X we mean the mapping $\varphi : P(X) \to P(X)$ such that $\text{int} A \subseteq A^{\varphi}$ for all A in the power set $P(X)$. The identity operation on the class of all ordinary operations $O_{(P(X),T)}$ on X will be denoted by $i_{P(X)}$, where $i_{P(X)}(A) = A$ for all $A \in P(X)$.

φ **-Open L- sets**. Let a topological L-space (X,τ) be fixed and $\varphi \in O_{(L^X,\tau)}$. The L-set $\mu : X \to L$ is called φ-open L- set if $\mu \le \mu^{\varphi}$ holds. We will denote the class of all φ-open L- sets on X by $\varphi OF(X)$. The L-set μ is called φ-closed if its complement $co\,\mu$ is φ-open. The two operations $\varphi, \psi \in O_{(L^X,\tau)}$ are equivalent and written $\varphi \sim \psi$ if $\varphi OF(X) = \psi OF(X)$.

$\varphi_{1,2}$ **-interior L- sets**. Let a topological L-space (X,τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the $\varphi_{1,2}$-interior of the L-set $\mu : X \to L$ is the mapping $\varphi_{1,2}.\text{int}\,\mu : X \to L$ defined by:

$$\varphi_{1,2}.\text{int}\,\mu = \bigvee_{\rho \in \varphi_1 OF(X),\, \rho^{\varphi_2} \le \mu} \rho \tag{2.1}$$

$\varphi_{1,2}.\text{int}\,\mu$ is the greatest φ_1-open L-set ρ such that ρ^{φ_2} less than or equal to μ ([1]). The L- set μ is said to be $\varphi_{1,2}$-open if $\mu \le \varphi_{1,2}.\text{int}\,\mu$. The class of all $\varphi_{1,2}$-open L- sets on X will be denoted by $\varphi_{1,2}OF(X)$. The complement $co\,\mu$ of a $\varphi_{1,2}$-open L-subset μ will be called $\varphi_{1,2}$-closed, the class of all $\varphi_{1,2}$-closed L-subsets of X will be denoted by $\varphi_{1,2}CF(X)$. In the classical case of $L = \{0,1\}$, the topological L-space (X,τ) is up to identification by the ordinary topological space (X,T) and $\varphi_{1,2}.\text{int}\,\mu$ is the classical one. Hence, in this case the ordinary subset A of X is $\varphi_{1,2}$-open if $A \subseteq \varphi_{1,2}.\text{int} A$. The complement of a $\varphi_{1,2}$-open subset A of X will be called $\varphi_{1,2}$-closed. The class of all $\varphi_{1,2}$-open and the class of all $\varphi_{1,2}$-closed subsets of X will be denoted by $\varphi_{1,2}O(X)$ and $\varphi_{1,2}C(X)$, respectively. Clearly, F is $\varphi_{1,2}$-closed if and only if $\varphi_{1,2}.\text{cl}_T\, F = F$.

Proposition 2.3 [1] If (X,τ) be a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then, the mapping $\varphi_{1,2}.\text{int}\,\mu : X \to L$ fulfills the following axioms:

(i) If $\varphi_2 \ge 1_{L^X}$, then $\varphi_{1,2}.\text{int}\,\mu \le \mu$ holds.

(ii) $\varphi_{1,2}.\text{int}\,\mu$ is isotone, i.e, if $\mu \le \rho$ then $\varphi_{1,2}.\text{int}\,\mu \le \varphi_{1,2}.\text{int}\,\rho$ holds for all $\mu, \rho \in L^X$.

(iii) $\varphi_{1,2}.\text{int}\,\overline{1} = \overline{1}$.

(iv) If $\varphi_2 \ge 1_{L^X}$ is isotone operation and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $\varphi_{1,2}.\text{int}(\mu \wedge \rho) = \varphi_{1,2}.\text{int}\,\mu \wedge \varphi_{1,2}.\text{int}\,\rho$ for all $\mu, \rho \in L^X$.

(v) If φ_2 is isotone and idempotent operation, then $\varphi_{1,2}.\text{int}\,\mu \le \varphi_{1,2}.\text{int}(\varphi_{1,2}.\text{int}\,\mu)$ holds.

(vi) $\varphi_{1,2}.\text{int}(\bigvee_{i \in I} \mu_i) = \bigvee_{i \in I} \varphi_{1,2}.\text{int}\,\mu_i$ for all $\mu_i \in \varphi_{1,2}OF(X)$.

Proposition 2.4 [1] Let (X,τ) be a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the following are fulfilled:

(i) If $\varphi_2 \geq 1_{L^X}$, then the class $\varphi_{1,2}OF(X)$ of all $\varphi_{1,2}$ -open L-sets on X forms an extended L- topology on X , denoted by $\tau^{\varphi_{1,2}}$ ([15]).

(ii) If $\varphi_2 \geq 1_{L^X}$, then the class $\varphi_{1,2}OF(X)$ of all $\varphi_{1,2}$ -open L-sets on X forms a supra L- topology on X , denoted by $\overline{\tau}^{\varphi_{1,2}}$ ([15]).

(iii) If $\varphi_2 \geq 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $\varphi_{1,2}OF(X)$ is a pre L-topology on X , denoted by $\tau^{\wedge}_{\varphi_{1,2}}$ ([15]).

(iv) If $\varphi_2 \geq 1_{L^X}$ is isotone and idempotent operation and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $\varphi_{1,2}OF(X)$ forms an L- topology on X , denoted by $\tau_{\varphi_{1,2}}$ ([8, 16]).

From Propositions 2.3 and 2.4, if the topological L-space (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then

$$\varphi_{1,2}OF(X) = \{\mu \in L^X \mid \mu \leq \varphi_{1,2}.\text{int}\, \mu\} \tag{2.2}$$

and the following conditions are fulfilled:

(I1) If $\varphi_2 \geq 1_{L^X}$, then $\varphi_{1,2}.\text{int}\, \mu \leq \mu$ holds for all $\mu \in L^X$.

(I2) If $\mu \leq \rho$ then $\varphi_{1,2}.\text{int}\, \mu \leq \varphi_{1,2}.\text{int}\, \rho$ holds for all $\mu, \rho \in L^X$.

(I3) $\varphi_{1,2}.\text{int}\, \overline{1} = \overline{1}$.

(I4) If $\varphi_2 \geq 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $\varphi_{1,2}.\text{int}(\mu \wedge \rho) = \varphi_{1,2}.\text{int}\, \mu \wedge \varphi_{1,2}.\text{int}\, \rho$ for all $\mu, \rho \in L^X$.

(I5) If $\varphi_2 \geq 1_{L^X}$ is isotone and idempotent operation, then $\varphi_{1,2}.\text{int}(\varphi_{1,2}.\text{int}\, \mu) = \varphi_{1,2}.\text{int}\, \mu$ for all $\mu \in L^X$.

Characterized L-spaces. Independently on the L- topologies, the notion of $\varphi_{1,2}$ -interior operator for L- sets can be defined as a mapping $\varphi_{1,2}.\text{int} : L^X \to L^X$ which fulfills (I1) to (I5). It is well-known that (2.1) and (2.2) give a one-to-one correspondence between the class of all $\varphi_{1,2}$ -open L- sets and these operators, that is, $\varphi_{1,2}OF(X)$ can be characterized by $\varphi_{1,2}$ -interior operators. In this case the pair $(X, \varphi_{1,2}.\text{int})$ as will as the pair $(X, \varphi_{1,2}OF(X))$ will be called characterized L- space ([1]) of $\varphi_{1,2}$ -open L- subsets of X . If $(X, \varphi_{1,2}.\text{int})$ and $(X, \psi_{1,2}.\text{int})$ are two characterized L-spaces, then $(X, \varphi_{1,2}.\text{int})$ is said to be finer than $(X, \psi_{1,2}.\text{int})$ and denoted by $\varphi_{1,2}.\text{int} \leq \psi_{1,2}.\text{int}$ provided $\varphi_{1,2}.\text{int}\, \mu \geq \psi_{1,2}.\text{int}\, \mu$ holds for all $\mu \in L^X$. The characterized L-space $(X, \varphi_{1,2}.\text{int})$ of all $\varphi_{1,2}$ -open L-sets is said to be stratified if and only if $\varphi_{1,2}.\text{int}\, \overline{\alpha} = \overline{\alpha}$ for all $\alpha \in L$. As shown in [1], the characterized L-space $(X, \varphi_{1,2}.\text{int})$ is stratified if the related L- topology is stratified. Moreover, the characterized L-space $(X, \varphi_{1,2}.\text{int})$ is said to have the weak infimum property ([15]) provided for all $\mu \in L^X$ and $\alpha \in L$. The characterized L-space $(X, \varphi_{1,2}.\text{int})$ is said to be strongly stratified ([15]) provided $\varphi_{1,2}.\text{int}$ is stratified and have the weak infimum property.

If $\varphi_1 = \text{int}$ and $\varphi_2 = 1_{L^X}$, then the class $\varphi_{1,2}OF(X)$ of all $\varphi_{1,2}$ -open L-set of X coincide with τ which is defined in [8, 16] and hence the characterized L- space $(X, \varphi_{1,2}.\text{int})$ coincide with the topological L-space (X, τ) .

$\varphi_{1,2}$ **L-neighborhood filters**. An important notion in the characterized L-space $(X, \varphi_{1,2}.\text{int})$ is that of a $\varphi_{1,2}$ L-neighborhood filter at the point and at the ordinary subset in this space. Let (X, τ) be a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. As follows by (I1) to (I5) for each $x \in X$, the mapping $\mathcal{N}_{\varphi_{1,2}}(x): L^X \to L$ which is defined by:

$$\mathcal{N}_{\varphi_{1,2}}(x)(\mu) = (\varphi_{1,2}.\text{int}\, \mu)(x) \tag{2.3}$$

for all $\mu \in L^X$ is L-filter, called $\varphi_{1,2}$ L-neighborhood filter at x ([1]). If $\varphi \neq F \subseteq P(X)$, then the $\varphi_{1,2}$ L-neighborhood filter at F will be denoted by $\mathcal{N}_{\varphi_{1,2}}(F)$ and it will be defined by:

$$\mathcal{N}_{\varphi_{1,2}}(F) = \bigvee_{x \in F} \mathcal{N}_{\varphi_{1,2}}(x).$$

Since $\mathcal{N}_{\varphi_{1,2}}(x)$ is L-filter for all $x \in X$, then $\mathcal{N}_{\varphi_{1,2}}(F)$ is also L-filter on X. Moreover, because of $[\chi_F] = \bigvee_{x \in F} \dot{x}$, then we have $\mathcal{N}_{\varphi_{1,2}}(F) \geq [\chi_F]$ holds.

If the related $\varphi_{1,2}$-interior operator fulfill the axioms (I1) and (I2) only, then the mapping $\mathcal{N}_{\varphi_{1,2}}(x): L^X \to L$, which is defined by (2.3) is an L-stack ([15]), called $\varphi_{1,2}$ L- neighborhood stack at x. Moreover, if the $\varphi_{1,2}$-interior operator fulfill the axioms (I1), (I2) and (I4) such that in (I4) instead of $\rho \in L^X$ we take $\overline{\alpha}$, then the mapping $\mathcal{N}_{\varphi_{1,2}}(x): L^X \to L$, is an L-stack with the cutting property, called here $\varphi_{1,2}$ L- neighborhood stack with the cutting property at x. Obviously, the $\varphi_{1,2}$ L-neighborhood filters fulfill the following axioms:

(N1) $\dot{x} \leq \mathcal{N}_{\varphi_{1,2}}(x)$ holds for all $x \in X$.

(N2) $\mathcal{N}_{\varphi_{1,2}}(x)(\mu) \leq \mathcal{N}_{\varphi_{1,2}}(x)(\rho)$ holds for all $\mu, \rho \in L^X$ and $\mu \leq \rho$.

(N3) $\mathcal{N}_{\varphi_{1,2}}(x)(y \mapsto \mathcal{N}_{\varphi_{1,2}}(y)(\mu)) = \mathcal{N}_{\varphi_{1,2}}(x)(\mu)$, for all $x \in X$ and $\mu \in L^X$.

Clearly, $y \mapsto \mathcal{N}_{\varphi_{1,2}}(y)(\mu)$ is the L-set $\varphi_{1,2}.\text{int}\, \mu$.

The characterized L-space $(X, \varphi_{1,2}.\text{int})$ of all $\varphi_{1,2}$-open L-subsets of a set X is characterized as a filter pre L-topology ([1]), that is, as a mapping $\mathcal{N}_{\varphi_{1,2}}(x): X \to \mathcal{F}_L X$ such that the axioms (N1) to (N3) are fulfilled.

$\varphi_{1,2}\alpha$ **L-neighborhoods.** Let (X, τ) be a topological L-spaces and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then for each $\alpha \in L_0$ and each $x \in X$, the L-set $\mu \in L^X$ will be called $\varphi_{1,2}\ \alpha$ L-neighborhood at x if $\alpha \leq (\varphi_{1,2}.\text{int}\, \mu)(x)$ holds. Because of Proposition 2.1, the L-set $\mu \in L^X$ is $\varphi_{1,2}\ \alpha$ L-neighborhood at x if and only if $\mu \in \alpha\text{-pr}\, \mathcal{N}_{\varphi_{1,2}}(x)$, where $\mathcal{N}_{\varphi_{1,2}}(x)$ be given by (2.3). For each $\alpha \in L_0$ and each $x \in X$ let $N_\alpha(x)$ be the set of all $\varphi_{1,2}\ \alpha$ L-neighborhood at the point x, that is, $N_\alpha(x) = \{\mu \in L^X : \alpha \leq (\varphi_{1,2}.\text{int}\, \mu)(x)\}$, then the family $(N_\alpha(x))_{\alpha \in L_0}$ is the large valued L-filter base of $\mathcal{N}_{\varphi_{1,2}}(x)$.

$\varphi_{1,2}$ **L-convergence**. Let a topological L-spaces (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. If x is a point in the characterized L-space $(X, \varphi_{1,2}.\text{int})$, $F \subseteq X$ and $\mathcal{M}$ is L-filter on X. Then $\mathcal{M}$ is said to be $\varphi_{1,2}$ L-convergence ([1]) to x and written $\mathcal{M} \xrightarrow[\varphi_{1,2}.\text{int}]{} x$, provided $\mathcal{M}$ is finer than the $\varphi_{1,2}$- neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(x)$. Moreover, $\mathcal{M}$ is said to be $\varphi_{1,2}$-convergence to F and written $\mathcal{M} \xrightarrow[\varphi_{1,2}.\text{int}]{} F$, provided $\mathcal{M}$ is finer than the $\varphi_{1,2}$ L-neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(x)$ for all $x \in F$, that is, $\mathcal{M}$ is finer than the $\varphi_{1,2}$ L-neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(F)$.

$\varphi_{1,2}$-closure L-sets and $\varphi_{1,2}$-closure operators. Let a topological L-space (X,τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. The $\varphi_{1,2}$-closure of the L-set $\mu : X \to L$ is the mapping $\varphi_{1,2}.\text{cl}\,\mu : X \to L$ defined by:

$$(\varphi_{1,2}.\text{cl}\,\mu)(x) = \bigvee_{\mathcal{M} \le \mathcal{N}_{\varphi_{1,2}}(x)} \mathcal{M}(\mu)$$

for all $x \in X$. The L-filter $\mathcal{M}$ my have additional properties, e.g, we may assume that is homogeneous or even that is ultra. Obviously, $\varphi_{1,2}.\text{cl}\,\mu \ge \mu$ holds for all $\mu \in L^X$.The mapping $\varphi_{1,2}.\text{cl} : \mathcal{F}_L X \to \mathcal{F}_L X$ which assigns $\varphi_{1,2}.\text{cl}\,\mathcal{M}$ to each L-filter $\mathcal{M}$ on X, that is,

$$(\varphi_{1,2}.\text{cl}\,\mathcal{M})(\mu) = \bigvee_{\varphi_{1,2}.\text{cl}\,\rho \le \mu} \mathcal{M}(\rho)$$

is called $\varphi_{1,2}$-closure operator ([3]) of the characterized L-space $(X, \varphi_{1,2}.\text{int})$ with respect to the related L-topology τ. Obviously, the $\varphi_{1,2}$-closure operator $\varphi_{1,2}.\text{cl}$ is isotone hull operator, that is, for all $\mathcal{M}, \mathcal{N} \in \mathcal{F}_L X$ we have

$$\mathcal{M} \le \mathcal{N} \quad \text{implies} \quad \varphi_{1,2}.\text{cl}\,\mathcal{M} \le \varphi_{1,2}.\text{cl}\,\mathcal{N}$$

and that $\mathcal{M} \le \varphi_{1,2}.\text{cl}\,\mathcal{M}$.

$\varphi_{1,2}\ \psi_{1,2}$ L-continuous and $\varphi_{1,2}\ \psi_{1,2}$ L-open mappings. In the following let a topological L-spaces (X,τ_1) and (Y,τ_2) are fixed, $\varphi_1, \varphi_2 \in O_{(L^X,\tau_1)}$ and $\psi_1, \psi_2 \in O_{(L^Y,\tau_2)}$. A mapping $f : (X, \varphi_{1,2}.\text{int}) \to (Y, \psi_{1,2}.\text{int})$ is said to be $\varphi_{1,2}\ \psi_{1,2}$ L-continuous ([1]) if and only if

$$(\psi_{1,2}.\text{int}\,\eta) \circ f \le \varphi_{1,2}.\text{int}\,(\eta \circ f) \tag{2.4}$$

holds for all $\eta \in L^Y$. If an order reversing involution $\alpha \mapsto \alpha'$ of L is given, then we have that f is $\varphi_{1,2}\ \psi_{1,2}$ L-continuous if and only if $\varphi_{1,2}.\text{cl}(\eta \circ f) \le (\psi_{1,2}.\text{cl}\,\eta) \circ f$ for all $\eta \in L^Y$, where $\varphi_{1,2}.\text{cl}$ and $\psi_{1,2}.\text{cl}$ are the closure operators related to $\varphi_{1,2}.\text{int}$ and $\psi_{1,2}.\text{int}$, respectively. Obviously if f is $\varphi_{1,2}\ \psi_{1,2}$ L-continuity mapping, then the inverse mapping $f^{-1} : (Y, \psi_{1,2}.\text{int}) \to (X, \varphi_{1,2}.\text{int})$ is $\psi_{1,2}\ \varphi_{1,2}$ L-continuous mapping, that is, $(\varphi_{1,2}..\text{int}\,\mu) \circ f^{-1} \le \psi_{1,2}.\text{int}\,(\mu \circ f^{-1})$ holds for all $\mu \in L^X$. By means of the $\varphi_{1,2}$ L-neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(x)$ of $\varphi_{1,2}.\text{int}$ at x and the $\psi_{1,2}$ L-neighborhood filter $\mathcal{N}_{\psi_{1,2}}(x)$ of $\psi_{1,2}.\text{int}$ at x, the $\varphi_{1,2}\ \psi_{1,2}$ L-continuity of f is also characterized as follows:

A mapping $f : (X, \varphi_{1,2}.\text{int}) \to (Y, \psi_{1,2}.\text{int})$ is $\varphi_{1,2}\ \psi_{1,2}$ L-continuous if for each $x \in X$ the inequality

$$\mathcal{N}_{\psi_{1,2}}(f(x)) \ge \mathcal{F}_L f\,(\mathcal{N}_{\varphi_{1,2}}(x)) \tag{2.5}$$

holds. Obviously, in case of $L = \{0,1\}$, $\varphi_1 = \psi_1 = \text{int}$, $\varphi_2 = 1_{L^X}$ and $\psi_2 = 1_{L^Y}$ the $\varphi_{1,2}\ \psi_{1,2}$ L-continuity of f is coincides with the usual continuity.

Proposition 2.5 [1] Let $f : (X, \varphi_{1,2}.\text{int}) \to (Y, \psi_{1,2}.\text{int})$ be a mapping between the characterized L-spaces $(X, \varphi_{1,2}.\text{int})$ and $(Y, \psi_{1,2}.\text{int})$. Then the following are equivalent:

(1) f is $\varphi_{1,2}\ \psi_{1,2}$ L-continuous.

(2) For each L-filter $\mathcal{M}$ on X and each $x \in X$ such that $\mathcal{M} \xrightarrow[\varphi_{1,2}.\mathrm{int}]{} x$ we have $\mathcal{F}_L f(\mathcal{M}) \xrightarrow[\psi_{1,2}.\mathrm{int}]{} f(x)$.

(3) For each $x \in X$, $\alpha \in L_0$ and $\psi_{1,2}\ \alpha$ L-neighborhood η at $f(x)$, we have $\eta \circ f$ is an $\varphi_{1,2}\ \alpha$ L-neighborhood at x.

(4) $f^{-1}(\eta) \in \varphi_{1,2}OF(X)$ for all $\eta \in \beta_{\psi_{1,2}.\mathrm{int}}$, where $\beta_{\psi_{1,2}.\mathrm{int}}$ is a base of $(Y, \psi_{1,2}.\mathrm{int})$.

We will denoted by **CRL-Sp, SCRL-Sp** and **CR-Sp** to the categories of all characterized L- spaces, stratified characterized L- spaces and the ordinary characterized spaces with the $\varphi_{1,2}\ \psi_{1,2}$ L-continuity and $\varphi_{1,2}\ \psi_{1,2}$-continuity as a morphismes between them, respectively. The objects in these categories are characterized L-spaces, stratified characterized L-spaces and characterizet spaces and will be dented by $(X, \varphi_{1,2}.\mathrm{int})$, $(X, \varphi_{1,2}.\mathrm{int}^S)$ and $(X, \varphi_{1,2}.\mathrm{int}^O)$ respectively.

The mapping $f : (X, \varphi_{1,2}.\mathrm{int}) \to (Y, \psi_{1,2}.\mathrm{int})$ is said to be $\varphi_{1,2}\ \psi_{1,2}$ L-open if and only if

$$f \circ (\varphi_{1,2}.\mathrm{int}\,\mu) \circ f \leq \psi_{1,2}.\mathrm{int}(f \circ \mu) \tag{2.6}$$

holds for all $\mu \in L^X$. If an order reversing involution $\alpha \mapsto \alpha'$ of L is given, then we have that f is $\varphi_{1,2}\ \psi_{1,2}$ L-open if and only if $\varphi_{1,2}.\mathrm{cl}(f \circ \mu) \leq f \circ (\psi_{1,2}.\mathrm{cl}\,\mu)$ for all $\mu \in L^X$. The mapping $f : (X, \varphi_{1,2}.\mathrm{int}) \to (Y, \psi_{1,2}.\mathrm{int})$ is said to be $\varphi_{1,2}\ \psi_{1,2}$ L-homeomorphism if and only if it is bijective $\varphi_{1,2}\ \psi_{1,2}$ L-continuous and $\varphi_{1,2}\ \psi_{1,2}$ L-open mapping.

Proposition 2.6 Let $f : (X, \varphi_{1,2}.\mathrm{int}) \to (Y, \psi_{1,2}.\mathrm{int})$ be a mapping between the characterized L-spaces $(X, \varphi_{1,2}.\mathrm{int})$ and $(Y, \psi_{1,2}.\mathrm{int})$. Then the following are equivalent:

(1) f is $\varphi_{1,2}\ \psi_{1,2}$ L-open.

(2) For each L-filter $\mathcal{N}$ on Y and each $y \in Y$ such that $\mathcal{N} \xrightarrow[\psi_{1,2}.\mathrm{int}]{} y$ we have $\mathcal{F}_L^- f(\mathcal{N}) \xrightarrow[\varphi_{1,2}.\mathrm{int}]{} f^{-1}(y)$, where $\mathcal{F}_L^- f(\mathcal{N})$ is the preimage of $\mathcal{N}$.

(3) For each $y \in Y$, $\alpha \in L_0$ and $\varphi_{1,2}\ \alpha$ L-neighborhood μ at $f^{-1}(y)$, we have $\mu \circ f^{-1}$ is an $\psi_{1,2}\ \alpha$ L-neighborhood at y.

(4) $f(\mu) \in \psi_{1,2}OF(Y)$ for all $\mu \in \beta_{\varphi_{1,2}.\mathrm{int}}$, where $\beta_{\varphi_{1,2}.\mathrm{int}}$ is a base of $(X, \varphi_{1,2}.\mathrm{int})$.

Proof. Similar to the proof of Proposition 3.2 in [1]. □

3. Special functors and initial characterized L-spaces

At first we are going to introduce and study some new types of functors between the categories **CRL-Sp, SCRL-Sp** and **CR-Sp** of all characterized L-spaces, stratified characterized L-spaces and characterized spaces. We make at first the relation between the farness on L-sets and the finer relation between characterized spaces. So, we define the α-level and initial characterized spaces for an L-topological space (X, τ) by means of the functors ω and i. For an ordinary topological space (X, T), the induced characterized L-space is also introduced by applied the functor ω. The functors ω and i are extended for any complete distributive lattice L to the functors functors ω_L and i_L. We further notions related to the notion of characterized L-spaces are e.g. those of characterized L-subspace and characterized product L-space are investigated as special cases from the notions of initial characterized L-spaces. By the initial left in **CRL-Sp** we show that the category **CRL-Sp** is topological category in sense of [1,23] and it is also complete and co-complete category, that is, all limits and all co-limits in **CRL-Sp** exist. Finally, we show that the category **SCRL-Sp** is bireflective subcategory of the category **CRL-Sp** and it is also topological.

Some functors. Let (X,τ) be a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the α-level characterized space and the initial characterized space of (X,τ) are denoted by $(X,\varphi_{1,2}.\mathrm{int}_\alpha)$ and $(X,\varphi_{1,2}.\mathrm{int}_i)$, respectively where $\varphi_{1,2}.\mathrm{int}_\alpha$ and $\varphi_{1,2}.\mathrm{int}_i$ are the $\varphi_{1,2}$-interior operators generate the two classes $(\varphi_{1,2}OF(X))_\alpha$ and $i(\varphi_{1,2}OF(X))$ which are defined as follows:

$$(\varphi_{1,2}OF(X))_\alpha = \{\mathrm{S}_\alpha(\mu) \in P(X) : \mu \in (\varphi_{1,2}OF(X))\}$$

and

$$i(\varphi_{1,2}OF(X)) = \inf\{(\varphi_{1,2}OF(X))_\alpha : \alpha \in L_1\},$$

respectively, where inf is the infimum with respect to the finer relation on characterized spaces. On other hand if (X,T) is ordinary topological space and $\varphi_1, \varphi_2 \in O_{(P(X),T)}$, then the induced characterized L-space on X will be denoted by $(X,\varphi_{1,2}.\mathrm{int}_\omega)$, where $\varphi_{1,2}.\mathrm{int}_\omega$ is the $\varphi_{1,2}$-interior operator generates the class $\omega(\varphi_{1,2}O(X))$ defined as follows:

$$\omega(\varphi_{1,2}O(X)) = \{\mu \in L^X : \quad \mathrm{S}_\alpha(\mu) \in \varphi_{1,2}O(X) \text{ for all } \alpha \in L_1\}.$$

ω and i are functors in sense of Lowen in [20] in special case of $L = I$. These functors can be extended for any completely distributive complete lattice L as follows:

Let (X,τ) be a topological L-space, $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$ and $\psi_1, \psi_2 \in O_{(L^L,T)}$. Then, $(X,\varphi_{1,2}.\mathrm{int}_{i_L})$ is initial characterized space on X and $(X,\varphi_{1,2}.\mathrm{int}_{\omega_L})$ is induced characterized L-space on X. In this case $\varphi_{1,2}.\mathrm{int}_{i_L}$ and $\varphi_{1,2}.\mathrm{int}_{\omega_L}$ are the $\varphi_{1,2}$-interior operators generate the two classes $i_L(\varphi_{1,2}OF(X))$ and $\omega_L(\varphi_{1,2}O(X))$ given by the following forms:

$$i_L(\varphi_{1,2}OF(X)) = \inf\{\mu^{-1}\big(\mathrm{UP}(\psi_{1,2}OF(L)\big) : \mu \in \varphi_{1,2}OF(X)\}$$

and

$$\omega_L(\varphi_{1,2}O(X)) = \ << C\big((X,\varphi_{1,2}O(X)), (L, \mathrm{UP}(\psi_{1,2}OF(L))\big) >>$$

where, $C\big((X,\varphi_{1,2}O(X)), (L,\mathrm{UP}(\psi_{1,2}OF(L))\big)$ is the $\varphi_{1,2}\ \psi_{1,2}$-continuous mappings between $(X,\varphi_{1,2}O(X))$ and $(L,\mathrm{UP}(\psi_{1,2}OF(L)))$ and $\mathrm{UP}(\psi_{1,2}OF(L))$ is the upper $\psi_{1,2}$-open L-set generated by the set $L \setminus \downarrow(\mathrm{a})$ for $\downarrow(\mathrm{a}) = \{x \in L : x \le \mathrm{a}\}$. If $\varphi_1 = \mathrm{int}$ and $\varphi_2 = 1_{L^X}$, then the initial characterized space $(X,\varphi_{1,2}.\mathrm{int}_{i_L})$ and the induced characterized L-space $(X,\varphi_{1,2}.\mathrm{int}_{\omega_L})$ are coincide with the initial topological space $(X, i(\tau))$ and the induced topological L-space $(X,\omega(T))$ which is defined in [7]. Other special choices for the operations φ_1 and φ_2 obtained in Table (1).

The mappings ω_L and i_L are fulfill the following result:

Proposition 3.1 For each completely distributive complete lattice L, the mapping ω_L : **CR - Sp** $\rightarrow$ **CRL - Sp** which assignment each object $(X,\varphi_{1,2}.\mathrm{int}^{\mathrm{O}})$ to the object $(X,\varphi_{1,2}.\mathrm{int}_{\omega_L})$ and which leaves mappings unchanged is concrete functor and the mapping i_L: **CRL - Sp** $\rightarrow$ **CR - Sp** which preserves mappings too, and assignment each object $(X,\varphi_{1,2}.\mathrm{int})$ to the object $(X,\varphi_{1,2}.\mathrm{int}_{i_L})$ is also concrete functor.

Proof. Similar to the proof of Proposition 3.3 in [1]. □

In the following we show that the category **SCRL-Sp** is bireflective subcategory of the category **CRL-Sp**.

Proposition 3.2 Let $f:(X,\varphi_{1,2}.\text{int})\to(Y,\psi_{1,2}.\text{int})$ be an $\varphi_{1,2}\psi_{1,2}$ L-continuous mapping from a stratified characterized L-space $(X,\varphi_{1,2}.\text{int})$ into a characterized L-space $(Y,\psi_{1,2}.\text{int})$. Then the mapping $f:(X,\varphi_{1,2}.\text{int})\to(Y,\psi_{1,2}.\text{int}^{S})$ is also $\varphi_{1,2}\psi_{1,2}$ L-continuous, where $\psi_{1,2}.\text{int}^{S}\,\eta=\psi_{1,2}.\text{int}\,\eta\vee\overline{\inf\eta}$ for all $\eta\in L^{Y}$ is the coarsest stratified $\psi_{1,2}$ -interior operator finer than $\psi_{1,2}.\text{int}$.

Proof. Consider $\eta\in L^{Y}$. Since $\inf\eta\le\inf(\eta\circ f)$ holds for all $\eta\in L^{Y}$, then by using the definition of $\psi_{1,2}.\text{int}^{S}$ it follows that $(\psi_{1,2}.\text{int}^{S}\,\eta)\circ f\le\varphi_{1,2}.\text{int}(\eta\circ f)\vee\overline{\inf(\eta\circ f)}$ holds. Because of the $\varphi_{1,2}\psi_{1,2}$ L-continuity of f we have $(\psi_{1,2}.\text{int}\,\eta)\circ f\le\varphi_{1,2}.\text{int}(\eta\circ f)$ and therefore $(\psi_{1,2}.\text{int}^{S}\,\eta)\circ f\le\varphi_{1,2}.\text{int}(\eta\circ f)\vee\overline{\inf(\eta\circ f)}$. Since $(X,\varphi_{1,2}.\text{int})$ is stratified, then we have $\overline{\inf(\eta\circ f)}\le\varphi_{1,2}.\text{int}(\eta\circ f)$ and therefore $(\psi_{1,2}.\text{int}^{S}\,\eta)\circ f\le\varphi_{1,2}.\text{int}(\eta\circ f)$ holds for all $\eta\in L^{Y}$. Thus, $f:(X,\varphi_{1,2}.\text{int})\to(Y,\psi_{1,2}.\text{int}^{S})$ is $\varphi_{1,2}\psi_{1,2}$ L-continuous. □

Corollary 3.1 The category **SCRL-Sp** is bireflective subcategory of **CRL-Sp**, and for each object $(X,\varphi_{1,2}.\text{int})$ of **CRL-Sp** the $\varphi_{1,2}\psi_{1,2}$ L-continuous mapping 1_{X} from the stratification $(X,\varphi_{1,2}.\text{int}^{S})$ of $(X,\varphi_{1,2}.\text{int})$ into $(X,\varphi_{1,2}.\text{int})$ is bi-coreflection of $(X,\varphi_{1,2}.\text{int})$.

Proof. Immediate from Proposition 3.2. □

Proposition 3.3 For each completely distributive complete lattice L, the mapping $S_{L}:\textbf{CRL-Sp}\to\textbf{SCRL-Sp}$ which assignment each object $(X,\varphi_{1,2}.\text{int})$ to the object $(X,\varphi_{1,2}.\text{int}^{O})$ and which leaves mappings unchanged is concrete functor.

Proof. Easy consequences from Proposition 3.2. □

Initial characterized L-spaces. Consider a family of characterized L-spaces $\big((X_i,\psi_{1,2}.\text{int}_i)\big)_{i\in I}$ and for each $i\in I$, let $f_i:X\to X_i$ be a mapping from X into X_i. By an initial characterized L-space of the family $\big((X_i,\psi_{1,2}.\text{int}_i)\big)_{i\in I}$ with respect to $(f_i)_{i\in I}$, we mean the characterized L-space $(X,\varphi_{1,2}.\text{int})$ for which the following conditions are fulfilled:

1. All the mappings $f_i:(X,\varphi_{1,2}.\text{int})\to(X_i,\psi_{1,2}.\text{int}_i)$ are $\varphi_{1,2}\,\psi_{1,2}$ L-continuous.
2. For an characterized L-space $(Y,\delta_{1,2}.\text{int})$ and a mapping $f:Y\to X$, the mapping $f:(Y,\delta_{1,2}.\text{int})\to(X,\varphi_{1,2}.\text{int})$ is $\delta_{1,2}\,\varphi_{1,2}$ L- continuous if all the mappings $f_i\circ f:(Y,\delta_{1,2}.\text{int})\to(X_i,\psi_{1,2}.\text{int}_i)$ are $\delta_{1,2}\,\psi_{1,2}$ L-continuous for all $i\in I$,

(See Fig. 3.1)

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ & {\scriptstyle f_i\circ f}\searrow & \downarrow{\scriptstyle f_i} \\ & & X_i \end{array}$$

Fig. 3.1

In the following proposition we show that the initial characterized L-space $(X, \varphi_{1,2}.\mathrm{int})$ for a family $((X_i, \psi_{1,2}.\mathrm{int}_i))_{i\in I}$ of characterized L-spaces with respect to the family $(f_i)_{i\in I}$ of mappings exists and will be defined.

Proposition 3.4 The initial characterized L-space $(X, \varphi_{1,2}.\mathrm{int})$ for the family of characterized L-spaces $((X_i, \psi_{1,2}.\mathrm{int}_i))_{i\in I}$ with respect to the family of mappings $(f_i)_{i\in I}$ always exists and it is given by:

$$\varphi_{1,2}.\mathrm{int}\,\mu = \bigvee_{\mu_i\circ f_i\le\mu, i\in I} (\psi_{1,2}.\mathrm{int}_i\ \mu_i)\circ f_i \tag{3.1}$$

for all $\mu \in L^X$.

Proof. Let $\varphi_{1,2}.\mathrm{int}$ be the operator defined (3.1). For each $\mu \in L^X$ and for all $\mu_i \in L^{X_i}, i\in I$ with $\mu_i \circ f_i \le \mu$ we have $(\psi_{1,2}.\mathrm{int}_i\ \mu_i)\circ f_i \le \mu$ and therefore $\varphi_{1,2}.\mathrm{int}\,\mu \le \mu$. Hence, $\varphi_{1,2}.\mathrm{int}$ fulfills condition (I1). For condition (I2), let $\mu,\eta \in L^X$ with $\mu\le\eta$, then $\mu_i\circ f_i \le \mu$ implies $\mu_i \circ f_i \le \eta$ and therefore $\varphi_{1,2}.\mathrm{int}\,\mu \le \bigvee_{\mu_i\circ f_i\le\eta, i\in I}(\psi_{1,2}.\mathrm{int}_i\ \mu_i)\circ f_i \le \bigvee_{\eta_i\circ f_i\le\eta, i\in I}(\psi_{1,2}.\mathrm{int}_i\ \eta_i)\circ f_i = \varphi_{1,2}.\mathrm{int}\,\eta$. Thus, condition (I2) is fulfilled. For all $\mu_i \in L^{X_i}, i\in I$ with $\mu_i\circ f_i \le \overline{1}$ we have $(\psi_{1,2}.\mathrm{int}_i\ \mu_i)\circ f_i \le \overline{1}$ and therefore $\varphi_{1,2}.\mathrm{int}\,\overline{1} = \overline{1}$. Hence, $\varphi_{1,2}.\mathrm{int}$ fulfill condition (I3). Now, let $\mu,\eta\in L^X$ and $\mu_i,\eta_i \in L^{X_i}, i\in I$ such that $\mu_i\circ f_i\le\mu$ and $\eta_i\circ f_i\le\eta$. Then from the distributivety of L, we have

$$\varphi_{1,2}.\mathrm{int}\,\mu \wedge \varphi_{1,2}.\mathrm{int}\,\eta = \bigvee_{\substack{\mu_i\circ f_i\le\mu,\\ \eta_i\circ f_i\le\eta, i\in I}} \left((\psi_{1,2}.\mathrm{int}_i\ \mu_i)\circ f_i \wedge (\psi_{1,2}.\mathrm{int}_i\ \eta_i)\circ f_i\right)$$

$$\le \bigvee_{\substack{\mu_i\circ f_i\le\mu,\\ \eta_i\circ f_i\le\eta, i\in I}} \left(\psi_{1,2}.\mathrm{int}_i\,(\mu_i\wedge\eta_i)\circ f_i\right) = \varphi_{1,2}.\mathrm{int}(\mu\wedge\eta).$$

Thus, $\varphi_{1,2}.\mathrm{int}$ fulfills condition (I4). Clearly, $\varphi_{1,2}.\mathrm{int}$ is idempotent, that is, condition (I5) is fulfilled. Hence, $(X, \varphi_{1,2}.\mathrm{int})$ is characterized L-space. Since for all $i\in I$ and $\mu_i\in L^{X_i}$ we have $(\psi_{1,2}.\mathrm{int}_i\ \mu_i)\circ f_i \le \varphi_{1,2}.\mathrm{int}(\mu_i\circ f_i)$, then all the mappings $f_i : (X, \varphi_{1,2}.\mathrm{int}) \to (X_i, \psi_{1,2}.\mathrm{int}_i)$ are $\varphi_{1,2}\ \psi_{1,2}$ L-continuous. Hence, condition (1) is fulfilled.

Now, let $(Y, \delta_{1,2}.\mathrm{int})$ is a characterized L-space and $f : Y \to X$ be a mapping such that the mappings $f_i\circ f : (Y, \delta_{1,2}.\mathrm{int}) \to (X_i, \psi_{1,2}.\mathrm{int}_i)$ are $\delta_{1,2}\ \psi_{1,2}$ L-continuous for all $i\in I$. Then, $(\psi_{1,2}.\mathrm{int}_i\ \mu_i)\circ(f_i\circ f) \le \delta_{1,2}.\mathrm{int}(\mu_i\circ f_i\circ f)$ holds for all $\mu_i\in L^{X_i}$, therefore because of (3.1) we have that $(\varphi_{1,2}.\mathrm{int}\,\mu)\circ f = \bigvee_{\mu_i\circ f_i\le\mu, i\in I}(\psi_{1,2}.\mathrm{int}_i\ \mu_i)\circ(f_i\circ f) \le \bigvee_{\mu_i\circ f_i\le\mu, i\in I}\delta_{1,2}.\mathrm{int}(\mu_i\circ f_i\circ f) \le \delta_{1,2}.\mathrm{int}(\mu\circ f)$ holds for all $\mu\in L^X$. Hence, the mapping $f : (Y, \delta_{1,2}.\mathrm{int}) \to (X, \varphi_{1,2}.\mathrm{int})$ is $\delta_{1,2}\ \varphi_{1,2}$ L- continuous, that is, condition (2) is also fulfilled. Consequently, $(X, \varphi_{1,2}.\mathrm{int})$ is initial characterized L-space of the family $((X_i, \psi_{1,2}.\mathrm{int}_i))_{i\in I}$ of characterized L-spaces with respect to $(f_i)_{i\in I}$. □

Because of Proposition 3.4, all the initial lefts and all the initial characterized L-spaces are exist uniquely in the category **CRL-Sp** and hence **CRL-Sp** is a topological category over the category **SET** of all sets.

Corollary 3.2 For each $i\in I$, let $\mathcal{N}^i_{\psi_{1,2}} : X \to \mathcal{F}_L X_i$ be the representation of the $\psi_{1,2}$-interior operators for the family $((X_i, \psi_{1,2}.\mathrm{int}_i))_{i\in I}$ as an L-filter pretopology. Then the mapping $\mathcal{N}_{\varphi_{1,2}} : X \to \mathcal{F}_L X$ which is defined by taking:

$$\mathcal{N}_{\varphi_{1,2}}(x)(\mu) = \bigvee_{\mu_i \circ f_i \le \mu, i \in I} \mathcal{N}^i_{\psi_{1,2}}(f_i(x))(\mu_i)$$

for all $x \in X$ and $\mu \in L^X$, is the representation of the $\varphi_{1,2}$ -interior operator for the initial characterized L-space $(X, \varphi_{1,2}.\text{int})$ for the family $((X_i, \psi_{1,2}.\text{int}_i))_{i\in I}$ with respect to the family of mappings $(f_i)_{i\in I}$.

Proposition 3.5 The initial characterized L-space $(X, \varphi_{1,2}.\text{int})$ for the family of characterized L-spaces $((X_i, \psi_{1,2}.\text{int}_i))_{i\in I}$ with respect to the family of mappings $(f_i)_{i\in I}$ is stratified if and only if $(X_i, \psi_{1,2}.\text{int}_i)$ is stratified for some $i \in I$.

Proof. Assume that $(X_j, \psi_{1,2}.\text{int}_j)$ is stratified for $j \in I$. Then because of (3.1), we have that $\varphi_{1,2}.\text{int}\,\overline{\alpha} = \bigvee_{\mu_j \circ f_j \le \overline{\alpha}} (\psi_{1,2}.\text{int}_j\ \mu_j) \circ f_j \ge \bigvee_{\tilde{\alpha}_j \circ f_j \le \overline{\alpha}} (\psi_{1,2}.\text{int}_j\ \tilde{\alpha}_j) \circ f_j = \overline{\alpha}$ holds for all $\alpha \in L$, where $\overline{\alpha}$ and $\tilde{\alpha}_j$ are the constant mappings on X and X_j hose value α and α_j , respectively. Hence, $(X, \varphi_{1,2}.\text{int})$ is stratified.

Conversely, let $(X, \varphi_{1,2}.\text{int})$ is stratified. Then, $\varphi_{1,2}.\text{int}\,\overline{\alpha} = \overline{\alpha}$ for all $\alpha \in L$ and therefore $\bigvee_{\tilde{\alpha}_i \circ f_i \le \overline{\alpha}, i\in I} (\psi_{1,2}.\text{int}_i\ \tilde{\alpha}_i) \circ f_i = \overline{\alpha}$. Hence, there is $j \in I$ such that $\psi_{1,2}.\text{int}_j\ \tilde{\alpha}_j \ge \overline{\alpha}$ and $\overline{\alpha} \ge \tilde{\alpha}_j \circ f \ge \tilde{\alpha}_j$, therefore $\psi_{1,2}.\text{int}_j\ \tilde{\alpha}_j = \tilde{\alpha}_j$ for some $j \in I$. Hence, $(X_j, \psi_{1,2}.\text{int}_j)$ is stratified for $j \in I$. □

In the following we consider some special cases from the initial characterized L-spaces such as characterized L-subspaces and characterized product L-spaces.

Characterized L-subspaces. Let A be non-empty subset of a characterized L-space $(X, \varphi_{1,2}.\text{int})$ and $i_A : A \to X$ be the inclusion mapping of A into X . Then the mapping $\varphi_{1,2}.\text{int}_A : L^A \to L^A$ which is defined by:

$$\varphi_{1,2}.\text{int}_A\ \sigma = \bigvee_{\mu \circ i_A \le \sigma} (\varphi_{1,2}.\text{int}\,\mu) \circ i_A \tag{3.2}$$

for all $\sigma \in L^A$ is initial $\varphi_{1,2}$ -operator of $\varphi_{1,2}.\text{int}$ with respect to the inclusion mapping $i_A : A \to X$, called the induced $\varphi_{1,2}$ -operator of $\varphi_{1,2}.\text{int}$ on the subset A of X and $(A, \varphi_{1,2}.\text{int}_A)$ is initial characterized L-space called characterized L-subspace of the characterized L-space $(X, \varphi_{1,2}.\text{int})$.

Proposition 3.6 Let A be non-empty subset of a characterized L-space $(X, \varphi_{1,2}.\text{int})$. Then the characterized L-subspaces $(A, \varphi_{1,2}.\text{int}_A)$ of $(X, \varphi_{1,2}.\text{int})$ always exists and the initial $\varphi_{1,2}$ -operator $\varphi_{1,2}.\text{int}_A$ is given by (3.2). If $(X, \varphi_{1,2}.\text{int})$ is stratified, then $(A, \varphi_{1,2}.\text{int}_A)$ also is.

Proof. Let $\sigma \in L^A$ and $\mu \in L^X$ such that $\mu \circ i_A \le \sigma$ holds, then $(\varphi_{1,2}.\text{int}\,\mu) \circ i_A \le \sigma$ is also holds and therefore $\varphi_{1,2}.\text{int}_A\ \sigma \le \sigma$ holds for all $\sigma \in L^A$. Hence, $\varphi_{1,2}.\text{int}_A$ fulfills condition (I1). For condition (I2), let $\sigma, \eta \in L^A$ with $\sigma \le \eta$, then $\mu \circ i_A \le \sigma$ implies that $\mu \circ i_A \le \eta$ holds for all $\mu \in L^X$ and therefore $\varphi_{1,2}.\text{int}_A\ \sigma = \bigvee_{\mu \circ i_A \le \sigma} (\varphi_{1,2}.\text{int}\,\mu) \circ i_A \le \bigvee_{\mu \circ i_A \le \eta} (\varphi_{1,2}.\text{int}\,\mu) \circ i_A = \varphi_{1,2}.\text{int}_A\ \eta$. Thus, condition (I2) is fulfilled. Since $\varphi_{1,2}.\text{int}\,\overline{1} = \overline{1}$ and $\mu \circ i_A \le \overline{1}$ for all $\mu \in L^X$, then we have

$$\varphi_{1,2}.\text{int}_A\ \overline{1} = \bigvee_{\mu \circ i_A \le \overline{1}} (\varphi_{1,2}.\text{int}\,\mu) \circ i_A \ge \bigvee_{\overline{1} \circ i_A \le \overline{1}} (\varphi_{1,2}.\text{int}\,\overline{1}) \circ i_A = \overline{1}.$$

Hence, $\varphi_{1,2}.\mathrm{int}_A$ fulfills condition (I3). Now, let $\sigma,\eta \in L^A$ and $\mu_1, \mu_2 \in L^X$ such that $\mu_1 \circ i_A \leq \sigma$ and $\mu_2 \circ i_A \leq \eta$. Then from the distributives of L and (3.2), we have that

$$\varphi_{1,2}.\mathrm{int}_A\, \sigma \wedge \varphi_{1,2}.\mathrm{int}_A\, \eta = \bigvee_{\substack{\mu_1\circ i_A \leq \sigma,\\ \mu_2\circ i_A \leq \eta}} \big((\varphi_{1,2}.\mathrm{int}\,\mu_1)\circ i_A \wedge (\varphi_{1,2}.\mathrm{int}\,\mu_2)\circ i_A\big)$$

$$\leq \bigvee_{(\mu_1\wedge\mu_2)\circ i_A \leq \sigma\wedge\eta} \big(\varphi_{1,2}.\mathrm{int}\,(\mu_1\wedge\mu_2)\circ i_A\big) = \varphi_{1,2}.\mathrm{int}_A(\sigma\wedge\eta).$$

Since $\varphi_{1,2}.\mathrm{int}_A$ is isotone, it follows $\varphi_{1,2}.\mathrm{int}_A\, \sigma \wedge \varphi_{1,2}.\mathrm{int}_A\, \eta = \varphi_{1,2}.\mathrm{int}_A(\sigma\wedge\eta)$. Thus, condition (I4) is also fulfilled. Clearly, $\varphi_{1,2}.\mathrm{int}_A$ is idempotent, that is, condition (I5) is fulfilled. Hence, $(A, \varphi_{1,2}.\mathrm{int}_A)$ is characterized L-space. Since for all $\sigma \in L^A$, we have $(\varphi_{1,2}.\mathrm{int}_A\, \sigma)\circ i_A \leq \varphi_{1,2}.\mathrm{int}(\sigma\circ i_A)$, then the mapping $i_A : (A, \varphi_{1,2}.\mathrm{int}_A) \to (X, \varphi_{1,2}.\mathrm{int})$ is $\varphi_{1,2}\ \varphi_{1,2}$ L-continuous. Hence, condition (1) is fulfilled.

Now, let $(Y, \delta_{1,2}.\mathrm{int})$ is a characterized L-space and $f : Y \to A$ be a mapping such that the mappings $i_A \circ f : (Y, \delta_{1,2}.\mathrm{int}) \to (X, \varphi_{1,2}.\mathrm{int})$ is $\delta_{1,2}\ \varphi_{1,2}$ L-continuous. Then, $(\varphi_{1,2}.\mathrm{int}\,\mu)\circ(i_A\circ f) \leq \delta_{1,2}.\mathrm{int}(\mu\circ i_A \circ f)$ holds for all $\mu \in L^X$, therefore because of (3.2) we have that $(\varphi_{1,2}.\mathrm{int}_A\, \sigma)\circ f = \bigvee_{\mu\circ i_A \leq \sigma} (\varphi_{1,2}.\mathrm{int}\,\mu)\circ(i_A\circ f) \leq \bigvee_{\mu\circ i_A \leq \sigma} \delta_{1,2}.\mathrm{int}(\mu\circ i_A\circ f) \leq \delta_{1,2}.\mathrm{int}(\sigma\circ f)$ holds for all $\sigma \in L^A$. Hence, the mapping $f : (Y, \delta_{1,2}.\mathrm{int}) \to (A, \varphi_{1,2}.\mathrm{int}_A)$ is $\delta_{1,2}\ \varphi_{1,2}$ L- continuous, that is, condition (2) is also fulfilled. Consequently, $(A, \varphi_{1,2}.\mathrm{int}_A)$ is initial characterized L-space.

Finally, let $(X, \varphi_{1,2}.\mathrm{int})$ is stratified. Then, $\varphi_{1,2}.\mathrm{int}\,\overline{\alpha} = \overline{\alpha}$ for all $\alpha \in L$ and therefore $\bigvee_{\overline{\alpha}\circ i_A \leq \tilde{\alpha}} (\varphi_{1,2}.\mathrm{int}\,\overline{\alpha})\circ i_A = \tilde{\alpha}$, where $\overline{\alpha}$ and $\tilde{\alpha}$ are the constant mappings on X and A hose value α , respectively. Because of (3.2), we have $\varphi_{1,2}.\mathrm{int}_A\, \tilde{\alpha} = \tilde{\alpha}$ for all $\alpha \in L$. Hence, $(A, \varphi_{1,2}.\mathrm{int}_A)$ is stratified. □

Characterized product L-spaces. Assume that for each $\mathrm{i} \in \mathrm{I}$, $(X_{\mathrm{i}}, \psi_{1,2}.\mathrm{int}_{\mathrm{i}})$ be the characterized L-space of $\psi_{1,2}$-open L -subset of X_{i} . Let X be the cartesian product $\prod_{\mathrm{i}\in\mathrm{I}} X_{\mathrm{i}}$ of the family $(X_{\mathrm{i}})_{\mathrm{i}\in\mathrm{I}}$ and $P_{\mathrm{i}} : X \to X_{\mathrm{i}}$ is the related projection. Then the mapping $\varphi_{1,2}.\mathrm{int} : L^X \to L^X$ which is defined by:

$$\varphi_{1,2}.\mathrm{int}\,\mu = \bigvee_{\mu_{\mathrm{i}}\circ P_{\mathrm{i}} \leq \mu} (\psi_{1,2}.\mathrm{int}_{\mathrm{i}}\, \mu_{\mathrm{i}})\circ P_{\mathrm{i}} \tag{3.3}$$

for all $\mu \in L^X$ is initial $\varphi_{1,2}$-operator of $\psi_{1,2}.\mathrm{int}_{\mathrm{i}}$ with respect to the projection mapping $P_{\mathrm{i}} : X \to X_{\mathrm{i}}$, called the $\varphi_{1,2}$-product operator of the $\psi_{1,2}$-interior operators $\psi_{1,2}.\mathrm{int}_{\mathrm{i}}$ and $(X, \varphi_{1,2}.\mathrm{int})$ is initial characterized L-space called characterized product L-space of the characterized L-spaces $(X_{\mathrm{i}}, \psi_{1,2}.\mathrm{int}_{\mathrm{i}})$ with respect to the family $(P_{\mathrm{i}} : X \to X_{\mathrm{i}})_{\mathrm{i}\in\mathrm{I}}$ of projections and will be denoted by $(\prod_{\mathrm{i}\in\mathrm{I}} X_{\mathrm{i}}, \prod_{\mathrm{i}\in\mathrm{I}} \psi_{1,2}.\mathrm{int}_{\mathrm{i}})$.

Initial lefts in CRL-Sp. For the general notion of initial left we refer the standard books of category theory which include the categorical topology, e.g. [1,23]. The notion of initial left is meant here with respect to the forgetful functor of **CRL-Sp** to **SET**. It can be defined as follows:

The family of one and the same domain $\big(f_{\mathrm{i}} : (X, \varphi_{1,2}.\mathrm{int}) \to (X_{\mathrm{i}}, \psi_{1,2}.\mathrm{int}_{\mathrm{i}})\big)_{\mathrm{i}\in\mathrm{I}}$, where I is any close of orphisms in the category **CRL-Sp** is called initial left of the family $\big(f_{\mathrm{i}} : X \to X_{\mathrm{i}}, \psi_{1,2}.\mathrm{int}_{\mathrm{i}}\big)_{\mathrm{i}\in\mathrm{I}}$ provided for any characterized L-space $(Y, \sigma_{1,2}.\mathrm{int})$ of $\sigma_{1,2}$-open subsets of Y, the mapping $f : (Y, \sigma_{1,2}.\mathrm{int}) \to (X, \varphi_{1,2}.\mathrm{int})$ is $\sigma_{1,2}\ \varphi_{1,2}$ L-continuous if all the compositions $f_{\mathrm{i}} \circ f : (Y, \sigma_{1,2}.\mathrm{int}) \to (X_{\mathrm{i}}, \psi_{1,2}.\mathrm{int}_{\mathrm{i}})$ are $\sigma_{1,2}\ \psi_{1,2}$ L-continuous.

Proposition 3.7 For each family $\left(f_{\mathrm{i}} : X \to X_{\mathrm{i}}, \psi_{1,2}.\mathrm{int}_{\mathrm{i}}\right)_{\mathrm{i}\in \mathrm{I}}$ consisting of the mappings $f_{\mathrm{i}} : X \to X_{\mathrm{i}}$ and of the $\psi_{1,2}$-interior operators $\psi_{1,2}.\mathrm{int}_{\mathrm{i}}$ on the co domains X_{i} of these mappings, the family $\left(f_{\mathrm{i}} : (X, \varphi_{1,2}.\mathrm{int}) \to (X_{\mathrm{i}}, \psi_{1,2}.\mathrm{int}_{\mathrm{i}})\right)_{\mathrm{i}\in \mathrm{I}}$ with the initial $\varphi_{1,2}$-interior operator $\varphi_{1,2}.\mathrm{int} : L^X \to L^X$ of $(\psi_{1,2}.\mathrm{int}_{\mathrm{i}})_{\mathrm{i}\in\mathrm{I}}$ with respect to $(f_{\mathrm{i}})_{\mathrm{i}\in\mathrm{I}}$ defined by (3.1) is an initial left.

Proof. Let a characterized L-space $(Y, \sigma_{1,2}.\mathrm{int})$ of $\sigma_{1,2}$-open subsets of Y and a mapping $f : Y \to X$ be fixed. If all the mappings $f_{\mathrm{i}} \circ f : (Y, \sigma_{1,2}.\mathrm{int}) \to (X_{\mathrm{i}}, \psi_{1,2}.\mathrm{int}_{\mathrm{i}})$ are $\sigma_{1,2}\ \psi_{1,2}$ L-continuous, that is, if $(\psi_{1,2}.\mathrm{int}_{\mathrm{i}}\ \mu_{\mathrm{i}}) \circ (f_{\mathrm{i}} \circ f) \le \sigma_{1,2}.\mathrm{int}(\mu_{\mathrm{i}} \circ f_{\mathrm{i}} \circ f)$ holds for all $\mu_i \in L^{X_i}$, then because of equation (3.1), we have that $(\varphi_{1,2}.\mathrm{int}\,\mu) \circ f = \bigvee_{\mu_i \circ f_i \le \mu, i\in I} (\psi_{1,2}.\mathrm{int}_i\ \mu_i) \circ (f_i \circ f) \le \bigvee_{\mu_i \circ f_i \le \mu, i\in I} \sigma_{1,2}.\mathrm{int}(\mu_i \circ f_i \circ f) \le \sigma_{1,2}.\mathrm{int}(\mu \circ f)$ holds for all $\mu \in L^X$. Hence, the mapping $f : (Y, \sigma_{1,2}.\mathrm{int}) \to (X, \varphi_{1,2}.\mathrm{int})$ is $\sigma_{1,2}\ \varphi_{1,2}$ L- continuous. Thus, the family $\left(f_{\mathrm{i}} : (X, \varphi_{1,2}.\mathrm{int}) \to (X_{\mathrm{i}}, \psi_{1,2}.\mathrm{int}_{\mathrm{i}})\right)_{\mathrm{i}\in\mathrm{I}}$ is an initial left of $(\psi_{1,2}.\mathrm{int}_{\mathrm{i}})_{\mathrm{i}\in\mathrm{I}}$ with respect to $(f_{\mathrm{i}})_{\mathrm{i}\in\mathrm{I}}$. □

Corollary 3.3 Consider $\left((X_{\mathrm{i}}, \psi_{1,2}.\mathrm{int}_{\mathrm{i}})\right)_{\mathrm{i}\in\mathrm{I}}$ and $\left((Y_{\mathrm{i}}, \delta_{1,2}.\mathrm{int}_{\mathrm{i}})\right)_{\mathrm{i}\in\mathrm{I}}$ are two families of characterized L-spaces, $(X, \varphi_{1,2}.\mathrm{int})$ and $(Y, \sigma_{1,2}.\mathrm{int})$ are the related characterized product L-spaces. Foe each $\mathrm{i} \in \mathrm{I}$, let $f_{\mathrm{i}} : (X_{\mathrm{i}}, \psi_{1,2}.\mathrm{int}_{\mathrm{i}}) \to (Y_{\mathrm{i}}, \delta_{1,2}.\mathrm{int}_{\mathrm{i}})$ be an $\psi_{1,2}\ \delta_{1,2}$ L- continuous (resp. $\psi_{1,2}\ \delta_{1,2}$ L- open) mapping, then the product mapping $f = \prod_{\mathrm{i}\in\mathrm{I}} f_{\mathrm{i}} : (X, \varphi_{1,2}.\mathrm{int}) \to (Y, \sigma_{1,2}.\mathrm{int})$ which is defined by $f\left((x_{\mathrm{i}})_{\mathrm{i}\in\mathrm{I}}\right) = \left(f_{\mathrm{i}}(x_{\mathrm{i}})\right)_{\mathrm{i}\in\mathrm{I}}$ for all $(x_{\mathrm{i}})_{\mathrm{i}\in\mathrm{I}} \in X = \prod_{\mathrm{i}\in\mathrm{I}} X_{\mathrm{i}}$ is $\varphi_{1,2}\ \sigma_{1,2}$ L- continuous (resp. $\varphi_{1,2}\ \sigma_{1,2}$ L-open).

Proof. Immediate from (3.3) and Proposition 3.7. □

4. Characterized L-topological Groups

This section is deviated to introduce and study the notion of characterized L-topological groups as a generalization of the weaken and stronger forms of the L-topological groups which introduced in [5,7,9,10,17,20]. It will be shown that the characterized L-topological group is an extension with respect to the functor ω_L. As an example we show that the α-level characterized space $\left(X, (\varphi_{1,2}OF(X))_\alpha\right)$, $\alpha \in L_1$ and the initial characterized space $\left(X, i_L(\varphi_{1,2}OF(X))\right)$ of an characterized L-topological group $(G, \varphi_{1,2}.\mathrm{int}_G)$ are characterized topological groups. Some examples in special choice for the operations φ_1 and φ_2 are given for characterized L-topological groups.

In the following let G is a multiplicative group. We denote, as usual, the identity element of G by e and the inverse of x in G by x^{-1}. Consider τ is an L-topology on G and $\varphi_1, \varphi_2 \in O_{(L^G,\tau)}$. Then the pair $(G, \varphi_{1,2}.\mathrm{int}_G)$ will be called an characterized L-topological group if and only if the mappings $\alpha : (G \times G, \varphi_{1,2}.\mathrm{int}_G \times \varphi_{1,2}.\mathrm{int}_G) \to (G, \varphi_{1,2}.\mathrm{int}_G)$ And $\beta : (G, \varphi_{1,2}.\mathrm{int}_G) \to (G, \varphi_{1,2}.\mathrm{int}_G)$ that defined by:

$$\alpha((x, y)) = x\,y \qquad \forall\ (x, y) \in G \times G \tag{4.1}$$

and

$$\beta(x) = x^{-1} \qquad \forall\ x \in G \tag{4.2}$$

are $\varphi_{1,2}\ \varphi_{1,2}$ L- continuous, respectively.

If $\varphi_1 = \text{int}$ and $\varphi_2 = 1_{L^X}$, then the characterized L-topological group $(G, \varphi_{1,2}.\text{int}_G)$ is coincide with the L-topological group (G, τ) which is defined in [5,7,9,10,17,20]. Other special choices for the operations φ_1 and φ_2 obtained in Table (1).

Example 4.1 Let G be a multiplicative group, T is ordinary topology on G and $\varphi_1, \varphi_2 \in O_{(P(G),T)}$. Then, the pair $(G, \varphi_{1,2}.\text{int}_G)$ is a characterized L-topological group for any lattice L, where $\varphi_1 = \text{int}$, $\varphi_2 = 1_{P(G)}$ and:

$$\varphi_{1,2}.\text{int}_G = \{\alpha \cdot 1_A : \alpha \in L \text{ and } A \in \varphi_{1,2}O(G)\} \text{ and } \mu_{1_A}(x) = 1 \text{ for all } x \in A.$$

Specially, if L is dense chain and $\psi_1, \psi_2 \in O_{(P(G),T)}$, then the pair $(G, \psi_{1,2}.\text{int}_G)$ is also characterized L-topological group, where $\psi_1 = \text{int}$, $\psi_2 = 1_{P(G)}$ and:

$$\psi_{1,2}.\text{int}_G = \{\eta \in L^G : \{x : \mu_\eta(x) \le \alpha\} \in \psi_{1,2}C(G) \text{ and } \alpha \in L\}.$$

Example 4.2 Let $\mathbb{R}$ be the set of all real numbers and $\varphi \ne \mathcal{U}_t^\xi \subseteq L^{\mathbb{R}}$ is defined by Rod-abaugh in [25] as follows:

$$\mathcal{U}_t^\xi(x) = J_\xi(x)(t) \quad \text{for all } x, t \in \mathbb{R},$$

Where $\{J_\xi\}_{\xi \in I}$ is non-empty subset of $L^{\mathbb{R}\times\mathbb{R}}$. The classe $\{\mathcal{U}_t^\xi\}_{(\xi,t)}$ is a sub base for an L-topology τ on $\mathbb{R}$, called the dual L-topology [25]. Consider $\varphi_1, \varphi_2 \in O_{(L^{\mathbb{R}},\tau)}$ for which $\varphi_1 = \text{int}$, $\varphi_2 = 1_{L^{\mathbb{R}}}$ and the usual addition $+$ is the binary operation on $\mathbb{R}$. Then, the pair $(\mathbb{R}, \varphi_{1,2}.\text{int}_{\mathbb{R}})$ is characterized L-topological group in our sense.

Example 4.3 Let $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ be the set of all real numbers except the zero number and the binary operation on $\mathbb{R}^*$ is the usual multiplication. Then, obviously $(\mathbb{R}^*, \cdot)$ is a group. Consider τ^* is the dual L-topology τ defined in Example 4.2. Consider $\varphi_1, \varphi_2 \in O_{(L^{\mathbb{R}^*},\tau^*)}$ for which $\varphi_1 = \text{int}$, $\varphi_2 = 1_{L^{\mathbb{R}}}$. Then, the pair $(\mathbb{R}^*, \varphi_{1,2}.\text{int}_{\mathbb{R}^*})$ is characterized L-topological group in our sense.

In the following proposition we will be given an equivalent definition for the characterized L-topological group

Proposition 4.1 Let G be a multiplicative group, τ is an L-topology on G and $\varphi_1, \varphi_2 \in O_{(L^G,\tau)}$. Then, $(G, \varphi_{1,2}.\text{int}_G)$ is characterized L-topological group if and only if the mapping $\gamma : (G \times G, \varphi_{1,2}.\text{int}_G \times \varphi_{1,2}.\text{int}_G) \to (G, \varphi_{1,2}.\text{int}_G)$ which is defined by:

$$\gamma(x, y) = x\, y^{-1} \quad \text{for all } (x,y) \in G \tag{4.3}$$

is $\varphi_{1,2}$ $\varphi_{1,2}$ L- continuous.

Proof. Let $(G, \varphi_{1,2}.\text{int}_G)$ is a characterized L-topological group and consider the mapping $f : (G \times G, \varphi_{1,2}.\text{int}_G \times \varphi_{1,2}.\text{int}_G) \to (G \times G, \varphi_{1,2}.\text{int}_G \times \varphi_{1,2}.\text{int}_G)$ defined as follo- ws $f(x, y) = (x, y^{-1})$ for all $(x,y) \in G$. Then, $f = 1_G \times \beta$, where 1_G is the identity mapping on G and β be the mapping defined by (4.2). Because of Corollary 3.3, f is $\varphi_{1,2}$ $\varphi_{1,2}$ L- continuous. Hence, $\gamma = \alpha \circ f$ is the composition of the $\varphi_{1,2}$ $\varphi_{1,2}$ L- continuous mappings α and f (See Fig. 4.1), that is, γ is $\varphi_{1,2}$ $\varphi_{1,2}$ L- continuous.

$$\begin{array}{ccc} G\times G & \xrightarrow{f} & G\times G \\ & {\scriptstyle\gamma}\searrow & \downarrow{\scriptstyle\alpha} \\ & & G \end{array}$$

Fig. 4.1

Conversely, let γ is $\varphi_{1,2}$ $\varphi_{1,2}$ L- continuous mapping and let the canonical injection mapping $i:(G,\varphi_{1,2}.\mathrm{int}_G)\to(G\times G,\ \varphi_{1,2}.\mathrm{int}_G\times\varphi_{1,2}.\mathrm{int}_G)$ defined by $i(x)=(e,x)$ for all $x\in G$. Then, $\beta=\gamma\circ i$ is the composition of the $\varphi_{1,2}$ $\varphi_{1,2}$ L- continuous mappings γ and , that is, β is $\varphi_{1,2}$ $\varphi_{1,2}$ L-continuous. Since $f=1_{G\times G}\circ\beta$ is the composition of the $\varphi_{1,2}$ $\varphi_{1,2}$ L- continuous mappings $1_{G\times G}$ and β, then f is $\varphi_{1,2}$ $\varphi_{1,2}$ L- continuous and therefore, $\alpha=\gamma\circ f$ is also $\varphi_{1,2}$ $\varphi_{1,2}$ L- continuous (See Fig. 4.2). Hence, $(G,\varphi_{1,2}.\mathrm{int}_G)$ is characterized L-topological group

$$\begin{array}{ccccc} G\times G & \xrightarrow{\beta} & G\times G & & \\ \searrow{\scriptstyle\beta} & \searrow & \downarrow 1_{G\times G} & & \\ {\scriptstyle\alpha}\searrow & & G\times G & & \\ & & \searrow & \downarrow{\scriptstyle\gamma} & \\ & & & G & \end{array}$$

. □

Fig. 4.2

Now, we applied Proposition 4.1 to introduce new examples for characterized L-topological groups. One of them is defined on the left L-Real line and the second is defined for any characterized L-topological groups.

Example 4.4 Let the left L-Real line $\mathbb{R}(L)$ with $L=\{0,1\}$ equipped with the L-addition $\oplus_L$ defined in [24] and τ_c is the canonical L-topology on $\mathbb{R}(L)$. Then, $(\mathbb{R}(L).\oplus_L)$ is a group. Consider $\varphi_1,\ \varphi_2\in O_{(L^{\mathbb{R}(L)},\tau_c)}$ for which $\varphi_1=\mathrm{int}$, $\varphi_2=1_{L^{\mathbb{R}(L)}}$. Then, because of Proposition 4.1, the pair $(\mathbb{R}(L),\varphi_{1,2}.\mathrm{int}_{\mathbb{R}(L)})$ is characterized L-topological group in our sense.

Example 4.5 Let $(G,\varphi_{1,2}.\mathrm{int}_G)$ be a characterized L-topological group. Then, $(G,\omega_L(\varphi_{1,2}O(G)))$ and $(G,i_L(\varphi_{1,2}OF(G)))$ are two examples of a characterized L-topological groups, called induced characterized L-topological group and initial characterized L-topological group, respectively.

If G is an multiplicative group and $a\in G$, then the left and the right translations are the homomorphism's denoted by $L_a:G\to G$ and $\mathrm{R}_a:G\to G$ and they will defined by $L_a(x)=a\,x$ and $\mathrm{R}_a(x)=x\,a$ for all $x\in G$, respectively. The left and the right translations in the characterized L-topological groups fulfill the following result.

Proposition 4.2 If $(G,\varphi_{1,2}.\mathrm{int}_G)$ is a characterized L-topological group and $a\in G$, then the left L_a and the right R_a translations are $\varphi_{1,2}$ $\varphi_{1,2}$ L- homeomorphisms.

Proof. Since L_a is the composition of the mapping α defined by (4.1) and the injection $i:(G,\varphi_{1,2}.\mathrm{int}_G)\to(G\times G,\ \varphi_{1,2}.\mathrm{int}_G\times\varphi_{1,2}.\mathrm{int}_G)$ defined by $i(x)=(a,x)$ for all $x\in G$, then L_a is injection $\varphi_{1,2}$ $\varphi_{1,2}$ L- continuous. Because of $L_a^{-1}(x)=L_{a^{-1}}(x)=a^{-1}x$ for all $x\in G$, then L_a^{-1} is also $\varphi_{1,2}$ $\varphi_{1,2}$ L-continuous. Therefore, L_a is $\varphi_{1,2}$ $\varphi_{1,2}$ L- homeomorphism. Similarly, one can prove that R_a is also $\varphi_{1,2}$ $\varphi_{1,2}$ L- homeomorphism. □

Lemma 4.1 Let $(G,\varphi_{1,2}.\text{int}_G)$ and $(H,\psi_{1,2}.\text{int}_H)$ are ordinary characterized topological groups, $\omega_L(\varphi_{1,2}O(G))$ and $\omega_L(\psi_{1,2}O(H))$ are the induced characterized L-topologies on G and H respectively. Then, $f:(G,\varphi_{1,2}.\text{int}_G)\to(H,\psi_{1,2}.\text{int}_H)$ is $\varphi_{1,2}\ \psi_{1,2}$-continuous if and only if $f:(G,\omega_L(\varphi_{1,2}O(G)))\to(H,\omega_L(\psi_{1,2}O(H)))$ is $\varphi_{1,2}\ \psi_{1,2}$ L-continuous.

Proof. Let $f:(G,\varphi_{1,2}.\text{int}_G)\to(H,\psi_{1,2}.\text{int}_H)$ is $\varphi_{1,2}\ \psi_{1,2}$-continuous. Because of Proposition 3.1, ω_L is a functor? Hence, $f:(G,\omega_L(\varphi_{1,2}O(G)))\to(H,\omega_L(\psi_{1,2}O(H)))$ is $\varphi_{1,2}\ \psi_{1,2}$ L-continuous.
Conversely, let $f:(G,\omega_L(\varphi_{1,2}O(G)))\to(H,\omega_L(\psi_{1,2}O(H)))$ is $\varphi_{1,2}\ \psi_{1,2}$ L-continuous and $B\in\psi_{1,2}O(H)$, where $\psi_{1,2}O(H)$ is the base of $(H,\omega_L(\psi_{1,2}O(H)))$. Then from the definition of $\omega_L(\varphi_{1,2}O(G))$ and $\omega_L(\psi_{1,2}O(H))$, it follows that $f^{-1}(B)\in\varphi_{1,2}O(G)$ and therefore $f:(G,\varphi_{1,2}.\text{int}_G)\to(H,\psi_{1,2}.\text{int}_H)$ is $\varphi_{1,2}\ \psi_{1,2}$-continuous. □

Now, we applied Lemma 4.1 in the following proposition to show that the notion of characterized L-topological group is an extension with respect to the concrete functor ω_L .

Proposition 4.3 Let G be a multiplicative group, T is an ordinary topology on G and $\varphi_1,\ \varphi_2\in O_{(P(G),T)}$. Then, the characterized space $(G,\varphi_{1,2}.\text{int}_G)$ is characterized topological group if and only if the induced characterized space $(G,\omega_L(\varphi_{1,2}O(G)))$ is characterized L-topological group.

Proof. Follows immediately from Lemma 4.1. □

Denote by **CRL-TopGrp** for the category of all characterized L-topological groups and all the $\varphi_{1,2}\ \varphi_{1,2}$ L-continuous homeomorphisms between them as morphisms. It is clear that **CRL-TopGrp** is concrete category over the category **Grp** of all groups. Also, denote by **CR-TopGrp** for the category of all characterized topological groups and all the $\varphi_{1,2}\ \varphi_{1,2}$-continuous homomorphism mappings between them as a morphisms.

Proposition 4.4 The concrete functor ω_L is covariant functor from the category **CR-TopGrp** of all characterized topological groups to the concrete category **CRL-TopGrp** of all characterized L-topological groups.

Proof. Follows directly from Lemma 4.1 and Proposition 4.3. □

Lemma 4.2 Let G and H are multiplicative groups, τ_1 and τ_2 are two L-topologies on G and H respectively, $\varphi_1,\ \varphi_2\in O_{(L^G,\tau_1)}$ and $\psi_1,\psi_2\in O_{(L^H,\tau_2)}$. Consider $(G,\varphi_{1,2}.\text{int}_G)$ and $(H,\varphi_{1,2}.\text{int}_H)$ are the related characterized L-topological groups. If $(G,(\varphi_{1,2}OF(G))_\alpha)$, $(G,i_L(\varphi_{1,2}OF(G)))$ and $(H,(\psi_{1,2}OF(H))_\alpha)$, $(H,i_L(\psi_{1,2}OF(H)))$ for $\alpha\in L_1$ are the related α-level and initial characterized spaces of $(G,\varphi_{1,2}.\text{int}_G)$ and $(H,\varphi_{1,2}.\text{int}_H)$ respectively, then the following statements:

(1) $f:(G,\varphi_{1,2}.\text{int}_G)\to(H,\varphi_{1,2}.\text{int}_H)$ is $\varphi_{1,2}\ \psi_{1,2}$ L-continuous.
(2) $f:(G,(\varphi_{1,2}OF(G))_\alpha)\to(H,(\psi_{1,2}OF(H))_\alpha)$ is $\varphi_{1,2}\ \psi_{1,2}$-continuous for all $\alpha\in L_1$.
(3) $f:(G,i_L(\varphi_{1,2}OF(G)))\to(H,i_L(\psi_{1,2}OF(H)))$ is $\varphi_{1,2}\ \psi_{1,2}$-continuous.

Fulfill the implications: $(1)\Rightarrow(2)\Rightarrow(3)$.

Proof. Follows directly from Proposition 3.1. □

In the following proposition we give the relation between characterized L-topological groups and its α -level and initial characterized spaces. This relation is immediately follows from Lemma 4.2.

Proposition 4.5 Let G be a multiplicative groups, τ is an L-topologies on G and $\varphi_1, \varphi_2 \in O_{(L^G,\tau)}$. If $(G,(\varphi_{1,2}OF(G))_\alpha)$, for $\alpha \in L_1$ and $(G, i_L(\varphi_{1,2}OF(G)))$ are the α -level and initial characterized spaces of $(G, \varphi_{1,2}.\text{int}_G)$, then the following statements:

(1) $(G, \varphi_{1,2}.\text{int}_G)$ is characterized L-topological group.

(2) $(G,(\varphi_{1,2}OF(G))_\alpha)$ is characterized topological group for all $\alpha \in L_1$.

(3) $(G, i_L(\varphi_{1,2}OF(G)))$ is characterized topological group.

Fulfill the implications: $(1) \Rightarrow (2) \Rightarrow (3)$.

Proof. Follows directly from Lemma 4.2. □

Corollary 4.1 The concrete functors i_L and i_α are covariant functors from the concrete category **CRL-TopGrp** of all characterized L-topological groups to the category **CR-TopGrp** of all characterized topological groups.

Proof. Follows directly from Lemma 4.2 and Proposition 4.5. □

5. Conclusion

In this paper, we introduced and studied the notions of initial characterized L-spaces and characterized L-topological groups. The properties of such spaces are deeply studied. We will show that the initial characterized L-space for characterized L-spaces exists. By this notion, the notions of characterized L-subspace and characterized product L-space are introduced and studied. We extend Lowen functors ω and i to the functors ω_L and i_L for an completely distributive complete lattice L and we show that for the notion of characterized L-topological groups the functors ω_L, i_L and i_α are concrete and covariant functors. Some sort of relationship were introduced, such as, the ordinary characterized space is characterized topological group if and only if its induced is characterized L-topological group. However, we show that for each characterized L-topological group, the α -level and the initial characterized spaces are characterized topological groups. Many new special classes from the initial characterized L-spaces characterized L-subspaces, characterized product L-spaces, induced characterized L-spaces and characterized L-topological groups are listed in Table (1).

	Operations	Initial characterized L-spaces	Characterized L-sub spaces	Characterized product L-spaces	Induced characterized L-spaces	Characterized L-topological groups
1	$\varphi_1 = \text{int}$ $\varphi_2 = 1_{L^X}$	Initial L-top. space [21]	L-sub space [21]	Product L-space [21]	Induced L-space [5,7]	L-topological group [5,7,9,10]
2	$\varphi_1 = \text{int}$ $\varphi_2 = \text{cl}$	Initial θ L-space	θ L-sub space	θ - product L-space	θ - induced L-space	θ L – topological group
3	$\varphi_1 = \text{int}$ $\varphi_2 = \text{int} \circ \text{cl}$	Initial δ L-space	δ L-sub space	δ - product L-space	δ - induced L-space	δ L - topological group
4	$\varphi_1 = \text{cl} \circ \text{int}$ $\varphi_2 = 1_{L^X}$	Initial semi L- space	Semi L-sub space	Semi - product L-space	Semi- induced L-space	Semi L-topological group
5	$\varphi_1 = \text{cl} \circ \text{int}$ $\varphi_2 = \text{cl}$	Initial $(\theta.S)$ L- space	$(\theta.S)$ L-sub space	$(\theta.S)$ - product L-space	$(\theta.S)$ - induced L-space	$(\theta.S)$ L-topological group
6	$\varphi_1 = \text{cl} \circ \text{int}$ $\varphi_2 = \text{int} \circ \text{cl}$	Initial $(\delta.S)$ L- space	$(\delta.S)$ L-sub space	$(\delta.S)$ - product L-space	$(\delta.S)$ - induced L-space	$(\delta.S)$ L-topological group
7	$\varphi_1 = \text{int} \circ \text{cl}$ $\varphi_2 = 1_{L^X}$	Initial pre L-space	Pre L-sub space	Pre product L-space	Pre induced L-space	Pre L-topological group
8	$\varphi_1 = \text{cl} \circ \text{int}$ $\varphi_2 = S.\text{cl}$	Initial $(S.\theta)$ L- space	$(S.\theta)$ L-sub space	$(S.\theta)$ - product L-space	$(S.\theta)$ - induced L-space	$(S.\theta)$ L-topological group
9	$\varphi_1 = \text{cl} \circ \text{int}$ $\varphi_2 = S.\text{int} \circ S.\text{cl}$	Initial $(S.\delta)$ L- space	$(S.\delta)$ L-sub space	$(S.\delta)$ - product L-space	$(S.\delta)$ - induced L-space	$(S.\delta)$ L-topological group
10	$\varphi_1 = \text{cl} \circ \text{int} \circ \text{cl}$ $\varphi_2 = 1_{L^X}$	Initial β L-space	β L-sub space	β - product L-space	β - induced L-space	β L-topological group
11	$\varphi_1 = \text{int} \circ \text{cl} \circ \text{int}$ $\varphi_2 = 1_{L^X}$	Initial λ L-space	λ L-sub space	λ - product L-space	λ - induced L-space	λ L-topological group
12	$\varphi_1 = S.\text{cl} \circ \text{int}$ $\varphi_2 = 1_{L^X}$	Initial feebly L- space	Feebly L-sub space	Feebly product L-space	Feebly induced L-space	Feebly L-topological group

Table (1) : Some special classes of initial characterized L-spaces, characterized L-subspaces, Characterized product L-spaces, Induced characterized L-spaces and Characterized L-topological groups.

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