The Degree of Coconvex Multi Polynomial Approximation

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Abstract

Let $f \in C[-1,1]^d$ change its convexity finitely many times in the interval , say s times , at Y_s : -1$< y_{11} < y_{21} < \cdots < y_{s1} < 1$,..., -1$< y_{1d} < y_{2d} < \cdots < y_{sd} < 1$. We estimate the degree of approximation of f by multi polynomials of degree n , which change convexity exactly at the points Y_s . We show that provided n is sufficiently large, depending on the location of the points Y_s , the rate of approximation is estimated by the multi third Ditzian –Totik moduli of smoothness of f multiplied by a constant $C(s)$.

Keywords: coconvex approximation, moduli of smoothness, multi approximation.

1. Introduction and Main Results .

Let $f \in C[-1,1]^d$ change its convexity finitely many times ,say s , at the points Y_s : -1$< y_{11} < y_{21} < \cdots < y_{s1} < 1$,..., -1$< y_{1d} < y_{2d} < \cdots < y_{sd} < 1$ in $[-1,1]^d$ where $C[-1,1]^d$ the space of all continuous functions on $[-1,1]^d$ such that $x = (x_1, \dots, x_d), x \in [-1,1]^d$. For later reference set $y_{0j} = -1$ and $y_{sj+1} = 1$ where $j = 1, \dots, d$. Note that if $f \in C^2[-1,1]^d$, then the above is equivalent to $f''(x_1, \dots, x_d) \prod_{i=1}^{s}\big((x_1 - y_{i1}) \dots (x_d - y_{id})\big) \geq 0$, in $[-1,1]^d$. We wish to approximate f by means of polynomials which are coconvex with f, that is , which change convexity exactly at the points $y_1 = (y_{11}, \dots, y_{1d})$,..., $y_s = (y_{s1}, \dots, y_{sd})$. The first Jackson estimate involving the Ditzian – Totik moduli of smoothness is due to Leviatan [3] . As was pointed out to us by the referee , the degree of coconvex polynomial approximation apparently was first discussed in the Ph. D dissertation of Diana C. Myers [4] where she obtained the Jackson estimate $\omega\left(f, \frac{1}{n}\right)$ for nearly coconvex approximation . The first estimates on the degree of coconvex approximation for a twice continuously differentiable function with an arbitrary finite number of convexity changes have recently been obtained by Kopotun [1] and the purpose of this note is to improve those result in that we do not assume the existence everywhere , and continuity of even the first derivative . We are going to make use of some special polynomials related to the function f which were constructed in that article [1] based upon the polynomials introduced by Shevchuk [5] . We remark that in the above mentioned paper , Kopotun was able to estimates on the simultaneous approximation of the function and its derivatives by the polynomials and their derivatives , thus obtaining simultaneously , coconvex approximation to f and comonotone approximation to its derivative . In order to state our main result we definition of the multi mth order Ditzian –Totik moduli of smoothness $\omega_m^{\varphi}(f, \delta)$. for $f \in C[-1,1]^d$, we set

$$\omega_m^{\varphi}(f, \delta) = \sup_{\substack{0 < h_j \leq \delta_j \\ j = 1, \dots, d}} \left\| \Delta_\eta^m \big(f; ((x_1, \dots, x_d))\big) \right\| ,$$

$h = (h_1, \dots, h_d)$, $\delta = (\delta_1, \dots, \delta_d)$, $\eta = (\eta_1, \dots, \eta_d) = \big(h_1\varphi(x_1), \dots, h_d\varphi(x_d)\big)$, where the norm is the max-norm , $\varphi(x_j) = \sqrt{1 - x_j^2}$, $j = 1, \dots d$ and

$$\Delta_\eta^m f\big((x_1, \dots, x_d)\big) = \begin{cases} \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} f\left(\left(x_1 - \frac{m}{2}\eta_1 + i\eta_1, \dots, x_d - \frac{m}{2}\eta_d + i\eta_d\right)\right) & if x \pm \frac{m}{2}\eta \in [-1,1]^d, \\ 0 , & \text{otherwise.} \end{cases}$$

is the symmetric multi mth difference.

We define $\left\|f\big((x_1, \dots, x_d)\big)\right\| = \sup_{\substack{x_j \in [-1,1] \\ j = 1, \dots, d}} \left|f\big((x_1, \dots, x_d)\big)\right|$, we also need the following moduli of smoothness

$$\omega_m(f,\delta) = \sup_{\substack{0<h_j\le\delta_j \\ j=1,\dots,d}} \left\| \Delta_h^m \left(f; \left((x_1,\dots,x_d) \right) \right) \right\|.$$

For $f \in C[-1,1]^d$, we denote by

$$E_n^{(2)}(f,Y_s) = inf\|f-p_n\|: p_n \in \Pi_n \cap \Delta^2(Y_s)$$

the degree of *coconvex approximation* of f by algebraic multi polynomials,

where p_n has the form

$$p_n\big((x_1,\dots,x_d)\big) = (a_{01}+a_{02}+\cdots+a_{0d}) + a_1(x_1+\cdots+x_d)$$

$$+a_2(x_1^2+\cdots+x_d^2)+\cdots+a_n(x_1^n+\cdots+x_d^n).$$

Our main result is the following :

Theorem : Let $f \in C[-1,1]^d$ have s changes of convexity at Y_s : $-1 < y_{11} < y_{21} < \cdots < y_{s1} < 1$,…, $-1 < y_{1d} < y_{2d} < \cdots < y_{sd} < 1$, and denote

$d(Y_s) = \min\left\{ \begin{matrix} \big((1+y_{11})\dots(1+y_{1d})\big), \big((y_{21}-y_{11})\dots(y_{2d}-y_{1d})\big),\dots, \\ \big((y_{s1}-y_{s1-1})\dots(y_{sd}-y_{sd-1})\big), \big((1-y_{s1})\dots(1-y_{sd})\big) \end{matrix} \right\}$. Then there exists a constant $A = A(s)$ which depends only on the number of convexity changes s , such that for $n > \frac{A(s)}{d(Y_s)}$ there is a multi polynomial p_n of degree not exceeding n , which is coconvex with f and satisfies

$$\|f-p_n\| \le \mathrm{C}(s)\omega_m^{\varphi}(f,\delta), \tag{1}$$

where $\delta = \left(\frac{1}{\mathrm{n}},\dots,\frac{1}{\mathrm{n}}\right)$.

In the sequel we will denote by C an absolute constant which may vary from one occurrence to another even in the same line . Similarly $\mathrm{C}(\cdot)$ will denote a constant which depends on a specific parameter but may change from one occurrence to another .

2. Proof of the Theorem :

If f has no change of convexity in $[-1,1]^d$, i.e. , $s = 0$ and f is convex in $[-1,1]^d$, then the theorem is valid , thus we will assume that $s \ge 1$. We first need to construct a smoother function at $y_1 = (y_{11},\dots,y_{1d})$. The function f is either concave or convex in $[-1,y_{11}]\times\dots\times[-1,y_{1d}]$, and each case will need a separate though similar construction. We will detail the construction for the case where f is concave in $[-1,y_{11}]\times\dots\times[-1,y_{1d}]$. For the sake of simplicity in notation in the sequel we write $\alpha_j = y_{1j}$ where $j = 1,\dots,d$. Now let

$x_r = (x_{r1},\dots,x_{rd})$, $x_{r,n} = \big(x_{r1,n},\dots,x_{rd,n}\big)$ and $\cos\frac{r\pi}{n} = \left(\cos\frac{r1\pi}{n},\dots\cos\frac{rd\pi}{n}\right)$,

such that $x_{rj} = x_{rj,n} = \cos\frac{rj\pi}{n}$, $r = 0,\dots,n$ and $j = 1,\dots,d$, be the chebyshev nodes; and denote $I_r = [x_{r1},x_{r1-1}]\times\dots\times[x_{rd},x_{rd-1}]$, $h_{rj} = h_{rj,n} = x_{rj-1} - x_{rj}$ and

$$\psi_r\big((x_1,\dots,x_d)\big) = \psi_{r,n}\big((x_1,\dots,x_d)\big) = \frac{(h_{r1}\dots h_{rd})}{|(x_1-x_{r1})\dots(x_d-x_{rd})| + (h_{r1}\dots h_{rd})} , \tag{2}$$

It is well know that $h_{rj\pm1} < 3h_{rj}$ and that for $x \in I_r$, $\Delta_n(x_{\mathrm{j}}) \le h_{rj} < 5\Delta_n(x_{\mathrm{j}})$,where $\Delta_n(x) = \big(\Delta_n(x_1),\dots,\Delta_n(x_d)\big)$ and $\Delta_n(x_{\mathrm{j}}) = \frac{\sqrt{1-x_j^2}}{n} + \frac{1}{\mathrm{n}^2}$. We assume that $\alpha = (\alpha_1,\dots,\alpha_d)$ such that $\alpha \in [x_{r01},x_{r01-1})\times\dots\times[x_{r0d},x_{r0d-1})$. Then, if $n > N_\alpha = max\left\{\frac{50}{(y_{21}-\alpha_1)\dots(y_{2d}-\alpha_d)}, \frac{50}{(1+\alpha_1)\dots(1+\alpha_d)}\right\}$, we are assured that $x_{r0j+3} \geq -1$ and that $x_{r0j-4} \leq \mathrm{y}_{2\mathrm{j}}$. Set $h_j = \mathrm{c}\Delta_n(\alpha_\mathrm{j}) < \frac{1}{6} h_{r0j}$, where c is chosen sufficiently small to guarantee the right inequality . Note that this implies $x_{r0j+1} < \alpha_\mathrm{j} - 2h_j < \alpha_\mathrm{j} + 2h_j < x_{r0j-2}$.

We are going to replace f on the interval $[\alpha_1 - h_1, \alpha_1 + h_1] \times \ldots \times [\alpha_\mathrm{d} - h_d, \alpha_\mathrm{d} + h_d]$ in a way that will keep us near the original function and at the same time the new function , will be smoother at $\alpha = (\alpha_1, \ldots, \alpha_\mathrm{d})$. When $s = 0$, then (1) holds for all $n \geq 2$. Thus , we proceed by induction. To this end we note that either $\Delta_h^2 f((\alpha_1, \ldots, \alpha_\mathrm{d})) \geq 0$, or $\Delta_h^2 f((\alpha_1, \ldots, \alpha_\mathrm{d})) < 0$.

In the first case, let $L_1((x_1, \ldots, x_d))$ denote the linear function interpolating f at $(\alpha_1 - h_1, \ldots, \alpha_\mathrm{d} - h_d)$ and $(\alpha_1, \ldots, \alpha_\mathrm{d})$. Then the function $g = f - L_1$ satisfies

$$g((\alpha_1 - h_1, \ldots, \alpha_\mathrm{d} - h_d)) = g((\alpha_1, \ldots, \alpha_\mathrm{d})) = 0 ,$$

$$g((\alpha_1 + h_1, \ldots, \alpha_\mathrm{d} + h_d)) \geq 0 \text{ and } g((x_1, \ldots, x_d)) \leq 0 , \ -1 \leq x_j < \alpha_\mathrm{j} - h_j .$$

Hence , for $\bar{J}_0 = [x_{r01+1}, x_{r01-2}] \times \ldots \times [x_{r0d+1}, x_{r0d-2}]$, we have ,

$$0 \leq g((\alpha_1 + h_1, \ldots, \alpha_\mathrm{d} + h_d))$$

$$\leq g((\alpha_1 + h_1, \ldots, \alpha_\mathrm{d} + h_d)) - g((\alpha_1 - 2h_1, \ldots, \alpha_\mathrm{d} - 2h_d))$$

$$= g((\alpha_1 + h_1, \ldots, \alpha_\mathrm{d} + h_d)) - 3g((\alpha_1, \ldots, \alpha_\mathrm{d})) + 3g((\alpha_1 + h_1, \ldots, \alpha_\mathrm{d} + h_d)) - g((\alpha_1 - 2h_1, \ldots, \alpha_\mathrm{d} - 2h_d))$$

$$\leq \omega_3(g, h, \bar{J}_0) = \omega_3(f, h, \bar{J}_0).$$

Similarly , in the latter case , let $L_1((x_1, \ldots, x_d))$ denote the linear function interpolating f at $(\alpha_1 + h_1, \ldots, \alpha_\mathrm{d} + h_d)$ and $(\alpha_1, \ldots, \alpha_\mathrm{d})$. Then the function $g = f - L_1$ satisfies

$$g((\alpha_1 + h_1, \ldots, \alpha_\mathrm{d} + h_d)) = g((\alpha_1, \ldots, \alpha_\mathrm{d})) = 0,$$

$$g((\alpha_1 - h_1, \ldots, \alpha_\mathrm{d} - h_d)) < 0, \text{ and } g((x_1, \ldots, x_d)) \geq 0 , \ \alpha_\mathrm{j} + h_j \leq x_j < y_{2j}.$$ Hence ,

$$0 < -g((\alpha_1 - h_1, \ldots, \alpha_\mathrm{d} - h_d))$$

$$\leq g((\alpha_1 + 2h_1, \ldots, \alpha_\mathrm{d} + 2h_d)) - g((\alpha_1 - h_1, \ldots, \alpha_\mathrm{d} - h_d))$$

$$= g((\alpha_1 + 2h_1, \ldots, \alpha_\mathrm{d} + 2h_d)) - 3g((\alpha_1 + h_1, \ldots, \alpha_\mathrm{d} + h_d))$$

$$+3g((\alpha_1, \ldots, \alpha_\mathrm{d})) - g((\alpha_1 - h_1, \ldots, \alpha_\mathrm{d} - h_d))$$

$$\leq \omega_3(g, h, \bar{J}_0) = \omega_3(f, h, \bar{J}_0).$$

Thus , in both case we have ,

$$max\{|g((\alpha_1 - h_1, \ldots, \alpha_\mathrm{d} - h_d))|, |g((\alpha_1, \ldots, \alpha_\mathrm{d}))|, |g((\alpha_1 + h_1, \ldots, \alpha_\mathrm{d} + h_d))|\} \leq C\omega_3(f, h, \bar{J}_0),$$

which in turn implies that the quadratic polynomial $L_2((x_1, \ldots, x_d))$, interpolating g at $(\alpha_1 - h_1, \ldots, \alpha_\mathrm{d} - h_d)$, $(\alpha_1, \ldots, \alpha_\mathrm{d})$ and $(\alpha_1 + h_1, \ldots, \alpha_\mathrm{d} + h_d)$ is bounded by the same on $[\alpha_1 - h_1, \alpha_1 + h_1] \times \ldots \times [\alpha_1 - h_1, \alpha_1 + h_1]$. Thus

$$|L_2(x)| \leq C \left(\frac{|(x_1 - \alpha_1) \ldots (x_d - \alpha_\mathrm{d})| + (h_{r01} \ldots h_{r0d})}{(h_{r01} \ldots h_{r0d})} \right)^2 \omega_3(f, h, \bar{J}_0) , \tag{3}$$

where $x \in [-1,1]^d$.

At the same time applying Whitney 's theorem for the multi approximation [2] we conclude that

$$|g(x) - L_2(x)| \le C(d)\omega_3(f,h,\bar{J}_0),$$

where $x \in [\alpha_1 - h_1, \alpha_1 + h_1] \times \ldots \times [\alpha_1 - h_1, \alpha_1 + h_1]$.
Since

$$|g(x) - L_2(x)| \le C\left(\frac{|x-\alpha|+h_{r0}}{h_{r0}}\right)^3 \omega_3(f,h,J_0), \quad [6]$$

where $x \in [-1,1]$ and $J_0 = [x_{r0+1}, x_{r0-2}]$.

Thus

$$|g(x) - L_2(x)| \le C\left(\frac{|(x_1-\alpha_1)\ldots(x_d-\alpha_d)| + (h_{r01}\ldots h_{r0d})}{(h_{r01}\ldots h_{r0d})}\right)^3 \omega_3(f,h,\bar{J}_0), \tag{4}$$

where $x \in [-1,1]^d$ and $\bar{J}_0 = [x_{r01+1}, x_{r01-2}] \times \ldots \times [x_{r0d+1}, x_{r0d-2}]$.

Since $\left|\Delta_h^3 f\big((x_1,\ldots,x_d)\big)\right| \le \left|\Delta_\eta^3 f\big((x_1,\ldots,x_d)\big)\right|$,

then $\omega_3(f,h,\bar{J}_0) \le C\omega_3^{\varphi}(f,\delta)$, $\delta = \left(\frac{1}{n},\ldots,\frac{1}{n}\right)$

we obtain by (3) and (4)

$$|g(x)| \le C\psi_{r0}^{-3}\omega_3^{\varphi}(f,\delta) \quad , \qquad x \in [-1,1]^d, \tag{5}$$

Now let

$$\tilde{g}(x) = \begin{cases} -g(x) , & x \in [-1,\alpha_1] \times \ldots \times [-1,\alpha_d] \\ g(x) , & otherwise , \end{cases}$$

and finally ,

$$\hat{g}(x) = \begin{cases} \tilde{g}(x) , & x \notin [\alpha_1 - h_1, \alpha_1 + h_1] \times \ldots \times [\alpha_1 - h_1, \alpha_1 + h_1], \\ max\{\tilde{g}(x), 0\} , & x \in [\alpha_1 - h_1, \alpha_1 + h_1] \times \ldots \times [\alpha_1 - h_1, \alpha_1 + h_1]. \end{cases}$$

By virtue of (5) we immediately have

$$\|\tilde{g} - \hat{g}\| \le C\,\omega_3^{\varphi}(f,\delta), \tag{6}$$

Also

$$\omega_3^{\varphi}(\hat{g},\delta) \le \omega_3^{\varphi}(\tilde{g},\delta) + C\omega_3^{\varphi}(f,\delta) \le C\omega_3^{\varphi}(f,\delta) , \tag{7}$$

where the first inequality follows from (6) , while the second is due to the fact that for $x \notin \bar{J}_0$,

$$\left|\Delta_{\ddot{\eta}}^2\left(\tilde{g}\big((x_1,\ldots,x_d)\big)\right)\right| = \left|\Delta_{\ddot{\eta}}^2\left(g\big((x_1,\ldots,x_d)\big)\right)\right| = \left|\Delta_{\ddot{\eta}}^2\left(f\big((x_1,\ldots,x_d)\big)\right)\right|,$$

where $\ddot{\eta} = \left(\frac{\varphi(x_1)}{n},\ldots,\frac{\varphi(x_d)}{n}\right)$ and for $x \in \bar{J}_0$ we apply (5) .

It is readily seen that $\hat{g} \in C[-1,1]^d$, that it is convex in $[-1,y_{21}] \times \ldots \times [-1,y_{2d}]$ and that it changes convexity at $Y_{s-1}^{'} = Y_s \setminus \{y_{11}\},\ldots,\{y_{1d}\}$. If on the other hand , f was convex in $[-1,y_{11}] \times \ldots \times [-1,y_{1d}]$, then $\hat{g}$ would be concave in $[-1,y_{21}] \times \ldots \times [-1,y_{2d}]$ and change convexity at $Y_{s-1}^{'}$.Thus in any case $\hat{g}$ has fewer convexity changes , so by induction , we may assume that for $n > \frac{A(s-1)}{d(Y_{s-1}^{'})}$, there exists an *nth* degree multi polynomial q_n which is coconvex with $\hat{g}$ and which satisfies the analogue of (1) . Namely by (7) ,

$$\|\hat{g}-q_n\| \le C(s-1)\omega_3^{\varphi}(\hat{g},\delta) \le C(s)\omega_3^{\varphi}(f,\delta), \tag{8}$$

Thus , fixing $n > max\left\{\frac{A(s-1)}{d(Y'_{s-1})}, N_\alpha\right\}$, readily leads to the definition of $A(s)$. Note that since $\hat{g}(\alpha_1,\dots,\alpha_d)=0$, we may assume that $q_n(\alpha_1,\dots,\alpha_d)=0$ constant in (8) .

Kopotun [1] has constructed for α and q_n two polynomial v_n and w_n of degree at most $20n(s+1)=4n\mu$, with the properties that for all $x\in[-1,1]$,

$$v_n(x)sgn(x-\alpha)\ge 0.$$

Thus let

$$V_n\big((x_1,\dots,x_d)\big)=v_n(x_1)+\cdots+v_n(x_d)$$

and

$$W_n\big((x_1,\dots,x_d)\big)=w_n(x_1)+\cdots+w_n(x_d)$$

where $$v_n(x_1)=a_{01}+a_1x_1+a_2x_1^2+\cdots+a_nx_1^n$$

$$\vdots$$

$$v_n(x_d)=a_{0d}+a_1x_d+a_2x_d^2+\cdots+a_nx_d^n$$

and

$$w_n(x_1)=c_{01}+c_1x_1+c_2x_1^2+\cdots+c_nx_1^n$$

$$\vdots$$

$$w_n(x_d)=c_{0d}+c_1x_d+c_2x_d^2+\cdots+c_nx_d^n$$

two multi polynomials of degree at most $20n(s+1)=4n\mu$ with the properties that for all $x\in[-1,1]^d$,

$$\big(v_n(x_1)+\cdots+v_n(x_d)\big)\big(sgn\ (x_1-\alpha_1)+\cdots+sgn\ (x_d-\alpha_d)\big)$$

$$\ge v_n(x_1)\,sgn\ (x_1-\alpha_1)+\cdots+v_n(x_d)sgn\ (x_d-\alpha_d)\ge 0$$

where $\operatorname{sgn}\big(x_j-\alpha_j\big)=\begin{cases}1 & x_j-\alpha_j>0\\ -1 & x_j-\alpha_j<0\end{cases}$

and $sgn\big((x_1-\alpha_1,\dots,x_d-\alpha_d)\big)=sgn\ (x_1-\alpha_1)+\cdots+sgn\ (x_d-\alpha_d)$,

$$V_n'\big((x_1,\dots,x_d)\big)q_n''\big((x_1,\dots,x_d)\big)\Big(q_n'\big((x_1,\dots,x_d)\big)\qquad -q_n'\big((\alpha_1,\dots,\alpha_d)\big)\Big)sgn\big((x_1-\alpha_1,\dots,x_d-\alpha_d)\big)\ge 0$$

$$\big|\big(v_n(x_1)+\cdots+v_n(x_d)\big)-\big(sgn\ (x_1-\alpha_1)+\cdots+sgn\ (x_d-\alpha_d)\big)\big|$$

$$\le|v_n(x_1)-sgn\ (x_1-\alpha_1)|+\cdots+|v_n(x_d)-sgn\ (x_d-\alpha_d)|$$

$$\le C(s)\,\psi_{r0}^{\mu}\big((x_1,\dots,x_d)\big), \tag{9}$$

$$\big|\big(w_n(x_1)+\cdots+w_n(x_d)\big)-\big(sgn\ (x_1-\alpha_1)+\cdots+sgn\ (x_d-\alpha_d)\big)\big|$$

$$\le|w_n(x_1)-sgn\ (x_1-\alpha_1)|+\cdots+|w_n(x_d)-sgn\ (x_d-\alpha_d)|$$

$$\le C(s)\,\psi_{r0}^{\mu}\big((x_1,\dots,x_d)\big), \tag{10}$$

$$\big|V_n'\big((x_1,\dots,x_d)\big)\big|=|v_n'(x_1)+\cdots+v_n'(x_d)|$$

$$\leq |v_n'(x_1)| + \cdots + |v_n'(x_d)|$$

$$\leq C(s)\, \psi_{r0}^{\mu}\big((x_1, \dots, x_d)\big)(h_{r01}^{-1} \dots h_{r0d}^{-1}), \qquad (11)$$

and finally

$$W_n'\big((x_1, \dots, x_d)\big) sgn q_n'\big((\alpha_1, \dots, \alpha_d)\big) \leq 0\,,$$

$x \in [y_{2r1}, y_{2r1+1}] \times \dots \times [y_{2rd}, y_{2rd+1}]$, $r = 0, \dots, [s/2]$, and

$$W_n'\big((x_1, \dots, x_d)\big) sgn q_n'\big((\alpha_1, \dots, \alpha_d)\big) \geq 0,$$

$x \in [y_{2r1+1}, y_{2r1+2}] \times \dots \times [y_{2rd+1}, y_{2rd+2}]$, $r = 0, \dots, [(s-1)/2]$.

We are ready to define the multi polynomial

$$p_n\big((x_1, \dots, x_d)\big) = \int_{\alpha_1}^{x_1} \dots \int_{\alpha_d}^{x_d} \Big[\Big(q_n'\big((u_1, \dots, u_d)\big) - q_n'\big((\alpha_1, \dots, \alpha_d)\big)\Big) V_n\big((u_1, \dots, u_d)\big) + q_n'\big((\alpha_1, \dots, \alpha_d)\big) W_n\big((u_1, \dots, u_d)\big)\Big]\, du_1 \dots du_d\,,$$

of degree at most $5nu$, which evidently is coconvex with f . Note that $P_n\big((x_1, \dots, x_d)\big) = L_1\big((x_1, \dots, x_d)\big) + p_n\big((x_1, \dots, x_d)\big)$ is of the same degree and it too is coconvex with f . Hence , we conclude the induction step by proving (1) for P_n.

We begin with

$$\big|f\big((x_1, \dots, x_d)\big) - P_n\big((x_1, \dots, x_d)\big)\big| = \big|g\big((x_1, \dots, x_d)\big) - p_n\big((x_1, \dots, x_d)\big)\big|$$

$$= \big|\tilde{g}\,\big((x_1, \dots, x_d)\big) sgn\big((x_1 - \alpha_1, \dots, x_d - \alpha_d)\big) - p_n\big((x_1, \dots, x_d)\big)\big|$$

$$\leq \|\tilde{g} - \hat{g}\| + \big|\hat{g}\,\big((x_1, \dots, x_d)\big) sgn\big((x_1 - \alpha_1, \dots, x_d - \alpha_d)\big) - p_n\big((x_1, \dots, x_d)\big)\big|$$

$$\leq C\omega_3^{\varphi}(f, \delta) + \Big|\Big(\hat{g}\,\big((x_1, \dots, x_d)\big) - q_n\big((x_1, \dots, x_d)\big)\Big) sgn\big((x_1 - \alpha_1, \dots, x_d - \alpha_d)\big)\Big|$$

$$+ \Big|q_n\big((x_1, \dots, x_d)\big) sgn\big((x_1 - \alpha_1, \dots, x_d - \alpha_d)\big) - \int_{\alpha_1}^{x_1} \dots \int_{\alpha_d}^{x_d} q_n'\big((u_1, \dots, u_d)\big) V_n\big((u_1, \dots, u_d)\big) du_1 \dots du_d\Big|$$

$$+ \Big|q_n'\big((\alpha_1, \dots, \alpha_d)\big) \int_{\alpha_1}^{x_1} \dots \int_{\alpha_d}^{x_d} \Big(V_n\big((u_1, \dots, u_d)\big) - W_n\big((u_1, \dots, u_d)\big)\Big) du_1 \dots du_d\Big|$$

$= E_1 + E_2 + E_3 + E_4$, say .

By virtue of (8) ,

$$E_2 \leq C(s)\omega_3^{\varphi}(f, \delta), \qquad (12)$$

Recalling that $q_n\big((\alpha_1, \dots, \alpha_d)\big) = 0$, integration by parts and (11) yield ,

$$E_3 = \left| q_n((x_1,\dots,x_d))\left(sgn((x_1-\alpha_1,\dots,x_d-\alpha_d)) - V_n((x_1,\dots,x_d))\right) + \int_{\alpha_1}^{x_1}\dots\int_{\alpha_d}^{x_d} q_n((u_1,\dots,u_d))V_n'((u_1,\dots,u_d))du_1\dots du_d \right|$$

$$\le C(s)\left(|q_n((x_1,\dots,x_d))|\psi_{r0}^{\mu}((x_1,\dots,x_d)) + \int_{\alpha_1}^{x_1}\dots\int_{\alpha_d}^{x_d} |q_n((u_1,\dots,u_d))|\psi_{r0}^{\mu}((u_1,\dots,u_d))(h_{r01}^{-1}\dots h_{r0d}^{-1})du_1\dots du_d \right)$$

$$\le C(s)\omega_3^{\varphi}(f,\delta), \qquad (13)$$

where the last inequality in (13) follows from (5), (6) and (7) ,

$$|q_n((x_1,\dots,x_d))| \le |\tilde{g}((x_1,\dots,x_d))| + |\tilde{g}((x_1,\dots,x_d)) - q_n((x_1,\dots,x_d))|$$

$$\le |g((x_1,\dots,x_d))| + \|\tilde{g}-\hat{g}\| + \|\hat{g}-q_n\|$$

$$\le C\psi_{r0}^{-3}\omega_3^{\varphi}(f,\delta),$$

and the easy inequality

$$\int_{\alpha_1}^{x_1}\dots\int_{\alpha_d}^{x_d} \psi_{r0}^{v}((u_1,\dots,u_d))du_1\dots du_d \le C(h_{r01}\dots h_{r0d}), \qquad v\ge 2, \qquad (14)$$

Finally , in order to estimate E_4 , we need an estimate on $q_n'((\alpha_1,\dots,\alpha_d))$. To this end we observe that since q_n is convex in $[-1,y_{21}]\times\dots\times[-1,y_{2d}]$, then q_n' is monotone increasing there .

If $q_n'((\alpha_1,\dots,\alpha_d)) \ge 0$, then by (5) , for some $\xi \in (\alpha_1,\alpha_1+h_{r01})\times\dots\times(\alpha_d,\alpha_d+h_{r0d})$,

$$0 \le q_n'((\alpha_1,\dots,\alpha_d)) \le q_n'((\xi_1,\dots,\xi_d))$$

$$= \frac{q_n((\alpha_1+h_{r01},\dots,\alpha_d+h_{r0d})) - q_n((\alpha_1,\dots,\alpha_d))}{(h_{r01}\dots h_{r0d})}$$

$$= (h_{r01}^{-1}\dots h_{r0d}^{-1})\, q_n((\alpha_1+h_{r01},\dots,\alpha_d+h_{r0d}))$$

$$\le C(h_{r01}^{-1}\dots h_{r0d}^{-1})\,\omega_3^{\varphi}(f,\delta).$$

And if $q_n'((\alpha_1,\dots,\alpha_d)) < 0$, then by (5), for some $\xi \in (\alpha_1-h_{r01},\alpha_1)\times\dots\times(\alpha_d-h_{r0d},\alpha_d)$,

$$0 \le -q_n'((\alpha_1,\dots,\alpha_d)) \le -q_n'((\xi_1,\dots,\xi_d))$$

$$= \frac{q_n((\alpha_1-h_{r01},\dots,\alpha_d-h_{r0d})) - q_n((\alpha_1,\dots,\alpha_d))}{(h_{r01}\dots h_{r0d})}$$

$$= (h_{r01}^{-1}\dots h_{r0d}^{-1})\, q_n((\alpha_1-h_{r01},\dots,\alpha_d-h_{r0d}))$$

$$\le C(h_{r01}^{-1}\dots h_{r0d}^{-1})\,\omega_3^{\varphi}(f,\delta).$$

Hence by (9) ,(10) and (14)

$$E_4 \leq \left|q_n'\big((\alpha_1, \dots, \alpha_d)\big)\right| \int_{\alpha_1}^{x_1} \dots \int_{\alpha_d}^{x_d} \left|V_n\big((u_1, \dots, u_d)\big) - W_n\big((u_1, \dots, u_d)\big)\right| \, du_1 \dots du_d$$

$$\leq C(h_{r01} \dots h_{r0d})\left|q_n'\big((\alpha_1, \dots, \alpha_d)\big)\right| \leq C\omega_3^{\varphi}(f, \delta), \quad (15)$$

Combining (12) ,(13) and (15) we see that

$$\|f - P_n\| \leq C(s)\omega_3^{\varphi}(f, \delta).$$

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