Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.4, No.3, 2014

On Initial and Final Characterized L- topological Groups

A. S. ABD-ALLAH *
Department of Mathematics, Faculty of Science, El-Mansoura University, El-Mansoura, Egypt

Abstract:
In this research work, new topological notions are proposed and investigated. The notions are named final characterized L-spaces and initial and final characterized L-topological groups. The properties of such notions are deeply studied. We show that all the final lefts and all the final characterized L-spaces are uniquely exist in the category **CRL-Sp** and hence **CRL-Sp** is topological category over the category **SET** of all sets. By the notion of final characterized L-space, the notions of characterized qoutien pre L-spaces and characterized sum L-spaces are introduced and studied. The characterized L-subspaces together with their related inclusion mappings and the characterized quotient pre L-spaces together with their related canonical surjections are the equalizers and co-equalizers, respectively in **CRL-Sp**. Moreover, we show that the initial and final lefts and then the initial and final characterized L-topological groups uniquely exist in the category **CRL-TopGrp.** Hence, the category **CRL-TopGrp** is topological category over the category **Grp** of all groups. By the notion of initial and final characterized L-topological groups, the notions of characterized L-subgroups, characterized product L-topological groups and characterized L-topological quotient groups are introduced and studied., we show that the category **CRL-TopGrp** is concrete and co-concrete category of the category **L-Top.** Finally, we show that the special faithful functors $\mathscr{F}: \mathbf{CRL-TopGrp} \to \mathbf{L-Top}$ and $\mathscr{F}^*: \mathbf{L-Top} \to \mathbf{CRL-TopGrp}$ are isomorphism, that is, the category **CRL-TopGrp** is algebraic and co-algebraic category over the category **L-Top** as in sense of [7].

Keywords: L-filter, topological L-space, operations, characterized L-space, categories **L-Top**, **Grp**, **CRL-Sp**, **SCRL-Sp**, **CR-Sp**, **CRL-TopGrp** and **CR-TopGrp,** $\varphi_{1,2}$ L- neighborhood filters, $\varphi_{1,2}\ \psi_{1,2}$ L-continuous, $\varphi_{1,2}\ \psi_{1,2}$ L-open, $\varphi_{1,2}\ \psi_{1,2}$ L-homeomorphism, $\varphi_{1,2}\ \psi_{1,2}$ L-homomorphism, final characterized L-space, characterized quotient pre L-space, characterized sum L-space, characterized L-topological group, characterized L-subgroup, characterized product L-topological group, characterized L-topological quotient group.

1. Introduction

The notion of L-filter has been introduced by Eklund *et al*. [10]. By means of this notion a point-based approach to L- topology related to the usual points has been developed. More general concept for L-filter introduced by G$\ddot{a}$hler in [11] and L-filters are classified by types. Because of the specific type of L-filter however the approach of Eklund is related only to L-topologies which are stratified, that is, all constant L-sets are open. The more specific L-filters considered in the former papers are called now homogeneous. The operation on the ordinary topological space (X, T) has been defined by Kasahara ([16]) as a mapping φ from T into 2^X such that, $A \subseteq A^\varphi$, for all $A \in T$. In. [5], Abd El-Monsef's *et al.*extended Kasahars's operation to the power set $P(X)$ of a set X. Kandil *et al*. ([15]), extended Kasahars's and Abd El-Monsef's operations by introducing an operation on the class of all L-sets endowed with an L-topology τ as a mapping $\varphi: L^X \to L^X$ such that $\text{int}\,\mu \leq \mu^\varphi$ for all $\mu \in L^X$, where μ^φ denotes the value of φ at μ. The notions of the L-filters and the operations on the class of all L-sets on X endowed with an L-topology τ are applied in [2,3,4] to introduce a more general theory including all the weaker and stronger forms of the L-topology. By means of these notions the notion of $\varphi_{1,2}$-interior of L-set, $\varphi_{1,2}$ L-convergence and $\varphi_{1,2}$ L-neighborhood filters are defined and applied to introduced many special classes of separation axioms. The notion of $\varphi_{1,2}$-interior operator for L-sets is defined as a mapping $\varphi_{1,2}.\text{int}: L^X \to L^X$ which fulfill (I1) to (I5) in [2]. There is a one-to-one correspondence between

* **Present address**: Department of Mathematics, Faculty of Science and hum anilities studies, Salman Bin Abdulaziz Univ., P. O. Box 132012, Code No. 11941 Hotat Bani Tamim, Saudi Arabia
E-mail address: asabdallah@hotmail.com and drhmsa1961@yahho.com

the class of all $\varphi_{1,2}$-open L-subsets of X and these operators, that is, the class $\varphi_{1,2}OF(X)$ of all $\varphi_{1,2}$-open L-subsets of X can be characterized by these operators. Then the triple $(X, \varphi_{1,2}.\text{int})$ as will as the triple $(X, \varphi_{1,2}OF(X))$ will be called the characterized L-space of $\varphi_{1,2}$-open L-subsets. The characterized L-spaces are characterized by many of characterizing notions in [2,3], for example by: $\varphi_{1,2}$ L-neighborhood filters, $\varphi_{1,2}$ L-interior of the L-filters and by the set of $\varphi_{1,2}$-inner points of the L-filters. Moreover, the notions of closeness and compactness in characterized L-spaces are introduced and studied in [4].

This paper is devoted to introduce and study the notions of final characterized L-spaces and initial and final characterized L-topological groups as a generalization of the weaker and stronger forms of the final topological L-space and initial and final L-topological group introduced in [8, 18]. **In section 2**, some definitions and notions related to L-sets, L-topologies, L-filters, operations on L-sets, characterized L-spaces, $\varphi_{1,2}$ L-neighborhood filters, $\varphi_{1,2}\alpha$ L-neighborhood, $\varphi_{1,2}$ $\psi_{1,2}$ L-contiuous mappings, $\varphi_{1,2}$ $\psi_{1,2}$ L-open mappings, $\varphi_{1,2}$ $\psi_{1,2}$ L-homeomorphism mappings and characterized L-topological groups are given. The categories of all characterized L-spaces, stratified characterized L- spaces and the characterized L-topological groups with the $\varphi_{1,2}$ $\psi_{1,2}$ L-continuity and $\varphi_{1,2}$ $\psi_{1,2}$-homomorphisms as a morphismeis between them are presented. **Section 3**, is devoted to introduce and study the notion of final characterized L-spaces. We show that all the final lefts and all the final characterized L-spaces are uniquely exist in the category **CRL-Sp**. Further notions related to the notion of characterized L-spaces are e.g. those of a characterized qoutient pre L-spaces and a characterized sum L-spaces are investigated as special cases for the notions of final characterized L-spaces. By the initial and final lefts in **CRL-Sp** we show that the category **CRL-Sp** is topological category over the category **SET** of all sets in sense of [7,19] and it is also complete and co-complete category, that is, all limits and all co-limits in **CRL-Sp** exist, which of course are unique up to isomorphisms. According to general procedure, we show that the characterized L-subspaces together with their related inclusion mappings and the characterized quotient pre L-spaces together with their related canonical surjections are equalizers and co-equalizers in **CRL-Sp**, respectively. **Section 4**, is deviated to introduce and study the notion of initial characterized L-topological groups as a generalization of the weaken and stronger forms of the initial L-topological groups which introduced in [8]. It will be shown that the initial lefts and then the initial characterized L-topological groups are uniquely exist in the category **CRL-TopGrp** and therefore, the category **CRL-TopGrp** is topological category over the category **Grp** of all groups. More generally, we show that the category **CRL-TopGrp** is concrete category of the category **L-Top** of all topological spaces and the faithful functor $\mathscr{F}: \mathbf{CRL-TopGrp} \to \mathbf{L-Top}$ is isomorphism. Thus, the category **CRL-TopGrp** is algebraic category over the category **L-Top** in sense of [7]. Finally, by the notion of initial characterized L-topological groups, the notions of characterized L-subgroups and characterized product L-topological groups are introduced and studied. **In section 5**, the notion of final characterized L-topological groups are introduced and studied as a generalization of the weaken and stronger forms of the final L-topological groups introduced in [8]. It will be shown that the final lefts and then the final characterized L-topological groups are uniquely exists in the category **CRL-TopGrp**. More generally, we show that the category **CRL-TopGrp** is co-concrete category of the category **L-Top** of all topological L-spaces and the faithful functor $\mathscr{F}^*: \mathbf{L-Top} \to \mathbf{CRL-TopGrp}$ is isomorphism. Thus, the category **CRL-TopGrp** is co-algebraic category over the category **L-Top** in sense of [7]. By the notion of final characterized L-topological groups, the notions of characterized L-topological quotient groups is introduced and studied. Finally, we present a relation between the characterized L-topological quotient groups and the characterized product L-topological groups.

2. Preliminaries

In this research work we consider L be a completely distributive complete lattice with different least and last elements 0 and 1, respectively. Consider $L_0 = L \setminus \{0\}$ and $L_1 = L \setminus \{1\}$. Sometimes we will assume more specially that L is complete chain, that is, L is a complete lattice whose partial ordering is a linear one. For a set X, let L^X be the set of all L-subsets of X, that is, of all mappings $f: X \to L$. Assume that an order-reversing involution $\alpha \mapsto \alpha'$ of L is fixed. For each L-set $\mu \in L^X$, let μ' denote the complement of μ and it is defined by: $\mu'(x) = \mu(x)'$ for all $x \in X$. Denote by $\overline{\alpha}$ the constant L-subset of X with value $\alpha \in L$. For all

$x \in X$. and for all $\alpha \in L_0$, the L-subset x_α of X whose value α at x and 0 otherwise is called an L-point in X. Now, we begin by recalling some facts on the L-filters.

L-filters. The L- filter on a set X ([11]) is a mapping $\mathcal{M} : L^X \to L$ such that the following conditions are fulfilled:

(F1) $\mathcal{M}(\overline{\alpha}) \le \alpha$ for all $\alpha \in L$ and $\mathcal{M}(\overline{1}) = 1$.

(F2) $\mathcal{M}(\mu \wedge \rho) = \mathcal{M}(\mu) \wedge \mathcal{M}(\rho)$ for all $\mu, \rho \in L^X$.

The L-filter $\mathcal{M}$ is called homogeneous ([11]) if $\mathcal{M}(\overline{\alpha}) = \alpha$ for all $\alpha \in L$. For each $x \in X$, the mapping $\dot{x} : L^X \to L$ defined by $\dot{x}(\mu) = \mu(x)$ for all $\mu \in L^X$ is a homogeneous L-filter on X. For each $\mu \in L^X$, the mapping $\dot{\mu} : L^X \to L$ defined by $\dot{\mu}(\eta) = \bigwedge_{0<\eta(x)} \eta(x)$ for all $\eta \in L^X$ is also homogeneous L-filter on X, called homogenous L- filter at the L-subset $\mu \in L^X$. Let $\mathcal{F}_L X$ and $\mathsf{F}_L X$ will be denote the sets of all L-filters and of all homogeneous L- filters on a set X, respectively. If $\mathcal{M}$ and $\mathcal{N}$ are L- filters on a set X, $\mathcal{M}$ is said to be finer than $\mathcal{N}$, denoted by $\mathcal{M} \le \mathcal{N}$, provided $\mathcal{M}(\mu) \ge \mathcal{N}(\mu)$ holds for all $\mu \in L^X$. Noting that if L is a complete chain then $\mathcal{M}$ is not finer than $\mathcal{N}$, denoted by $\mathcal{M} \not\le \mathcal{N}$, provided there exists $\mu \in L^X$ such that $\mathcal{M}(\mu) < \mathcal{N}(\mu)$ holds.

For each non-empty set $\mathcal{A}$ of the L- filters on X the supremum $\bigvee_{\mathcal{M} \in \mathcal{A}} \mathcal{M}$ exists ([11]) and given by:

$$(\bigvee_{\mathcal{M} \in \mathcal{A}} \mathcal{M})(\mu) = \bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M}(\mu)$$

for all $\mu \in L^X$. Whereas the infimum $\bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M}$ of $\mathcal{A}$ does not exists in general as an L-filter. If the infimum $\bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M}$ exists, then we have:

$$(\bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M})(\mu) = \bigvee_{\substack{\mu_1 \wedge \ldots \wedge \mu_n \le \mu, \\ \mathcal{M}_1, \ldots, \mathcal{M}_n \in \mathcal{A}}} (\mathcal{M}_1(\mu_1) \wedge \ldots \wedge \mathcal{M}_n(\mu_n))$$

For all $\mu \in L^X$, where n is a positive integer, $\mu_1, \ldots, \mu_n$ is a collection such that $\mu_1 \wedge \ldots \wedge \mu_n \le \mu$ and $\mathcal{M}_1, \ldots, \mathcal{M}_n$ are L- filters from $\mathcal{A}$. Let X be a set and $\mu \in L^X$, then the homogeneous L- filter $\dot{\mu}$ at $\mu \in L^X$ is the L-filter on X given by:

$$\dot{\mu} = \bigvee_{0 < \mu(x)} \dot{x}$$

L- filter bases. A family $(\mathcal{B}_\alpha)_{\alpha \in L_0}$ of non-empty subsets of L^X is called a valued L- filter base ([11]) if the following conditions are fulfilled:

(V1) $\mu \in \mathcal{B}_\alpha$ implies $\alpha \le \sup \mu$.

(V2) For all $\alpha, \beta \in L_0$ with $\alpha \wedge \beta \in L_0$ and all $\mu \in \mathcal{B}_\alpha$ and $\rho \in \mathcal{B}_\beta$ there are $\gamma \ge \alpha \wedge \beta$ and $\eta \ge \mu \wedge \sigma$ such that $\eta \in \mathcal{B}_\gamma$.

Each valued base $(\mathcal{B}_\alpha)_{\alpha \in L_0}$ defines the L-filter $\mathcal{M}$ on X ([11]) by $\mathcal{M}(\mu) = \bigvee_{\rho \in \mathcal{B}_\alpha, \rho \le \mu} \alpha$ for all $\mu \in L^X$. Conversely, each L- filter $\mathcal{M}$ can be generated by a valued base, e.g. by $(\alpha\text{-pr}\,\mathcal{M})_{\alpha \in L_0}$ with $\alpha\text{-pr}\,\mathcal{M} = \{\mu \in L^X \mid \alpha \le \mathcal{M}(\mu)\}$. The family $(\alpha\text{-pr}\,\mathcal{M})_{\alpha \in L_0}$ is a family of prefilters on X and is called the large valued base of $\mathcal{M}$. Recall that a prefilter on X ([16]) is a non-empty proper subset $\mathcal{F}$ of L^X such that:

(1) $\mu, \rho \in \mathcal{F}$ Implies $\mu \wedge \rho \in \mathcal{F}$ and (2) from $\mu \in \mathcal{F}$ and $\mu \le \rho$ it follows $\rho \in \mathcal{F}$.

Topological L-spaces. By an L-topology on a set X ([9, 14]), we mean a subset of $\mu \in L^X$ which is closed with respect to all suprema and all finite infima and contains the constant L-sets $\overline{0}$ and $\overline{1}$. A set X equipped with an L-topology τ on X is called topological L-space. For each topological L-space (X,τ), the elements of τ are called open L-subsets of this space. If τ_1 and τ_2 are L-topologies on a set X, τ_2 is said to be finer than τ_1 and τ_1 is said to be coarser than τ_2 provided $\tau_1 \subseteq \tau_2$ holds. For each L-set $\mu \in L^X$, the strong α-cut and the weak α-cut of μ are ordinary subsets of X defined by the subsets $\mathrm{S}_\alpha(\mu) = \{x \in X : \mu(x) > \alpha\}$ and $\mathrm{W}_\alpha(\mu) = \{x \in X : \mu(x) \geq \alpha\}$, respectively. For each complete chain L, the α-level topology and the initial topology ([17]) of an L-topology τ on X are defined as follows:

$$\tau_\alpha = \{\mathrm{S}_\alpha(\mu) \in P(X) : \mu \in \tau\} \text{ and } i(\tau) = \inf\{\tau_\alpha : \alpha \in L_1\},$$

respectively, where inf is the infimum with respect to the finer relation on topologies. On other hand if (X,T) is ordinary topological space, then the induced L-topology on X is defined by Lowen in [17] as the set $\omega(T) = \{\mu \in L^X : \mathrm{S}_\alpha(\mu) \in T \text{ for all } \alpha \in L_1\}$. Lowen in [17], show that ω and i are functors in special case of $L = I$. The topological L-space (X,τ) and also τ are said to be stratified provided $\overline{\alpha} \in \tau$ holds for all $\alpha \in L$, that is, all constant L-sets are open ([17]). Denote by **L-Top** and **Top** to the categories of all L-topological spaces and all ordinary topological spaces, respectively.

Operation on L-sets. In the sequel, let a topological L-space (X,τ) be fixed. By the operation ([15]) on a set X we mean a mapping $\varphi : L^X \to L^X$ such that $\operatorname{int}\mu \leq \mu^\varphi$ holds, for all $\mu \in L^X$, where μ^φ denotes the value of φ at μ. The class of all operations on X will be denoted by $O_{(L^X,\tau)}$. The constant operation on $O_{(L^X,\tau)}$ is the operation $c_{L^X} : L^X \to L^X$ such that $c_{L^X}(\mu) = \overline{1}$, for all $\mu \in L^X$. By identity operation on $O_{(L^X,\tau)}$, we mean the operation $1_{L^X} : L^X \to L^X$ such that $1_{L^X}(\mu) = \mu$, for all $\mu \in L^X$. In case of $L = \{0,1\}$, the identity operation on the class of all ordinary operations $O_{(P(X),T)}$ on X will be denoted by $i_{P(X)}$, and it is defined by $i_{P(X)}(A) = A$ for all $A \in P(X)$. If $\leq$ is a partially ordered relation on $O_{(L^X,\tau)}$ defined as follows: $\varphi_1 \leq \varphi_2 \Leftrightarrow \mu^{\varphi_1} \leq \mu^{\varphi_2}$ for all $\mu \in L^X$, then obviously, $O_{(L^X,\tau)}$ is a completely distributive lattice. As an application on the partially ordered relation $\leq$ on the set X, we classified the operatins by names as listed, the operation $\varphi : L^X \to L^X$ will be called:

(i) Isotone if $\mu \leq \rho$ implies $\mu^\varphi \leq \rho^\varphi$, for all $\mu, \rho \in L^X$.

(ii) Weakly finite intersection preserving (wfip, for short) with respect to $\mathcal{A} \subseteq L^X$ if $\rho \wedge \mu^\varphi \leq (\rho \wedge \mu)^\varphi$ holds, for all $\rho \in \mathcal{A}$ and $\mu \in L^X$.

(iii) Idempotent if $\mu^\varphi = (\mu^\varphi)^\varphi$, for all $\mu \in L^X$.

φ-open L- sets. Let a topological L-space (X,τ) be fixed and $\varphi \in O_{(L^X,\tau)}$. The L-set $\mu : X \to L$ will be called φ-open L- set if $\mu \leq \mu^\varphi$ holds. We will denote the class of all φ-open L- sets on X by $\varphi OF(X)$. The L- set μ is called φ-closed if its complement $co\,\mu$ is φ-open. The two operations $\varphi, \psi \in O_{(L^X,\tau)}$ are equivalent and written $\varphi \sim \psi$ if $\varphi OF(X) = \psi OF(X)$.

$\varphi_{1,2}$-interior of L- sets. Let a topological L-space (X,τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the $\varphi_{1,2}$-interior of the L-set $\mu : X \to L$ is the mapping $\varphi_{1,2}.\operatorname{int}\mu : X \to L$ defined by:

$$\varphi_{1,2}.\operatorname{int}\mu = \bigvee_{\rho \in \varphi_1 OF(X),\, \rho^{\varphi_2} \leq \mu} \rho \tag{2.1}$$

$\varphi_{1,2}.\text{int}\,\mu$ is the greatest φ_1-open L-set ρ such that ρ^{φ_2} less than or equal to μ ([2]). The L- set μ is said to be $\varphi_{1,2}$-open if $\mu \le \varphi_{1,2}.\text{int}\,\mu$. The class of all $\varphi_{1,2}$-open L- sets on X will be denoted by $\varphi_{1,2}OF(X)$. The complement $co\,\mu$ of a $\varphi_{1,2}$-open L-subset μ will be called $\varphi_{1,2}$-closed, the class of all $\varphi_{1,2}$-closed L-subsets of X will be denoted by $\varphi_{1,2}CF(X)$. In the classical case of $L = \{0,1\}$, the topological L-space (X,τ) is up to identification by the ordinary topological space (X,T) and $\varphi_{1,2}.\text{int}\,\mu$ is the classical one. Hence, in this case the ordinary subset A of X is $\varphi_{1,2}$-open if $A \subseteq \varphi_{1,2}.\text{int}\,A$. The complement of a $\varphi_{1,2}$-open subset A of X will be called $\varphi_{1,2}$-closed. The class of all $\varphi_{1,2}$-open and the class of all $\varphi_{1,2}$-closed subsets of X will be denoted by $\varphi_{1,2}O(X)$ and $\varphi_{1,2}C(X)$, respectively. Clearly, F is $\varphi_{1,2}$-closed if and only if $\varphi_{1,2}.\text{cl}_T\, F = F$.

Proposition 2.1 [2] If (X,τ) is a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then, the mapping $\varphi_{1,2}.\text{int}\,\mu : X \to L$ fulfills the following axioms:

(i) If $\varphi_2 \ge 1_{L^X}$, then $\varphi_{1,2}.\text{int}\,\mu \le \mu$ holds.

(ii) $\varphi_{1,2}.\text{int}\,\mu$ is isotone, i.e, if $\mu \le \rho$ then $\varphi_{1,2}.\text{int}\,\mu \le \varphi_{1,2}.\text{int}\,\rho$ holds for all $\mu, \rho \in L^X$.

(iii) $\varphi_{1,2}.\text{int}\,\overline{1} = \overline{1}$.

(iv) If $\varphi_2 \ge 1_{L^X}$ is isotone operation and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $\varphi_{1,2}.\text{int}(\mu \wedge \rho) = \varphi_{1,2}.\text{int}\,\mu \wedge \varphi_{1,2}.\text{int}\,\rho$ for all $\mu, \rho \in L^X$.

(v) If φ_2 is isotone and idempotent operation, then $\varphi_{1,2}.\text{int}\,\mu \le \varphi_{1,2}.\text{int}(\varphi_{1,2}.\text{int}\,\mu)$ holds.

(vi) $\varphi_{1,2}.\text{int}(\bigvee_{i\in I} \mu_i) = \bigvee_{i\in I} \varphi_{1,2}.\text{int}\,\mu_i$ for all $\mu_i \in \varphi_{1,2}OF(X)$.

Proposition 2.2 [2] Let (X,τ) be a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the following are fulfilled:

(i) If $\varphi_2 \ge 1_{L^X}$, then the class $\varphi_{1,2}OF(X)$ of all $\varphi_{1,2}$-open L-sets on X forms an extended L- topology on X, denoted by $\tau^{\varphi_{1,2}}$ ([13]).

(ii) If $\varphi_2 \ge 1_{L^X}$, then the class $\varphi_{1,2}OF(X)$ of all $\varphi_{1,2}$-open L-sets on X forms a supra L- topology on X, denoted by $\overline{\tau}^{\varphi_{1,2}}$ ([13]).

(iii) If $\varphi_2 \ge 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $\varphi_{1,2}OF(X)$ is a pre L-topology on X, denoted by $\tau^{\wedge}_{\varphi_{1,2}}$ ([13]).

(iv) If $\varphi_2 \ge 1_{L^X}$ is isotone and idempotent operation and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $\varphi_{1,2}OF(X)$ forms an L- topology on X, denoted by $\tau_{\varphi_{1,2}}$ ([9, 14]).

From Propositions 2.1 and 2.2, if the topological L-space (X,τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then

$$\varphi_{1,2}OF(X) = \{\mu \in L^X \mid \mu \le \varphi_{1,2}.\text{int}\,\mu\} \tag{2.2}$$

and the following conditions are fulfilled:

(I1) If $\varphi_2 \ge 1_{L^X}$, then $\varphi_{1,2}.\text{int}\,\mu \le \mu$ holds for all $\mu \in L^X$.

(I2) If $\mu \le \rho$ then $\varphi_{1,2}.\text{int}\,\mu \le \varphi_{1,2}.\text{int}\,\rho$ holds for all $\mu, \rho \in L^X$.

(I3) $\varphi_{1,2}.\text{int}\,\overline{1} = \overline{1}$.

(I4) If $\varphi_2 \geq 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $\varphi_{1,2}.\text{int}(\mu \wedge \rho) = \varphi_{1,2}.\text{int}\,\mu \wedge \varphi_{1,2}.\text{int}\,\rho$ for all $\mu, \rho \in L^X$.

(I5) If $\varphi_2 \geq 1_{L^X}$ is isotone and idempotent, then $\varphi_{1,2}.\text{int}(\varphi_{1,2}.\text{int}\,\mu) = \varphi_{1,2}.\text{int}\,\mu$ for all $\mu \in L^X$.

Characterized L-spaces. Independently on the L- topologies, the notion of $\varphi_{1,2}$-interior operator for L- sets can be defined as a mapping $\varphi_{1,2}.\text{int}: L^X \to L^X$ which fulfills (I1) to (I5). It is well-known that (2.1) and (2.2) give a one-to-one correspondence between the class of all $\varphi_{1,2}$-open L- sets and these operators, that is, $\varphi_{1,2}OF(X)$ can be characterized by $\varphi_{1,2}$-interior operators. In this case the pair $(X, \varphi_{1,2}.\text{int})$ as will as the pair $(X, \varphi_{1,2}OF(X))$ will be called characterized L- space ([2]) of $\varphi_{1,2}$-open L- subsets of X. If $(X, \varphi_{1,2}.\text{int})$ and $(X, \psi_{1,2}.\text{int})$ are two characterized L-spaces, then $(X, \varphi_{1,2}.\text{int})$ is said to be finer than $(X, \psi_{1,2}.\text{int})$ and denoted by $\varphi_{1,2}.\text{int} \leq \psi_{1,2}.\text{int}$ provided $\varphi_{1,2}.\text{int}\,\mu \geq \psi_{1,2}.\text{int}\,\mu$ holds for all $\mu \in L^X$. The characterized L-space $(X, \varphi_{1,2}.\text{int})$ of all $\varphi_{1,2}$-open L-sets is said to be stratified if and only if $\varphi_{1,2}.\text{int}\,\overline{\alpha} = \overline{\alpha}$ for all $\alpha \in L$. As shown in [2], the characterized L-space $(X, \varphi_{1,2}.\text{int})$ is stratified if the related L- topology is stratified. Moreover, the characterized L-space $(X, \varphi_{1,2}.\text{int})$ is said to have the weak infimum property ([13]) provided for all $\mu \in L^X$ and $\alpha \in L$. The characterized L-space $(X, \varphi_{1,2}.\text{int})$ is said to be strongly stratified ([13]) provided $\varphi_{1,2}.\text{int}$ is stratified and have the weak infimum property.

If $\varphi_1 = \text{int}$ and $\varphi_2 = 1_{L^X}$, then the class $\varphi_{1,2}OF(X)$ of all $\varphi_{1,2}$-open L-set of X coincide with τ which is defined in [9,14] and hence the characterized L- space $(X, \varphi_{1,2}.\text{int})$ coincide with the topological L-space (X, τ).

$\varphi_{1,2}$ **L-neighborhood filters**. An important notion in the characterized L-space $(X, \varphi_{1,2}.\text{int})$ is that of a $\varphi_{1,2}$ L-neighborhood filter at the point and at the ordinary subset in this space. Let (X, τ) be a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. As follows by (I1) to (I5) for each $x \in X$, the mapping $\mathcal{N}_{\varphi_{1,2}}(x): L^X \to L$ which is defined by:

$$\mathcal{N}_{\varphi_{1,2}}(x)(\mu) = (\varphi_{1,2}.\text{int}\,\mu)(x) \tag{2.3}$$

for all $\mu \in L^X$ is L-filter, called $\varphi_{1,2}$ L-neighborhood filter at x ([2]). If $\varphi \neq F \subseteq P(X)$, then the $\varphi_{1,2}$ L-neighborhood filter at F will be denoted by $\mathcal{N}_{\varphi_{1,2}}(F)$ and it will be defined by:

$$\mathcal{N}_{\varphi_{1,2}}(F) = \bigvee_{x \in F} \mathcal{N}_{\varphi_{1,2}}(x).$$

Since $\mathcal{N}_{\varphi_{1,2}}(x)$ is L-filter for all $x \in X$, then $\mathcal{N}_{\varphi_{1,2}}(F)$ is also L-filter on X. Moreover, because of $[\chi_F] = \bigvee_{x \in F} \dot{x}$, then we have $\mathcal{N}_{\varphi_{1,2}}(F) \geq [\chi_F]$ holds.

If the related $\varphi_{1,2}$-interior operator fulfill the axioms (I1) and (I2) only, then the mapping $\mathcal{N}_{\varphi_{1,2}}(x): L^X \to L$, which is defined by (2.3) is an L-stack ([15]), called $\varphi_{1,2}$ L- neighborhood stack at x. Moreover, if the $\varphi_{1,2}$-interior operator fulfill the axioms (I1), (I2) and (I4) such that in (I4) instead of $\rho \in L^X$ we choice $\overline{\alpha}$, then the mapping $\mathcal{N}_{\varphi_{1,2}}(x): L^X \to L$, is an L-stack with the cutting property, called here $\varphi_{1,2}$ L- neighborhood stack with the cutting property at x. Obviously, the $\varphi_{1,2}$ L-neighborhood filters fulfill the following axioms:

(N1) $\dot{x} \le \mathcal{N}_{\varphi_{1,2}}(x)$ holds for all $x \in X$.

(N2) $\mathcal{N}_{\varphi_{1,2}}(x)(\mu) \le \mathcal{N}_{\varphi_{1,2}}(x)(\rho)$ holds for all $\mu, \rho \in L^X$ and $\mu \le \rho$.

(N3) $\mathcal{N}_{\varphi_{1,2}}(x)(y \mapsto \mathcal{N}_{\varphi_{1,2}}(y)(\mu)) = \mathcal{N}_{\varphi_{1,2}}(x)(\mu),$ for all $x \in X$ and $\mu \in L^X$.

Clearly, $y \mapsto \mathcal{N}_{\varphi_{1,2}}(y)(\mu)$ is the L-set $\varphi_{1,2}.\text{int}\,\mu$.

The characterized L-space $(X, \varphi_{1,2}.\text{int})$ of all $\varphi_{1,2}$ -open L-subsets of a set X is characterized as a filter pre L-topology ([2]), that is, as a mapping $\mathcal{N}_{\varphi_{1,2}}(x) : X \to \mathcal{F}_L X$ such that the axioms (N1) to (N3) are fulfilled.

$\varphi_{1,2}\alpha$ **L-neighborhoods.** Let (X, τ) be a topological L-spaces and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then for each $\alpha \in L_0$ and each $x \in X$, the L-set $\mu \in L^X$ will be called $\varphi_{1,2}$ α L-neighborhood at x if $\alpha \le (\varphi_{1,2}.\text{int}\,\mu)(x)$ holds. Because of Proposition 2.1, the L-set $\mu \in L^X$ is $\varphi_{1,2}$ α L-neighborhood at x if and only if $\mu \in \alpha\text{-pr}\,\mathcal{N}_{\varphi_{1,2}}(x)$, where $\mathcal{N}_{\varphi_{1,2}}(x)$ be given by (2.3). For each $\alpha \in L_0$ and each $x \in X$ let $N_\alpha(x)$ be the set of all $\varphi_{1,2}$ α L-neighborhood at x , that is, $N_\alpha(x) = \{\mu \in L^X : \alpha \le (\varphi_{1,2}.\text{int}\,\mu)(x)\}$, then the family $(N_\alpha(x))_{\alpha \in L_0}$ is the large valued L-filter base of $\mathcal{N}_{\varphi_{1,2}}(x)$.

$\varphi_{1,2}$ **L-convergence**. Let a topological L-spaces (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. If x is a point in the characterized L-space $(X, \varphi_{1,2}.\text{int})$, $F \subseteq X$ and $\mathcal{M}$ is L-filter on X . Then $\mathcal{M}$ is said to be $\varphi_{1,2}$ L-convergence ([2]) to x and written $\mathcal{M} \xrightarrow[\varphi_{1,2}.\text{int}]{} x$, provided $\mathcal{M}$ is finer than the $\varphi_{1,2}$ - neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(x)$. Moreover, $\mathcal{M}$ is said to be $\varphi_{1,2}$ -convergence to F and written $\mathcal{M} \xrightarrow[\varphi_{1,2}.\text{int}]{} F$, provided $\mathcal{M}$ is finer than the $\varphi_{1,2}$ L-neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(x)$ for all $x \in F$, that is, $\mathcal{M}$ is finer than the $\varphi_{1,2}$ L-neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(F)$.

$\varphi_{1,2}$ **-closure L-sets**. Let a topological L-space (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. The $\varphi_{1,2}$ -closure of the L-set $\mu : X \to L$ is the mapping $\varphi_{1,2}.\text{cl}\,\mu : X \to L$ defined by:

$$(\varphi_{1,2}.\text{cl}\,\mu)(x) = \bigvee_{\mathcal{M} \le \mathcal{N}_{\varphi_{1,2}}(x)} \mathcal{M}(\mu)$$

for all $x \in X$. The L-filter $\mathcal{M}$ my have additional properties, e.g, we may assume that is homogeneous or even that is ultra. Obviously, $\varphi_{1,2}.\text{cl}\,\mu \ge \mu$ holds for all $\mu \in L^X$.

$\varphi_{1,2}$ $\psi_{1,2}$ **L-continuous and** $\varphi_{1,2}$ $\psi_{1,2}$ **L-open mappings.** In the following let a topological L-spaces (X, τ_1) and (Y, τ_2) are fixed, $\varphi_1, \varphi_2 \in O_{(L^X, \tau_1)}$ and $\psi_1, \psi_2 \in O_{(L^Y, \tau_2)}$. The mapping $f : (X, \varphi_{1,2}.\text{int}) \to (Y, \psi_{1,2}.\text{int})$ is said to be $\varphi_{1,2}$ $\psi_{1,2}$ L-continuous ([2]) if and only if

$$(\psi_{1,2}.\text{int}\,\eta) \circ f \le \varphi_{1,2}.\text{int}\,(\eta \circ f) \tag{2.4}$$

holds for all $\eta \in L^Y$. If an order reversing involution $\alpha \mapsto \alpha'$ of L is given, then we have that f is $\varphi_{1,2}$ $\psi_{1,2}$ L-continuous if and only if $\varphi_{1,2}.\text{cl}(\eta \circ f) \le (\psi_{1,2}.\text{cl}\,\eta) \circ f$ for all $\eta \in L^Y$, where $\varphi_{1,2}.\text{cl}$ and $\psi_{1,2}.\text{cl}$ are the closure operators related to $\varphi_{1,2}.\text{int}$ and $\psi_{1,2}.\text{int}$, respectively. Obviously if f is $\varphi_{1,2}$ $\psi_{1,2}$ L-continuity mapping, then the inverse mapping $f^{-1} : (Y, \psi_{1,2}.\text{int}) \to (X, \varphi_{1,2}.\text{int})$ is

$\psi_{1,2}$ $\varphi_{1,2}$ L-continuous mapping, that is, $(\varphi_{1,2}..\mathrm{int}\,\mu)\circ f^{-1} \le \psi_{1,2}.\mathrm{int}(\mu\circ f^{-1})$ holds for all $\mu\in L^X$. By means of the $\varphi_{1,2}$ L-neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(x)$ of $\varphi_{1,2}.\mathrm{int}$ at x and the $\psi_{1,2}$ L-neighborhood filter $\mathcal{N}_{\psi_{1,2}}(x)$ of $\psi_{1,2}.\mathrm{int}$ at x, the $\varphi_{1,2}$ $\psi_{1,2}$ L-continuity of f is also characterized as follows:

A mapping $f:(X,\varphi_{1,2}.\mathrm{int})\to(Y,\psi_{1,2}.\mathrm{int})$ is $\varphi_{1,2}$ $\psi_{1,2}$ L-continuous if for each $x\in X$ the inequality

$$\mathcal{N}_{\psi_{1,2}}(f(x))\ge \mathcal{F}_L f(\mathcal{N}_{\varphi_{1,2}}(x))$$

holds. Obviously, in the case of $L=\{0,1\}$, $\varphi_1=\psi_1=\mathrm{int}$, $\varphi_2=1_{L^X}$ and $\psi_2=1_{L^Y}$, the $\varphi_{1,2}$ $\psi_{1,2}$ L-continuity of f is coincides with the usual L-continuity.

Proposition 2.3 [2] Let $f:(X,\varphi_{1,2}.\mathrm{int})\to(Y,\psi_{1,2}.\mathrm{int})$ be a mapping between the characterized L-spaces $(X,\varphi_{1,2}.\mathrm{int})$ and $(Y,\psi_{1,2}.\mathrm{int})$. Then the following are equivalent:

(1) f is $\varphi_{1,2}$ $\psi_{1,2}$ L-continuous.

(2) For each L-filter $\mathcal{M}$ on X and each $x\in X$ such that $\mathcal{M}\xrightarrow[\varphi_{1,2}.\mathrm{int}]{} x$ we have $\mathcal{F}_L f(\mathcal{M})\xrightarrow[\psi_{1,2}.\mathrm{int}]{} f(x)$.

(3) For each $x\in X$, $\alpha\in L_0$ and $\psi_{1,2}$ α L-neighborhood η at $f(x)$, we have $\eta\circ f$ is an $\varphi_{1,2}$ α L-neighborhood at x.

(4) $f^{-1}(\eta)\in\beta_{\varphi_{1,2}.\mathrm{int}}$ for all $\eta\in\beta_{\psi_{1,2}.\mathrm{int}}$, where $\beta_{\varphi_{1,2}.\mathrm{int}}$ and $\beta_{\psi_{1,2}.\mathrm{int}}$ are the bases of $(X,\varphi_{1,2}.\mathrm{int})$ and $(Y,\psi_{1,2}.\mathrm{int})$, respectively.

We will denoted by **CRL-Sp, SCRL-Sp** and **CR-Sp** to the categories of all characterized L- spaces, stratified characterized L- spaces and the ordinary characterized spaces with the $\varphi_{1,2}$ $\psi_{1,2}$ L-continuity and $\varphi_{1,2}$ $\psi_{1,2}$-continuity as a morphismes between them, respectively. The objects in these categories are characterized L-spaces, stratified characterized L-spaces and characterizet spaces and will be dented by $(X,\varphi_{1,2}.\mathrm{int})$, $(X,\varphi_{1,2}.\mathrm{int}^S)$ and $(X,\varphi_{1,2}.\mathrm{int}_O)$, respectively.

The mapping $f:(X,\varphi_{1,2}.\mathrm{int})\to(Y,\psi_{1,2}.\mathrm{int})$ is said to be $\varphi_{1,2}$ $\psi_{1,2}$ L-open if and only if

$$f\circ(\varphi_{1,2}.\mathrm{int}\,\mu)\circ f \le \psi_{1,2}.\mathrm{int}(f\circ\mu) \tag{2.5}$$

holds for all $\mu\in L^X$. If an order reversing involution $\alpha\mapsto\alpha'$ of L is given, then we have that f is $\varphi_{1,2}$ $\psi_{1,2}$ L-open if and only if $\varphi_{1,2}.\mathrm{cl}(f\circ\mu)\le f\circ(\psi_{1,2}.\mathrm{cl}\,\mu)$ for all $\mu\in L^X$. The mapping $f:(X,\varphi_{1,2}.\mathrm{int})\to(Y,\psi_{1,2}.\mathrm{int})$ is said to be $\varphi_{1,2}$ $\psi_{1,2}$ L-homeomorphism if and only if it is bijective $\varphi_{1,2}$ $\psi_{1,2}$ L-continuous and $\varphi_{1,2}$ $\psi_{1,2}$ L-open mapping.

Proposition 2.4 [1] Let $f:(X,\varphi_{1,2}.\mathrm{int})\to(Y,\psi_{1,2}.\mathrm{int})$ be a mapping between the characterized L-spaces $(X,\varphi_{1,2}.\mathrm{int})$ and $(Y,\psi_{1,2}.\mathrm{int})$. Then the following are equivalent:

(1) f is $\varphi_{1,2}$ $\psi_{1,2}$ L-open.

(2) For each L-filter $\mathcal{N}$ on Y and each $y\in Y$ such that $\mathcal{N}\xrightarrow[\psi_{1,2}.\mathrm{int}]{} y$ we have $\mathcal{F}_L^- f(\mathcal{N})\xrightarrow[\varphi_{1,2}.\mathrm{int}]{} f^{-1}(y)$, where $\mathcal{F}_L^- f(\mathcal{N})$ is the preimage of $\mathcal{N}$.

(3) For each $y\in Y$, $\alpha\in L_0$ and $\varphi_{1,2}$ α L-neighborhood μ at $f^{-1}(y)$, we have $\mu\circ f^{-1}$ is an $\psi_{1,2}$ α L-neighborhood at y.

(4) $f(\mu) \in \psi_{1,2} OF(Y)$ for all $\mu \in \beta_{\varphi_{1,2}.\mathrm{int}}$, where $\beta_{\varphi_{1,2}.\mathrm{int}}$ is a base of $(X, \varphi_{1,2}.\mathrm{int})$.

Characterized L-topological groups. In the following let G is a multiplicative group. We denote, as usual, the identity element of G by e and the inverse of x in G by x^{-1}. Consider τ is an L-topology on G and $\varphi_1, \varphi_2 \in O_{(L^G, \tau)}$. Then the pair $(G, \varphi_{1,2}.\mathrm{int}_G)$ will be called an characterized L-topological group ([1]) if and only if the mappings:
$\alpha : (G \times G, \varphi_{1,2}.\mathrm{int}_G \times \varphi_{1,2}.\mathrm{int}_G) \to (G, \varphi_{1,2}.\mathrm{int}_G)$ and $\beta : (G, \varphi_{1,2}.\mathrm{int}_G) \to (G, \varphi_{1,2}.\mathrm{int}_G)$ that defined by:

$$\alpha((x, y)) = x\, y \qquad \forall\, (x, y) \in G \times G \tag{2.6}$$

and

$$\beta(x) = x^{-1} \qquad \forall\, x \in G \tag{2.7}$$

are $\varphi_{1,2}\ \varphi_{1,2}$ L- continuous, respectively.

If $\varphi_1 = \mathrm{int}$ and $\varphi_2 = 1_{L^X}$, then the characterized L-topological group $(G, \varphi_{1,2}.\mathrm{int}_G)$ is coincide with the L-topological group (G, τ) which is defined in [6,8]. As shown in [1], the characterized L-topological groups are characterized by an equivalent definition as will as in the following proposition:

Proposition 2.5 Let G be a multiplicative group, τ is an L-topology on G and $\varphi_1, \varphi_2 \in O_{(L^G, \tau)}$. Then, $(G, \varphi_{1,2}.\mathrm{int}_G)$ is characterized L-topological group if and only if the mapping $\gamma : (G \times G, \varphi_{1,2}.\mathrm{int}_G \times \varphi_{1,2}.\mathrm{int}_G) \to (G, \varphi_{1,2}.\mathrm{int}_G)$ which is defined by:

$$\gamma(x, y) = x\, y^{-1} \quad \text{for all } (x,y) \in G \tag{2.8}$$

is $\varphi_{1,2}\ \varphi_{1,2}$ L- continuous.

Denote by **CRL-TopGrp** and **CR-TopGrp** for the categories of all characterized L-topological groups and all characterized topological groups with all the $\varphi_{1,2}\ \varphi_{1,2}$ L-continuous homeomorphisms and with all the $\varphi_{1,2}\ \varphi_{1,2}$ -continuous homomorphism as morphisms mappings between them, respectively. As shown in [1], the category **CRL-TopGrp** is concrete category over the category **Grp** of all groups.

3. Initial and final characterized L-spaces

We make at first the relation between the farness on L-sets and the finer relation between characterized spaces to define the α -level and initial characterized spaces for an L-topological space (X, τ) by means of the functors ω and i. For an ordinary topological space (X, T), the induced characterized L-space is also introduced by using the functor ω. The functors ω and i are extended for any complete distributive lattice L to the functors functors ω_{L} and i_{L}. We further notions related to the notion of characterized L-spaces are e.g. those of characterized L-subspace, characterized product L-space, characterized quotient pre L-space and characterized sum L-space are investigated as special cases from the notions of initial and final characterized L-spaces. By the initial and final lefts in **CRL-Sp** we show that the category **CRL-Sp** is topological category in sense of [7,19] and it is also complete and co-complete category, that is, all limits and all co-limits in **CRL-Sp** exist, which of course are unique up to isomorphisms. Moreover, the category **SCRL-Sp** is bireflective subcategory of the category **CRL-Sp** and it is also topological category ([1]). Spacial cases we already described using the standard specifications, namly the characterized product and coproduct L-spaces. The latter type here is called characterized sum L-space. According to general procedure [6,12], the characterized L-subspaces together with their related inclusion mappings and the characterized quotient pre L-spaces together with their related canonical surjections are the equalizers and co-equalizers, respectively in **CRL-Sp**.

Let (X, τ) be a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then the α-level and the initial characterized spaces ([1]) of the characterized L-space $(X, \varphi_{1,2}.\text{int})$ will be denoted by $(X, \varphi_{1,2}.\text{int}_\alpha)$ and $(X, \varphi_{1,2}.\text{int}_i)$, respectively where $\varphi_{1,2}.\text{int}_\alpha$ and $\varphi_{1,2}.\text{int}_i$ are the $\varphi_{1,2}$-interior operators generates the two classes $(\varphi_{1,2}OF(X))_\alpha$ and $i(\varphi_{1,2}OF(X))$ which are given by

$(\varphi_{1,2}OF(X))_\alpha = \{\text{S}_\alpha(\mu) \in P(X) : \mu \in \varphi_{1,2}OF(X)\}$ and $i(\varphi_{1,2}OF(X)) = \inf\{(\varphi_{1,2}OF(X))_\alpha : \alpha \in L_1\}$, respectively, where inf is the infimum with respect to the finer relation on characterized spaces. On other hand if (X, T) is ordinary topological space and $\varphi_1, \varphi_2 \in O_{(P(X),T)}$, then the induced characterized L-space on X ([1]) will be denoted by $(X, \varphi_{1,2}.\text{int}_\omega)$, where $\varphi_{1,2}.\text{int}_\omega$ is the $\varphi_{1,2}$-interior operator generates the class $\omega(\varphi_{1,2}O(X))$ which is defined as follows:

$$\omega(\varphi_{1,2}O(X)) = \{\mu \in L^X : \quad \text{S}_\alpha(\mu) \in \varphi_{1,2}O(X) \text{ for all } \alpha \in L_1\}.$$

ω and i are functors in sense of Lowen in [17] in special case of $L = I$. These functors extended for any completely distributive complete lattice L in [1] as follows:

Let (X, τ) be a topological L-space, $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$ and $\psi_1, \psi_2 \in O_{(L^L, T)}$. Then, the characterizd spaces $(X, \varphi_{1,2}.\text{int}_{i_L})$ and $(X, \varphi_{1,2}.\text{int}_{\omega_L})$ are called initial characterized space and induced characterized L-space on X, respectively where $\varphi_{1,2}.\text{int}_{i_L}$ and $\varphi_{1,2}.\text{int}_{\omega_L}$ are the $\varphi_{1,2}$-interior operators generates the classes $i_L(\varphi_{1,2}OF(X))$ and $\omega_L(\varphi_{1,2}O(X))$ which are defined by the formulas:

$$i_L(\varphi_{1,2}OF(X)) = \inf\{\mu^{-1}\left(\text{UP}(\psi_{1,2}OF(L)\right) : \ \mu \in \varphi_{1,2}OF(X)\}$$

and

$$\omega_L(\varphi_{1,2}O(X)) = \ << C\left((X, \varphi_{1,2}O(X)), (L, \text{UP}(\psi_{1,2}OF(L))\right) >>$$

$C\left((X, \varphi_{1,2}O(X)), (L, \text{UP}(\psi_{1,2}OF(L))\right)$ is the set of all $\varphi_{1,2}\ \psi_{1,2}$-continuous mappings between $(X, \varphi_{1,2}O(X))$ and $(L, \text{UP}(\psi_{1,2}OF(L)))$, where $\text{UP}(\psi_{1,2}OF(L))$ is the upper $\psi_{1,2}$-open L-set generated by the set $L \setminus \downarrow(\text{a})$ for $\downarrow(\text{a}) = \{x \in L : x \le \text{a}\}$. If $\varphi_1 = \text{int}$ and $\varphi_2 = 1_{L^X}$, then the initial characterized space $(X, \varphi_{1,2}.\text{int}_{i_L})$ and the induced characterized L-space $(X, \varphi_{1,2}.\text{int}_{\omega_L})$ are coincide with the initial topological space $(X, i(\tau))$ and the induced topological L-space $(X, \omega(T))$ which are defined in [8]. As shown in [1], the functors ω_L : **CR - Sp** $\to$ **CRL - Sp** , i_L : **CRL - Sp** $\to$ **CR - Sp** and S_L : **CRL - Sp** $\to$ **SCRL - Sp** are concrete functors. Moreover, the category **SCRL-Sp** is bireflective subcategory of the category **CRL-Sp** and for each object $(X, \varphi_{1,2}.\text{int})$ of **CRL-Sp** the $\varphi_{1,2}\psi_{1,2}$ L-continuous mapping 1_X from the stratification $(X, \varphi_{1,2}.\text{int}^{\text{S}})$ of $(X, \varphi_{1,2}.\text{int})$ into $(X, \varphi_{1,2}.\text{int})$ is bi-coreflection of $(X, \varphi_{1,2}.\text{int})$.

Initial characterized L-spaces. Consider a family of characterized L-spaces $((X_i, \psi_{1,2}.\text{int}_i))_{i \in I}$ and for each $i \in I$, let $f_i : X \to X_i$ be a mapping from X into X_i. By an initial characterized L-space ([1]) of the family $((X_i, \psi_{1,2}.\text{int}_i))_{i \in I}$ with respect to $(f_i)_{i \in I}$, we mean the characterized L-space $(X, \varphi_{1,2}.\text{int})$ for which the following conditions are fulfilled:

(1) All the mappings $f_i : (X, \varphi_{1,2}.\text{int}) \to (X_i, \psi_{1,2}.\text{int}_i)$ are $\varphi_{1,2}\ \psi_{1,2}$ L-continuous.

(2) For an characterized L-space $(Y, \delta_{1,2}.\text{int})$ and a mapping $f: Y \to X$, the mapping $f: (Y, \delta_{1,2}.\text{int}) \to (X, \varphi_{1,2}.\text{int})$ is $\delta_{1,2}$ $\varphi_{1,2}$ L- continuous if all the mappings $f_i \circ f: (Y, \delta_{1,2}.\text{int}) \to (X_i, \psi_{1,2}.\text{int}_i)$ are $\delta_{1,2}$ $\psi_{1,2}$ L-continuous for all $i \in I$.

The initial characterized L-space $(X, \varphi_{1,2}.\text{int})$ for a family $((X_i, \psi_{1,2}.\text{int}_i))_{i \in I}$ of characterized L-spaces with respect to the family $(f_i)_{i \in I}$ of mappings exists and will be given by

$$\varphi_{1,2}.\text{int}\,\mu = \bigvee_{\mu_i \circ f_i \leq \mu, i \in I} (\psi_{1,2}.\text{int}_i\, \mu_i) \circ f_i \tag{3.1}$$

for all $\mu \in L^X$.

As showen in [1], the initial lefts and then the initial characterized L-spaces are uniquely exist in the category **CRL-Sp.** Hence, the category **CRL-Sp** is topological category over the category **SET** of all sets. Moreover, the initial characterized L-space $(X, \varphi_{1,2}.\text{int})$ for a family of characterized L-spaces $((X_i, \psi_{1,2}.\text{int}_i))_{i \in I}$ with respect to a family of mappings $(f_i)_{i \in I}$ is stratified if and only if $(X_i, \psi_{1,2}.\text{int}_i)$ is stratified for some $i \in I$. In the following we consider some special cases for the initial characterized L-spaces

Characterized L-subspaces. Let A be non-empty subset of a characterized L-space $(X, \varphi_{1,2}.\text{int})$ and $i_A: A \to X$ be the inclusion mapping of A into X. Then the mapping $\varphi_{1,2}.\text{int}_A: L^A \to L^A$ which is defined by:

$$\varphi_{1,2}.\text{int}_A\, \sigma = \bigvee_{\mu \circ i_A \leq \sigma} (\varphi_{1,2}.\text{int}\,\mu) \circ i_A \tag{3.2}$$

for all $\sigma \in L^A$ is initial $\varphi_{1,2}$-operator of $\varphi_{1,2}.\text{int}$ with respect to the inclusion mapping $i_A: A \to X$, called the induced $\varphi_{1,2}$-operator of $\varphi_{1,2}.\text{int}$ on the subset A of X and $(A, \varphi_{1,2}.\text{int}_A)$ is initial characterized L-space called characterized L-subspace ([1]) of the characterized L-space $(X, \varphi_{1,2}.\text{int})$. As showen in [1], the characterized L-subspaces $(A, \varphi_{1,2}.\text{int}_A)$ of the characterized L-spaces $(X, \varphi_{1,2}.\text{int})$ always exist and the related initial $\varphi_{1,2}$-operator of them is given by (3.2). Moreover, $(A, \varphi_{1,2}.\text{int}_A)$ is stratified if $(X, \varphi_{1,2}.\text{int})$ is stratified.

Characterized product L-spaces. Assume that for each $\mathrm{i} \in \mathrm{I}$, $(X_\mathrm{i}, \psi_{1,2}.\text{int}_\mathrm{i})$ be the characterized L-space of $\psi_{1,2}$-open L-subset of X_i. Let X be the cartesian product $\prod_{\mathrm{i} \in \mathrm{I}} X_\mathrm{i}$ of the family $(X_\mathrm{i})_{\mathrm{i} \in \mathrm{I}}$ and $P_\mathrm{i}: X \to X_\mathrm{i}$ is the related projection. Then the mapping $\varphi_{1,2}.\text{int}: L^X \to L^X$ which is defined by:

$$\varphi_{1,2}.\text{int}\,\mu = \bigvee_{\mu_\mathrm{i} \circ P_\mathrm{i} \leq \mu} (\psi_{1,2}.\text{int}_\mathrm{i}\, \mu_\mathrm{i}) \circ P_\mathrm{i} \tag{3.3}$$

for all $\mu \in L^X$ is initial $\varphi_{1,2}$-operator of $\psi_{1,2}.\text{int}_\mathrm{i}$ with respect to the projection mapping $P_\mathrm{i}: X \to X_\mathrm{i}$, called the $\varphi_{1,2}$-product operator of the $\psi_{1,2}$-interior operators $\psi_{1,2}.\text{int}_\mathrm{i}$ and $(X, \varphi_{1,2}.\text{int})$ is initial characterized L-space called characterized product L-space ([1]) of the characterized L-spaces $(X_\mathrm{i}, \psi_{1,2}.\text{int}_\mathrm{i})$ with respect to the family $(P_\mathrm{i}: X \to X_\mathrm{i})_{\mathrm{i} \in \mathrm{I}}$ of projections and will be denoted by $(\prod_{\mathrm{i} \in \mathrm{I}} X_\mathrm{i}, \prod_{\mathrm{i} \in \mathrm{I}} \psi_{1,2}.\text{int}_\mathrm{i})$.

Initial lefts in CRL-Sp. For the general notion of initial left we refer the standard books of category theory which include the categorical topology, e.g. [7,19]. The notion of initial left is meant here with respect to the forgetful functor of **CRL-Sp** to **SET**. It can be defined as follows:

The family of one and the same domain $\left(f_{\mathrm{i}}:(X,\varphi_{1,2}.\mathrm{int})\rightarrow(X_{\mathrm{i}},\psi_{1,2}.\mathrm{int}_{\mathrm{i}})\right)_{\mathrm{i}\in\mathrm{I}}$, where I is any classe in the category **CRL-Sp** is called initial left ([1]) of the family $\left(f_{\mathrm{i}}:X\rightarrow X_{\mathrm{i}},\psi_{1,2}.\mathrm{int}_{\mathrm{i}}\right)_{\mathrm{i}\in\mathrm{I}}$ provided for any characterized L-space $(Y,\sigma_{1,2}.\mathrm{int})$ of the $\sigma_{1,2}$ -open L-subsets of the set Y, the mapping $f:(Y,\sigma_{1,2}.\mathrm{int})\rightarrow(X,\varphi_{1,2}.\mathrm{int})$ is $\sigma_{1,2}$ $\varphi_{1,2}$ L-continuous if all the compositions $f_{\mathrm{i}}\circ f:(Y,\sigma_{1,2}.\mathrm{int})\rightarrow(X_{\mathrm{i}},\psi_{1,2}.\mathrm{int}_{\mathrm{i}})$ are $\sigma_{1,2}$ $\psi_{1,2}$ L-continuous. As showen in [1], for each family $\left(f_{\mathrm{i}}:X\rightarrow X_{\mathrm{i}},\psi_{1,2}.\mathrm{int}_{\mathrm{i}}\right)_{\mathrm{i}\in\mathrm{I}}$ of the mappings $f_{\mathrm{i}}:X\rightarrow X_{\mathrm{i}}$ and of $\psi_{1,2}$ -interior operators $\psi_{1,2}.\mathrm{int}_{\mathrm{i}}$ defined on the co-domains X_{i} of these mappings, the family $\left(f_{\mathrm{i}}:(X,\varphi_{1,2}.\mathrm{int})\rightarrow(X_{\mathrm{i}},\psi_{1,2}.\mathrm{int}_{\mathrm{i}})\right)_{\mathrm{i}\in\mathrm{I}}$ is initial left, where the initial $\varphi_{1,2}$ -interior operator $\varphi_{1,2}.\mathrm{int}$ defined by (3.1).

Lemma 3.1 [1] Let $(X,\varphi_{1,2}.\mathrm{int})$ and $(Y,\sigma_{1,2}.\mathrm{int})$ are the characterized product L-spaces for the families $\left((X_{\mathrm{i}},\psi_{1,2}.\mathrm{int}_{\mathrm{i}})\right)_{\mathrm{i}\in\mathrm{I}}$ and $\left((Y_{\mathrm{i}},\delta_{1,2}.\mathrm{int}_{\mathrm{i}})\right)_{\mathrm{i}\in\mathrm{I}}$ of characterized L-spaces. Then if foe each $\mathrm{i}\in\mathrm{I}$, the mapping $f_{\mathrm{i}}:(X_{\mathrm{i}},\psi_{1,2}.\mathrm{int}_{\mathrm{i}})\rightarrow(Y_{\mathrm{i}},\delta_{1,2}.\mathrm{int}_{\mathrm{i}})$ is $\psi_{1,2}$ $\delta_{1,2}$ L-continuous (resp. $\psi_{1,2}$ $\delta_{1,2}$ L- open) mapping, then the product mapping $f=\prod_{\mathrm{i}\in\mathrm{I}}f_{\mathrm{i}}:(X,\varphi_{1,2}.\mathrm{int})\rightarrow(Y,\sigma_{1,2}.\mathrm{int})$, which is defined by $f\left((x_{\mathrm{i}})_{\mathrm{i}\in\mathrm{I}}\right)=\left(f_{\mathrm{i}}(x_{\mathrm{i}})\right)_{\mathrm{i}\in\mathrm{I}}$ for all $(x_{\mathrm{i}})_{\mathrm{i}\in\mathrm{I}}\in X=\prod_{\mathrm{i}\in\mathrm{I}}X_{\mathrm{i}}$ is $\varphi_{1,2}$ $\sigma_{1,2}$ L- continuous (resp. $\varphi_{1,2}$ $\sigma_{1,2}$ L-open).

Final characterized L-spaces. It is well-known (cf.e.g [7,19]) that in a topological category all final lifts uniquely exiats and hence also all final structures exist. They are dually defined. In case of the category **CRL-Sp** the final structures can easily be given, as is shown in the following:

Let I be a class and for each $i\in I$, let $(X_i,\psi_{1,2}.\mathrm{int}_i)$ be a characterized L-space of $\psi_{1,2}$ -open L-subsets of X_i and $f_i:X_i\rightarrow X$ be a mapping from X_i into a set X. By a final characterized L-space of the family $\left((X_i,\psi_{1,2}.\mathrm{int}_i)\right)_{i\in I}$ with respect to the family $(f_i)_{i\in I}$, of mappings we mean the characterized L-space $(X,\varphi_{1,2}.\mathrm{int})$ for which the following conditions are fulfilled:

1. All the mappings $f_i:(X_i,\psi_{1,2}.\mathrm{int}_i)\rightarrow(X,\varphi_{1,2}.\mathrm{int})$ are $\psi_{1,2}$ $\varphi_{1,2}$ L-continuous.
2. For an characterized L-space $(Y,\delta_{1,2}.\mathrm{int})$ and a mapping $f:X\rightarrow Y$, the mapping $f:(X,\varphi_{1,2}.\mathrm{int})\rightarrow(Y,\delta_{1,2}.\mathrm{int})$ is $\varphi_{1,2}$ $\delta_{1,2}$ L- continuous if all the mappings $f\circ f_i:(X_i,\psi_{1,2}.\mathrm{int}_i)\rightarrow(Y,\delta_{1,2}.\mathrm{int})$ are $\psi_{1,2}$ $\delta_{1,2}$ L-continuous for all $i\in I$,

(See Fig. 3.1)

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ f_i\uparrow & \nearrow f\circ f_i & \\ X_i & & \end{array}$$

Fig.3.1

In the following proposition we show that the final characterized L-space $(X,\varphi_{1,2}.\mathrm{int})$ for a family $\left((X_i,\psi_{1,2}.\mathrm{int}_i)\right)_{i\in I}$ of characterized L-spaces with respect to the family $(f_i)_{i\in I}$ of mappings exists and will be defined.

Proposition 3.1 The final characterized L-space $(X,\varphi_{1,2}.\mathrm{int})$ for the family of characterized L-spaces $\left((X_i,\psi_{1,2}.\mathrm{int}_i)\right)_{i\in I}$ with respect to the family of mappings $(f_i)_{i\in I}$ always exists and it is given by:

$$(\varphi_{1,2}.\mathrm{int}\,\mu)(x)=\bigwedge_{x_i\in f_i^{-1}(\{x\}),i\in I}\psi_{1,2}.\mathrm{int}_i\,(\mu\circ f_i)(x_i)\wedge\mu(x) \qquad (3.4)$$

for all $x \in X$ and $\mu \in L^X$.

Proof. Let $\varphi_{1,2}.\text{int}$ be the operator defined (3.4). For each $x \in X$, $\mu \in L^X$ and for all $i \in I$ with $x_i \in f_i^{-1}(\{x\})$ we have $\bigwedge_{x_i \in f_i^{-1}(\{x\}), i \in I} \psi_{1,2}.\text{int}_i(\mu \circ f_i)(x_i) \wedge \mu(x) \geq \mu(x)$ and therefore $\varphi_{1,2}.\text{int}\,\mu \leq \mu$. Hence, $\varphi_{1,2}.\text{int}$ fulfills condition (I1). For condition (I2), let $\mu, \eta \in L^X$ with $\mu \leq \eta$, then $(\mu \circ f_i)(x) \geq (\eta \circ f_i)(x)$ and therefore $(\varphi_{1,2}.\text{int}\,\mu)(x) \geq \bigwedge_{x \in f_i^{-1}(\{x\}) \leq, i \in I} \psi_{1,2}.\text{int}_i(\mu \circ f_i)(x_i) \wedge \mu(x) \geq \bigvee_{x \in f_i^{-1}(\{x\}) \leq, i \in I} \psi_{1,2}.\text{int}_i(\eta \circ f_i)(x_i) \wedge \mu(x) = (\varphi_{1,2}.\text{int}\,\eta)(x)$ holds for all $x \in X$. Thus, condition (I2) is fulfilled. For all $x \in X$, $i \in I$ with $x_i \in f_i^{-1}(\{x\})$ we have $\bigwedge_{x_i \in f_i^{-1}(\{x\}), i \in I} \psi_{1,2}.\text{int}_i(\overline{1} \circ f_i)(x_i) \wedge \overline{1}(x) \leq \overline{1}(x)$ and therefore $\varphi_{1,2}.\text{int}\,\overline{1} = \overline{1}$. Hence, $\varphi_{1,2}.\text{int}$ fulfill condition (I3). Now, let $\mu, \eta \in L^X$ and $x \in X$, $i \in I$ such that $x_i \in f_i^{-1}(\{x\})$. Then from the distributives of L, we have that

$$
\begin{aligned}
(\varphi_{1,2}.\text{int}\,\mu \wedge \varphi_{1,2}.\text{int}\,\eta)(x) &= \bigwedge_{x_i \in f_i^{-1}(\{x\}), i \in I} \big(\psi_{1,2}.\text{int}(\mu \circ f_i) \wedge \psi_{1,2}.\text{int}(\eta \circ f_i)\big)(x_i) \wedge (\mu \wedge \eta)(x) \\
&\geq \bigwedge_{x_i \in f_i^{-1}(\{x\}), i \in I} \psi_{1,2}.\text{int}\big((\mu \wedge \eta) \circ f_i\big)(x_i) \wedge (\mu \wedge \eta)(x) \\
&= \varphi_{1,2}.\text{int}(\mu \wedge \eta)(x).
\end{aligned}
$$

Thus, $\varphi_{1,2}.\text{int}$ fulfills condition (I4). Clearly, $\varphi_{1,2}.\text{int}$ is idempotent, that is, condition (I5) is fulfilled. Hence, $(X, \varphi_{1,2}.\text{int})$ is characterized L-space. Since for all $i \in I$ with $f_i^{-1}(\{x\}) = \varphi$, we have $(\varphi_{1,2}.\text{int}\,\mu)(x) = \mu(x)$. Then, because of (3.4) for each $i \in I$ and $x_i \in X_i$, we have that the inequality $(\varphi_{1,2}.\text{int}\,\mu)(f_i(x_i)) \geq \psi_{1,2}.\text{int}(\mu \circ f_i)(x_i)$ holds and therefore, the inequality $(\varphi_{1,2}.\text{int}\,\mu) \circ f_i \leq \psi_{1,2}.\text{int}(\mu \circ f_i)$ is also holds. Hence, for each $i \in I$ all the mappings $f_i : (X_i, \psi_{1,2}.\text{int}_i) \to (X, \varphi_{1,2}.\text{int})$ are $\psi_{1,2}$ $\varphi_{1,2}$ L-continuous. Thus, condition (1) is fulfilled.

Now, let $(Y, \delta_{1,2}.\text{int})$ is a characterized L-space and $f : X \to Y$ be a mapping such that the mappings $f \circ f_i : (X_i, \psi_{1,2}.\text{int}_i) \to (Y, \delta_{1,2}.\text{int})$ are $\psi_{1,2}$ $\delta_{1,2}$ L-continuous for all $i \in I$. Then, we have that $(\delta_{1,2}.\text{int}\,\mu) \circ (f \circ f_i) \leq \psi_{1,2}.\text{int}_i(\mu \circ f \circ f_i)$ holds for all $\mu \in L^Y$ and because of (3.4) we have that $(\delta_{1,2}.\text{int}\,\mu)(f(x)) = \bigwedge_{x_i \in f_i^{-1}(\{x\}), i \in I} \psi_{1,2}.\text{int}_i(\mu \circ f_i)(f(x_i)) \wedge \mu(f(x)) \geq \bigwedge_{x_i \in f_i^{-1}(\{x\}), i \in I} \psi_{1,2}.\text{int}(\mu \circ f \circ f_i)(x_i) \wedge (\mu \circ f)(x) \geq \varphi_{1,2}.\text{int}(\mu \circ f)(x)$ is also holds for all $\mu \in L^Y$. Hence, the mapping $f : (X, \varphi_{1,2}.\text{int}) \to (Y, \delta_{1,2}.\text{int})$ is $\varphi_{1,2}$ $\delta_{1,2}$ L- continuous, that is, condition (2) is also fulfilled. Consequently, $(X, \varphi_{1,2}.\text{int})$ is final characterized L-space of the family $\big((X_i, \psi_{1,2}.\text{int}_i)\big)_{i \in I}$ of characterized L-spaces with respect to $(f_i)_{i \in I}$. □

Because of Proposition 3.1, all the final lefts and all the final characterized L-spaces are uniquely exist in the category **CRL-Sp** and hence **CRL-Sp** is a topological category over the category **SET** of all sets.

Proposition 3.2 The final characterized L-space $(X, \varphi_{1,2}.\text{int})$ for the family of characterized L-spaces $\big((X_i, \psi_{1,2}.\text{int}_i)\big)_{i \in I}$ with respect to the family of mappings $(f_i)_{i \in I}$ is stratified if and only if $(X_i, \psi_{1,2}.\text{int}_i)$ is stratified for some $i \in I$.

Proof. Assume that $(X_j, \psi_{1,2}.\text{int}_j)$ is stratified for $j \in I$. Then because of (3.4), we have that $(\varphi_{1,2}.\text{int}\,\overline{\alpha})(x) = \bigwedge_{x_j \in f_j^{-1}(\{x\}), j \in I} \psi_{1,2}.\text{int}_j(\tilde{\alpha}_j \circ f_j)(x_j) \wedge \overline{\alpha}(x) \leq \overline{\alpha}(x)$ holds for all $\alpha \in L$, where $\overline{\alpha}$ and $\tilde{\alpha}_j$ are the constant mappings on X and X_j hose value α and α_j, respectively. Hence, $\varphi_{1,2}.\text{int}\,\overline{\alpha} = \overline{\alpha}$ for all $\alpha \in L$ and therefore $(X, \varphi_{1,2}.\text{int})$ is stratified.
Conversely, let $(X, \varphi_{1,2}.\text{int})$ is stratified, that is $\varphi_{1,2}.\text{int}\,\overline{\alpha} = \overline{\alpha}$ for all $\alpha \in L$. Then $\bigwedge_{x_i \in f_i^{-1}(\{x\}), i \in I} \psi_{1,2}.\text{int}_i(\tilde{\alpha}_i \circ f_i)(x_i) \wedge \overline{\alpha}(x) = \overline{\alpha}(x)$ holds for all $x \in X$ and $i \in I$. Hence, there is $j \in I$ such that $(\psi_{1,2}.\text{int}_j\,\tilde{\alpha}_j)(x_j) \leq \overline{\alpha}(x)$ and $\overline{\alpha}(x) \leq (\tilde{\alpha}_j \circ f_j)(x_j) \leq \tilde{\alpha}_j(x_j)$, therefore $\psi_{1,2}.\text{int}_j\,\tilde{\alpha}_j = \tilde{\alpha}_j$ for some $j \in I$. Hence, $(X_j, \psi_{1,2}.\text{int}_j)$ is stratified for $j \in I$. □

In the following we consider the notions of a characterized quotient pre L-space and a characterized sum L-space as special cases from the final characterized L-spaces.

Characterized quotient pre L-spaces. Let A be non-empty L-subset of the characterized L-space $(X, \varphi_{1,2}.\text{int})$ and $f : X \to A$ is a surjective mapping of X into A. Then the mapping $\varphi_{1,2}.\text{int}_f : L^A \to L^A$ which is defined by:

$$(\varphi_{1,2}.\text{int}_f\,\mu)(a) = \bigwedge_{x \in f^{-1}(\{a\})} \varphi_{1,2}.\text{int}(\mu \circ f)(x) \tag{3.5}$$

for all $a \in A$ and $\mu \in L^A$ is final pre $\varphi_{1,2}$-interior operator of $\varphi_{1,2}.\text{int}$ with respect to the mapping $f : X \to A$ which is not idempotent, called the quotient pre $\varphi_{1,2}$-interior operator of $\varphi_{1,2}.\text{int}$ on the L-subset A and $(A, \varphi_{1,2}.\text{int}_f)$ is a final characterized L-space which is not idempotent called characterized quotient pre L-space of the characterized L-space $(X, \varphi_{1,2}.\text{int})$.
Note that in this case $\varphi_{1,2}.\text{int}$ is idempotent but $\varphi_{1,2}.\text{int}_f$ need not be. Even in the classical case of $L = \{0,1\}$ with choices $\varphi_1 = \text{int}$ and $\varphi_2 = 1_{L^X}$, we have that $\varphi_{1,2}.\text{int}$ is up to an identification the usual topology and $\varphi_{1,2}.\text{int}_f$ is up to an identification the usual pretopology which need not be idempotent. An example is given in [12] (p.234).

Proposition 3.3 Let A be non-empty subset of a characterized L-space $(X, \varphi_{1,2}.\text{int})$. Then the characterized quotient pre L-space $(A, \varphi_{1,2}.\text{int}_f)$ of $(X, \varphi_{1,2}.\text{int})$ always exists and the quotient $\varphi_{1,2}$-interior operator $\varphi_{1,2}.\text{int}_f$ is given by (3.5). If $(X, \varphi_{1,2}.\text{int})$ is stratified, then $(A, \varphi_{1,2}.\text{int}_f)$ also is.
Proof. Let $a \in A$ and $\mu \in L^A$ such that $x \in f^{-1}(\{a\})$ holds, then $\bigwedge_{x \in f^{-1}(\{a\})} \varphi_{1,2}.\text{int}(\mu \circ f)(x) \geq \mu(a)$ is also holds and therefore $\varphi_{1,2}.\text{int}_f\,\mu \leq \mu$ holds for all $\mu \in L^A$. Hence, $\varphi_{1,2}.\text{int}_f$ fulfills condition (I1). For condition (I2), let $a \in A$ and $\mu, \eta \in L^A$ with $\mu \leq \eta$ and $x \in f^{-1}(\{a\})$, then because of (3.5) we have $(\varphi_{1,2}.\text{int}_f\,\mu)(a) = \bigwedge_{x \in f^{-1}(\{a\})} \varphi_{1,2}.\text{int}(\mu \circ f)(x) \geq \bigwedge_{x \in f^{-1}(\{a\})} \varphi_{1,2}.\text{int}(\eta \circ f)(x) = (\varphi_{1,2}.\text{int}_f\,\eta)(a)$. Thus, condition (I2) is fulfilled. Since $\varphi_{1,2}.\text{int}\,\overline{1} = \overline{1}$ and $\mu \circ f \leq \overline{1}$ for all $\mu \in L^X$, then we have

$$(\varphi_{1,2}.\text{int}_f\,\overline{1})(a) = \bigwedge_{x \in f^{-1}(\{a\})} \varphi_{1,2}.\text{int}(\overline{1} \circ f)(x) \leq \bigwedge_{x \in f^{-1}(\{a\})} (\overline{1} \circ f)(x) = \overline{1}(a).$$

Hence, $\varphi_{1,2}.\text{int}_f$ fulfills condition (I3). Now, let $\mu, \eta \in L^A$ and $a \in A$ such that $x \in f^{-1}(\{a\})$. Then from the distributives of L and (3.5), we have that

$$(\varphi_{1,2}.\text{int}_f\ \mu \wedge \varphi_{1,2}.\text{int}_f\ \eta)(a) = \bigwedge_{x\in f^{-1}(\{a\})} (\varphi_{1,2}.\text{int}(\mu \circ f)(x) \wedge (\varphi_{1,2}.\text{int}(\eta \circ f)(x))$$
$$\geq \bigwedge_{x\in f^{-1}(\{a\})} (\varphi_{1,2}.\text{int}(\mu \wedge \eta) \circ f)(x) = \varphi_{1,2}.\text{int}_f(\mu \wedge \eta)(a).$$

Since $\varphi_{1,2}.\text{int}_f$ is isotone, it follows $\varphi_{1,2}.\text{int}_f\ \mu \wedge \varphi_{1,2}.\text{int}_f\ \eta = \varphi_{1,2}.\text{int}_f(\mu \wedge \eta)$. Thus, condition (I4) is also fulfilled. Hence, $(A, \varphi_{1,2}.\text{int}_f)$ is characterized pre L-space. Since for all $a \in A$ and $\mu \in L^A$, we have $(\varphi_{1,2}.\text{int}_f\ \mu \circ f)(a) \geq \varphi_{1,2}.\text{int}(\mu \circ f)(a)$, then the mapping $f:(X, \varphi_{1,2}.\text{int}) \to (A, \varphi_{1,2}.\text{int}_f)$ is $\varphi_{1,2}\ \varphi_{1,2}$ L-continuous. Hence, condition (1) is fulfilled.

Now, let $(Y, \delta_{1,2}.\text{int})$ is a characterized pre L-space and $g: A \to Y$ is a surjective mapping such that the composition $f \circ g:(A, \varphi_{1,2}.\text{int}_f) \to (Y, \delta_{1,2}.\text{int})$ is $\varphi_{1,2}\ \delta_{1,2}$ L-continuous mapping. Then, the inequality $(\delta_{1,2}.\text{int}\,\mu) \circ (f \circ g) \leq \varphi_{1,2}.\text{int}_f(\mu \circ f \circ g)$ holds for all $\mu \in L^Y$, therefore because of (3.5), the inequality $(\varphi_{1,2}.\text{int}_f\ \sigma)(f(a)) = \bigwedge_{x\in f^{-1}(\{a\})} \varphi_{1,2}.\text{int}(\sigma \circ g \circ f)(x) \geq \bigwedge_{x\in f^{-1}(\{a\})} \delta_{1,2}.\text{int}(\mu \circ g \circ f)(x) \geq$ $\delta_{1,2}.\text{int}(\sigma \circ f)(a)$ is also holds for all $a \in A$ and $\sigma \in L^A$. Hence, the mapping $f:(Y, \delta_{1,2}.\text{int}) \to (A, \varphi_{1,2}.\text{int}_f)$ is $\delta_{1,2}\ \varphi_{1,2}$ L- continuous, that is, condition (2) is also fulfilled. Consequently, $(A, \varphi_{1,2}.\text{int}_f)$ is initial characterized pre L-space.

Finally, let $(X, \varphi_{1,2}.\text{int})$ is stratified. Then, $\varphi_{1,2}.\text{int}\,\overline{\alpha} = \overline{\alpha}$ for all $\alpha \in L$ and therefore $\bigwedge_{x\in f^{-1}(\{a\})} \varphi_{1,2}.\text{int}(\overline{\alpha} \circ f)(x) = \tilde{\alpha}(a)$, where $\overline{\alpha}$ and $\tilde{\alpha}$ are the constant mappings on X and A hose value α, respectively. Because of (3.5), we have $\varphi_{1,2}.\text{int}_f\ \tilde{\alpha} = \tilde{\alpha}$ for all $\alpha \in L$. Hence, $(A, \varphi_{1,2}.\text{int}_f)$ is stratified. □

Characterized sum L-spaces. Assume that for each $i \in I$, $(X_i, \psi_{1,2}.\text{int}_i)$ be an characterized L-space of $\psi_{1,2}$-open L-subset of X_i. Let X be the disjoint union $\bigcup_{i\in I}(X_i \times \{i\})$ of the family $(X_i)_{i\in I}$ and for each $i \in I$, let $e_i: X_i \to X$ be the canonical injection of X_i into X given by $e_i(x_i) = (x_i, i)$. Then the mapping $\varphi_{1,2}.\text{int}: L^X \to L^X$ which is defined by:

$$(\varphi_{1,2}.\text{int}\,\mu)(a,i) = \psi_{1,2}.\text{int}_i(\mu \circ e_i)(a) \tag{3.6}$$

for all $i \in I$, $a \in X_i$ and $\mu \in L^X$ is final $\varphi_{1,2}$-interior operator of $(\psi_{1,2}.\text{int}_i)_{i\in I}$ with respect to the canonical injection $(e_i)_{i\in I}$. $\varphi_{1,2}.\text{int}$ will be called a sum $\varphi_{1,2}$- interior operator of the $\psi_{1,2}$-interior operators $(\psi_{1,2}.\text{int}_i)_{i\in I}$ and will be denoted by $\sum_{i\in I}\psi_{1,2}.\text{int}_i$. The pair $(X, \varphi_{1,2}.\text{int})$ is final characterized L-space called characterized sum L-space of the characterized L-spaces $(X_i, \psi_{1,2}.\text{int}_i)$ with respect to the family of canonical injection $(e_i)_{i\in I}$ and will be denoted by $\sum_{i\in I}(X_i, \psi_{1,2}.\text{int}_i)$ or $(X, \varphi_{1,2}.\text{int})$ for shorts.

Proposition 3.4 For each $i \in I$, let $(X_i, \psi_{1,2}.\text{int}_i)$ be a characterized L-space of $\psi_{1,2}$-open L-subset of X_i. Then the characterized sum L-prespace $\sum_{i\in I}(X_i, \psi_{1,2}.\text{int}_i)$ of $(X_i, \psi_{1,2}.\text{int}_i)$ always exists and the sum $\varphi_{1,2}$-interior operator $\varphi_{1,2}.\text{int}$ is given by (3.6). If $(X_i, \psi_{1,2}.\text{int}_i)$ stratified for each $i \in I$, then the characterized sum L-space $\sum_{i\in I}(X_i, \psi_{1,2}.\text{int}_i)$ is also stratified.

Proof. The first part is similar to that of Proposition 3.3. For the second part, let $\mathrm{i} \in I$, $a \in X_{\mathrm{i}}$ and $\alpha \in L^X$, where X is the disjoint union $\bigcup_{i \in I}(X_i \times \{i\})$ of the family $(X_{\mathrm{i}})_{\mathrm{i}\in \mathrm{I}}$. Because of (3.6) we have $(\varphi_{1,2}.\mathrm{int}\,\overline{\alpha})(a,i) = \psi_{1,2}.\mathrm{int}_{\mathrm{i}}(\overline{\alpha} \circ e_{\mathrm{i}})(a) = (\psi_{1,2}.\mathrm{int}_{\mathrm{i}}\,\overline{\alpha})(a,i) = \overline{\alpha}(a,i)$ and therefore $\varphi_{1,2}.\mathrm{int}\,\overline{\alpha} = \overline{\alpha}$. Hence, $\sum_{i \in I}(X_i, \psi_{1,2}.\mathrm{int}_{\mathrm{i}})$ is stratified. □

Final lefts in CRL-Sp. For the general notion of initial and final left we refer the standard books of category theory which include the categorical topology, e.g. [6,23]. The notion of final left is meant here with respect to the forgetful functor of **CRL-Sp** to **SET**. It can be defined as follows:
The family of one and the same co-domain $\big(f_{\mathrm{i}} : (X_{\mathrm{i}}, \psi_{1,2}.\mathrm{int}_{\mathrm{i}}) \to (X, \varphi_{1,2}.\mathrm{int})\big)_{\mathrm{i}\in\mathrm{I}}$, where I is any close of morphisms in the category **CRL-Sp** is called final left of the family $\big(f_{\mathrm{i}} : X_{\mathrm{i}} \to X, \psi_{1,2}.\mathrm{int}_{\mathrm{i}}\big)_{\mathrm{i}\in\mathrm{I}}$ provided for any characterized L-space $(Y, \sigma_{1,2}.\mathrm{int})$ of $\sigma_{1,2}$ -open subsets of Y , the mapping $f : (X, \varphi_{1,2}.\mathrm{int}) \to (Y, \sigma_{1,2}.\mathrm{int})$ is $\varphi_{1,2}$ $\sigma_{1,2}$ L-continuous if all the compositions mappings $f \circ f_{\mathrm{i}} : (X_{\mathrm{i}}, \psi_{1,2}.\mathrm{int}_{\mathrm{i}}) \to (Y, \sigma_{1,2}.\mathrm{int})$ are $\psi_{1,2}$ $\sigma_{1,2}$ L-continuous.

Proposition 3.7 For each family $\big(f_{\mathrm{i}} : X_{\mathrm{i}} \to X, \psi_{1,2}.\mathrm{int}_{\mathrm{i}}\big)_{\mathrm{i}\in\mathrm{I}}$ consisting of the mappings $f_{\mathrm{i}} : X_{\mathrm{i}} \to X$ and of the $\psi_{1,2}$ -interior operators $\psi_{1,2}.\mathrm{int}_{\mathrm{i}}$ on the domains X_{i} of these mappings, the family $\big(f_{\mathrm{i}} : (X_{\mathrm{i}}, \psi_{1,2}.\mathrm{int}_{\mathrm{i}}) \to (X, \varphi_{1,2}.\mathrm{int})\big)_{\mathrm{i}\in\mathrm{I}}$ with the final $\varphi_{1,2}$ -interior operator $\varphi_{1,2}.\mathrm{int} : L^X \to L^X$ of $(\psi_{1,2}.\mathrm{int}_{\mathrm{i}})_{\mathrm{i}\in\mathrm{I}}$ with respect to $(f_{\mathrm{i}})_{\mathrm{i}\in\mathrm{I}}$ defined by (3.4) is a final left.

Proof. Let a characterized L-space $(Y, \sigma_{1,2}.\mathrm{int})$ of $\sigma_{1,2}$ -open subsets of Y and a mapping $f : X \to Y$ be fixed. If all the mappings $f \circ f_{\mathrm{i}} : (X_{\mathrm{i}}, \psi_{1,2}.\mathrm{int}_{\mathrm{i}}) \to (Y, \sigma_{1,2}.\mathrm{int})$ are $\psi_{1,2}$ $\sigma_{1,2}$ L-continuous, that is, if $(\sigma_{1,2}.\mathrm{int}\,\eta) \circ (f \circ f_i) \le \psi_{1,2}.\mathrm{int}_i(\eta \circ f \circ f_i)$ holds for all $\eta \in L^Y$, then because of (3.4), we have that $(\sigma_{1,2}.\mathrm{int}\,\eta)(f(x)) = \bigwedge_{x_i \in f_i^{-1}(\{x\}), i \in I} \psi_{1,2}.\mathrm{int}_i(\eta \circ f_i)(f(x_i)) \wedge \eta(f(x)) \ge \bigwedge_{x_i \in f_i^{-1}(\{x\}), i \in I} \psi_{1,2}.\mathrm{int}(\eta \circ f \circ f_i)(x_i) \wedge (\eta \circ f)(x) \ge \varphi_{1,2}.\mathrm{int}(\eta \circ f)(x)$ holds for all $x \in X$ and $\eta \in L^Y$. Hence, the mapping $f : (X, \varphi_{1,2}.\mathrm{int}) \to (Y, \sigma_{1,2}.\mathrm{int})$ is $\varphi_{1,2}$ $\sigma_{1,2}$ L- continuous. Thus, the family $\big(f_{\mathrm{i}} : (X, \varphi_{1,2}.\mathrm{int}) \to (X_{\mathrm{i}}, \psi_{1,2}.\mathrm{int}_{\mathrm{i}})\big)_{\mathrm{i}\in\mathrm{I}}$ is a final left of $(\psi_{1,2}.\mathrm{int}_{\mathrm{i}})_{\mathrm{i}\in\mathrm{I}}$ with respect to $(f_{\mathrm{i}})_{\mathrm{i}\in\mathrm{I}}$. □

4. Initial characterized L-topological groups

In this section we show that the category **CRL-TopGrp** of all characterized L-topological groups is topological category over the category **Grp** of all groups and hence all initial characterized L-topological groups exist and can be characterized.

Consider a family of characterized L-topological groups $\big((G_i, \psi_{1,2}.\mathrm{int}_{G_i})\big)_{i \in I}$ and for each $i \in I$, let $f_i : G \to G_i$ be a homomorphism mapping from a group G into the groups G_i . Then for any characterized L-topological group $(G, \varphi_{1,2}.\mathrm{int}_G)$, the family $\big(f_i : (G, \varphi_{1,2}.\mathrm{int}_G) \to (G_i, \psi_{1,2}.\mathrm{int}_{G_i})\big)_{i \in I}$ is called initial lifts for the family $\big(f_i : G \to G_i, \psi_{1,2}.\mathrm{int}_{G_i}\big)_{i \in I}$ in the category **CRL-TopGrp** provided the following conditions are fulfilled:

(1) All the mappings $f_i : (G, \varphi_{1,2}.\mathrm{int}_G) \to (G_i, \psi_{1,2}.\mathrm{int}_{G_i})$ are $\varphi_{1,2}$ $\psi_{1,2}$ L-continuous homomorphism for all $i \in I$.

(2) For an characterized L-topological group $(H, \delta_{1,2}.\mathrm{int}_H)$ and a mapping $f : H \to G$, the mapping $f : (H, \delta_{1,2}.\mathrm{int}_H) \to (G, \varphi_{1,2}.\mathrm{int}_G)$ is $\delta_{1,2}$ $\varphi_{1,2}$ L- continuous homomorphism if and only if all the composition mappings $f_i \circ f : (H, \delta_{1,2}.\mathrm{int}_H) \to (G_i, \psi_{1,2}.\mathrm{int}_{G_i})$ are $\delta_{1,2}$ $\psi_{1,2}$ L-continuous homomorphism for all $i \in I$, (See Fig. 4.1)

$$\begin{array}{ccc} H & \xrightarrow{f} & G \\ & {\scriptstyle f_i \circ f} \searrow & \downarrow f_i \\ & & G_i \end{array}$$

Fig. 4.1

Hence, by an initial characterized L-topological group we mean the characterized L-topological group which provides the initial lifits in the category **CRL-TopGrp**.

To prove that all initial lifts and all initial characterized L-topological groups exist in the category **CRL-TopGrp** we need to prove at first that in case of $f_i : G \to G_i$ is an injective homomorphism for each $i \in I$, and $\varphi_{1,2}.\mathrm{int}_G$ is $\varphi_{1,2}$-interior operator for an initial characterized L-topology on a group G of $(\psi_{1,2}.\mathrm{int}_{G_i})_{i \in I}$ we get that $(G, \varphi_{1,2}.\mathrm{int}_G)$ is also characterized L-topological group. Now, we consider the case of I being a singleton.

Proposition 4.1 Let $(H, \delta_{1,2}.\mathrm{int}_H)$ be a characterized L-topological group and let $f : G \to H$ be an injective homomorphism from a group G into H. Then the initial characterized L-space $(G, f^{-1}(\delta_{1,2}.\mathrm{int}_H))$ of $(H, \delta_{1,2}.\mathrm{int}_H)$ with respect to f is characterized L-topological group.

Proof. Let at first $\gamma_G : (G \times G, f^{-1}(\delta_{1,2}.\mathrm{int}_H) \times f^{-1}(\delta_{1,2}.\mathrm{int}_H)) \to (G, f^{-1}(\delta_{1,2}.\mathrm{int}_H))$ and $\gamma_H : (H \times H, \delta_{1,2}.\mathrm{int}_H \times \delta_{1,2}.\mathrm{int}_H) \to (H, \delta_{1,2}.\mathrm{int}_H)$ are the mappings defined by (2.8) and let $\eta \in \beta_{f^{-1}(\delta_{1,2}.\mathrm{int}_H)}$, where $\beta_{f^{-1}(\delta_{1,2}.\mathrm{int}_H)}$ is the base of $(G, f^{-1}(\delta_{1,2}.\mathrm{int}_H))$ that generated by $f^{-1}(\delta_{1,2}.\mathrm{int}_H)$. Then, $\eta = f^{-1}(\rho)$ for some $\rho \in \beta_{\delta_{1,2}.\mathrm{int}_H}$. Since $(H, \delta_{1,2}.\mathrm{int}_H)$ is characterized L-topological group, then γ_H is $\delta_{1,2}$ $\delta_{1,2}$ L- continuous and therefore from Proposition 2.3, we have $\gamma_H^{-1}(\rho) \in \beta_{\delta_{1,2}.\mathrm{int}_H \times \delta_{1,2}.\mathrm{int}_H}$. Because of f is an injective homomorphism, then for all $x, y \in G$ we have

$$\begin{aligned} \gamma_G^{-1}\eta(x, y) &= (\rho \circ f \circ \gamma_G)(x, y) = (\rho \circ f)(x\, y^{-1}) \\ &= \rho(f(x) f(y^{-1})) = (\rho \circ \gamma_H)(f(x), f(y)) \\ &= (f \times f)^{-1}(\gamma_H^{-1}\rho)(x, y), \end{aligned}$$

that is, $\gamma_G^{-1}\eta = (f \times f)^{-1}(\gamma_H^{-1}\rho)$. Since $(G, f^{-1}(\delta_{1,2}.\mathrm{int}_H))$ is initial characterized L-space of $(H, \delta_{1,2}.\mathrm{int}_H)$ with respect to the mapping f, then $f : (G, f^{-1}(\delta_{1,2}.\mathrm{int}_H)) \to (H, \delta_{1,2}.\mathrm{int}_H)$ is $\delta_{1,2}$ $\delta_{1,2}$ L- continuous and from Lemma 3.1, it follows that the product mapping $f \times f : G \times G \to H \times H$ is $\delta_{1,2}$ $\delta_{1,2}$ L-continuous. Therefore, $(f \times f)^{-1}(\gamma_H^{-1}\rho) \in \beta_{(f \times f)^{-1}(\delta_{1,2}.\mathrm{int}_H \times \delta_{1,2}.\mathrm{int}_H)}$ and $\beta_{(f \times f)^{-1}(\delta_{1,2}.\mathrm{int}_H \times \delta_{1,2}.\mathrm{int}_H)} \subseteq \beta_{f^{-1}(\delta_{1,2}.\mathrm{int}_H) \times f^{-1}(\delta_{1,2}.\mathrm{int}_H)}$. Hence, $(f \times f)^{-1}(\gamma_H^{-1}\rho) \in \beta_{f^{-1}(\delta_{1,2}.\mathrm{int}_H) \times f^{-1}(\delta_{1,2}.\mathrm{int}_H)}$, that is, $\gamma_G^{-1}(\eta) \in \beta_{f^{-1}(\delta_{1,2}.\mathrm{int}_H) \times f^{-1}(\delta_{1,2}.\mathrm{int}_H)}$ and therefore from Proposition 2.3 it follows that γ_G is $\delta_{1,2}$ $\delta_{1,2}$ L-continuous. Hence, because of Proposition 2.5, $(G, f^{-1}(\delta_{1,2}.\mathrm{int}_H))$ is characterized L-topological group. □

Generally we consider the case of I is any class consistes of more than one elements .

Proposition 4.2 Let $\left((G_i, \psi_{1,2}.\text{int}_{G_i})\right)_{i\in I}$ be a family of characterized L-topological groups and for each $i \in I$, let $f_i : G \to G_i$ be an injective homomorphism from a group G into a group G_i . If $(G, \varphi_{1,2}.\text{int}_G)$ is the initial characterized L-space of the family $\left((G_i, \psi_{1,2}.\text{int}_{G_i})\right)_{i\in I}$ with respect to the family $(f_i)_{i\in I}$, then $(G, \varphi_{1,2}.\text{int}_G)$ is characterized L-topological group.

Proof. Let at first the mappings $\gamma_G : (G \times G, \varphi_{1,2}.\text{int}_G \times \varphi_{1,2}.\text{int}_G) \to (G, \varphi_{1,2}.\text{int}_G)$ and $\gamma_{G_i} : (G_i \times G_i, \psi_{1,2}.\text{int}_{G_i} \times \psi_{1,2}.\text{int}_{G_i}) \to (G_i, \psi_{1,2}.\text{int}_{G_i})$ are defined by (2.8). Since $f_i \circ \gamma_G = \gamma_{G_i} \circ (f_i \times f_i)$, f_i and γ_{G_i} are $\varphi_{1,2}\ \psi_{1,2}$ L-continuous and $\psi_{1,2}\ \psi_{1,2}$ L-continuous , respectively, then $f_i \circ \gamma_G$ is $\varphi_{1,2}\ \psi_{1,2}$ L-continuous . Because of condition of the initial lefts in the category **CRL-Top**, γ_G is $\varphi_{1,2}\ \varphi_{1,2}$ L-continuous and hence $(G, \varphi_{1,2}.\text{int}_G)$ is characterized L-topological group. □

In the following proposition we show that the initial lefts and then the initial characterized L-topological groups uniquely exist in the category **CRL-TopGrp.** Hence, the category **CRL-TopGrp** is topological category over the category **Grp** of all groups.

Proposition 4.3 Let $\left((G_i, \psi_{1,2}.\text{int}_{G_i})\right)_{i\in I}$ be a family of characterized L-topological groups and for each $i \in I$, let $f_i : G \to G_i$ be an injective homomorphism from a group G into a group G_i . If $(G, \varphi_{1,2}.\text{int}_G)$ is the initial characterized L-space of the family $\left((G_i, \psi_{1,2}.\text{int}_{G_i})\right)_{i\in I}$ with respect to the family of injective homomorphism mappings $(f_i)_{i\in I}$, then the family $\left(f_i : (G, \varphi_{1,2}.\text{int}_G) \to (G_i, \psi_{1,2}.\text{int}_{G_i})\right)_{i\in I}$ is an initial lift of $\left(f_i : G \to G_i,\ \psi_{1,2}.\text{int}_{G_i}\right)_{i\in I}$ in the category **CRL-TopGrp**.

Proof. Because of Propositions 4.1 and 4.2, $(G, \varphi_{1,2}.\text{int}_G)$ is characterized L-topological groip. From the definition of the initial lift in **CRL-Sp**, we get condition (1) from the definition of the initial lifit in **CRL-TopGrp** is fulfilled, that is, all mappings $f_i : (G, \varphi_{1,2}.\text{int}_G) \to (G_i, \psi_{1,2}.\text{int}_{G_i})$ are $\varphi_{1,2}\ \psi_{1,2}$ L-continuous homomorphism for all $i \in I$.
Let $(H, \delta_{1,2}.\text{int}_H)$ be a characterized L-topological group and a mapping $f : H \to G$ be a mapping. Then from the definition of the initial lift in **CRL-Sp**, we have that the mapping $f : (H, \delta_{1,2}.\text{int}_H) \to (G, \varphi_{1,2}.\text{int}_G)$ is $\delta_{1,2}\ \varphi_{1,2}$ L- continuous if and only if the composition mappings $f_i \circ f : (H, \delta_{1,2}.\text{int}_H) \to (G_i, \psi_{1,2}.\text{int}_{G_i})$ are $\delta_{1,2}\ \psi_{1,2}$ L-continuous for all $i \in I$. Now, let f is homomorphism. Since f_i is homomorphism for each $i \in I$, then $f_i \circ f$ is also homomorphism for all $i \in I$ .On other hand let $f_i \circ f$ is also homomorphism for all $i \in I$. Since f_i is homomorphism for each $i \in I$, then for all $a, b \in H$ we have

$$f_i(f(a\cdot b)) = (f_i \circ f)(a\cdot b) = f_i(f(a))\cdot f_i(f(b)) = f_i(f(a)\cdot f(b)).$$

Since f_i is injective for all $i \in I$, it follows that $f(a\cdot b) = f(a)\cdot f(b)$ for all $a, b \in H$, that is, f is homomorphism. Hence, $f : (H, \delta_{1,2}.\text{int}_H) \to (G, \varphi_{1,2}.\text{int}_G)$ is $\delta_{1,2}\ \varphi_{1,2}$ L- continuous homomorphism if and only if all the composition mappings $f_i \circ f : (H, \delta_{1,2}.\text{int}_H) \to (G_i, \psi_{1,2}.\text{int}_{G_i})$ are $\delta_{1,2}\ \psi_{1,2}$ L-continuous homomorphism for all $i \in I$. Thus, condition (2) from the definition of the initial lifit in **CRL-**

TopGrp is fulfilled. Consequently, $\big(f_i : (G, \varphi_{1,2}.\mathrm{int}_G) \to (G_i, \psi_{1,2}.\mathrm{int}_{G_i})\big)_{i \in I}$ is an initial lift of $\big(f_i : G \to G_i,\ \psi_{1,2}.\mathrm{int}_{G_i}\big)_{i \in I}$ in the category **CRL-TopGrp**. □

Because of Proposition 4.3, the characterized L-topological groups mentioned in Propositions 4.1 and 4.2 are coincide with the initial characterized L-topological groups, that is, if $\big((G_i, \psi_{1,2}.\mathrm{int}_{G_i})\big)_{i \in I}$ is a family of characterized L-topological groups and for each $i \in I$, the mapping $f_i : G \to G_i$ is an injective homomorphism and $(G, \varphi_{1,2}.\mathrm{int}_G)$ is the initial characterized L-space of the family $\big((G_i, \psi_{1,2}.\mathrm{int}_{G_i})\big)_{i \in I}$ with respect to the family of injective homomorphism mappings $(f_i)_{i \in I}$, then $(G, \varphi_{1,2}.\mathrm{int}_G)$ is initial characterized L-topological groups. Hence, the category **CRL-TopGrp** is concrete category of the category **L-Top** of all topological spaces and the faithful functor $\mathcal{F}$: **CRL – TopGrp →L – Top** is isomorphism. Thus, the category **CRL-TopGrp** is algebraic category over the category **L-Top** in sense of [7].

In the following we consider some special cases for the initial characterized L-topological groups.

Characterized L-subgroups. Let H be non-empty subgroup of a characterized L-topological group $(G, \varphi_{1,2}.\mathrm{int}_G)$ and $i_H : H \to G$ be the inclusion injective mapping of H into G. Then the mapping $\varphi_{1,2}.\mathrm{int}_H : L^H \to L^H$ which is defined by:

$$\varphi_{1,2}.\mathrm{int}_H\ \sigma = \bigvee_{\mu \circ i_H \le \sigma} (\varphi_{1,2}.\mathrm{int}_G\ \mu) \circ i_H \tag{4.1}$$

for all $\sigma \in L^H$ is initial $\varphi_{1,2}$-interior operator of $\varphi_{1,2}.\mathrm{int}_G$ with respect to the inclusion injective mapping $i_H : H \to G$, called an induced $\varphi_{1,2}$-interior operator of $\varphi_{1,2}.\mathrm{int}_G$ on the subgroup H of G and $(H, \varphi_{1,2}.\mathrm{int}_H)$ is initial characterized L-topological group called a characterized L-subgroup of the characterized L-topological group $(G, \varphi_{1,2}.\mathrm{int}_G)$.

Proposition 4.4 Let H be non-empty subgroup of a characterized L-topological group $(G, \varphi_{1,2}.\mathrm{int}_G)$. Then the characterized L-subgroup $(H, \varphi_{1,2}.\mathrm{int}_H)$ of $(G, \varphi_{1,2}.\mathrm{int}_G)$ always exists and the initial $\varphi_{1,2}$-interior operator $\varphi_{1,2}.\mathrm{int}_G$ is given by (4.1).

Proof. Immediate from Propositions 4.2 and 4.3. □

Characterized product L-topological groups. Assume that for each $i \in I$, $(G_i, \psi_{1,2}.\mathrm{int}_{G_i})$ be a characterized L-topological group and G be the cartesian product $\prod_{i \in I} G_i$ of the family $(G_i)_{i \in I}$ of groups. If $P_i : G \to G_i$ be the related injective projection, then the mapping $\varphi_{1,2}.\mathrm{int}_G : L^G \to L^G$ defined by:

$$\varphi_{1,2}.\mathrm{int}_G\ \mu = \bigvee_{\mu_i \circ P_i \le \mu} (\psi_{1,2}.\mathrm{int}_{G_i}\ \mu_i) \circ P_i \tag{4.2}$$

for all $\mu \in L^G$ is initial $\varphi_{1,2}$-interior operator of $\psi_{1,2}.\mathrm{int}_{G_i}$ with respect to the injective projection mapping $P_i : G \to G_i$, called product $\varphi_{1,2}$-interior operator of the $\psi_{1,2}$-interior operators $\psi_{1,2}.\mathrm{int}_{G_i}$ and $(G, \varphi_{1,2}.\mathrm{int}_G)$ is initial characterized L-topological group called characterized product L- topological group of the characterized L-topological groups $(G_i, \psi_{1,2}.\mathrm{int}_{G_i})$ with respect to the family $(P_i : G \to G_i)_{i \in I}$ of injective projections and will be denoted by $(\prod_{i \in I} G_i, \prod_{i \in I} \psi_{1,2}.\mathrm{int}_{G_i})$.

5. Final characterized L-topological groups

In this section we show that the final characterized L-topological group exists and it can be the final characterized L-spaces. Since the concrete category **CRL-TopGrp** of all characterized L-topological groups is topological category over the category **Grp** of all groups, then all final lifts also uniquely exist. This, even mean that also all final characterized L-topological groups exist.

Consider $\big((G_i, \psi_{1,2}.\mathrm{int}_{G_i})\big)_{i\in I}$ is a family of characterized L-topological groups and $(f_i)_{i\in I}$ be a family of homomorphism mappings from the groups G_i into the group G, indexed by the class I. Then for any characterized L-space $(G, \varphi_{1,2}.\mathrm{int}_G)$, the family $\big(f_i : (G_i, \psi_{1,2}.\mathrm{int}_{G_i}) \to (G, \varphi_{1,2}.\mathrm{int}_G)\big)_{i\in I}$ is called final lifts for the family $\big(f_i : G_i \to G,\ \psi_{1,2}.\mathrm{int}_{G_i}\big)_{i\in I}$ in the category **CRL-TopGrp**, provided $(G, \varphi_{1,2}.\mathrm{int}_G)$ is characterized L-topological group which fulfills the following conditions:

(1) All the mappings $f_i : (G_i, \psi_{1,2}.\mathrm{int}_{G_i}) \to (G, \varphi_{1,2}.\mathrm{int}_G)$ are $\psi_{1,2}\ \varphi_{1,2}$ L-continuous homomorphism for all $i \in I$.

(2) For an characterized L-topological group $(H, \delta_{1,2}.\mathrm{int}_H)$ and a mapping $f : G \to H$, the mapping $f : (G, \varphi_{1,2}.\mathrm{int}_G) \to (H, \delta_{1,2}.\mathrm{int}_H)$ is $\varphi_{1,2}\ \delta_{1,2}$ L- continuous homomorphism if and only if all the composition mappings $f \circ f_i : (G_i, \psi_{1,2}.\mathrm{int}_{G_i}) \to (H, \delta_{1,2}.\mathrm{int}_H)$ are $\psi_{1,2}\ \delta_{1,2}$ L-continuous homomorphism for all $i \in I$, (See Fig. 5.1)

$$\begin{array}{ccc} G & \xrightarrow{\ f\ } & H \\ f_i \uparrow & & \nearrow f \circ f_i \\ G_i & & \end{array}$$

Fig.5.1

Hence, by a final characterized L-topological group we mean the characterized L-topological group which provides the finl lifts in the category **CRL-TopGrp**.

To prove that all final lifts and all final characterized L-topological groups exist in the category **CRL-TopGrp** we need to prove that in case of $f_i : G_i \to G$ is an injective homomorphism for each $i \in I$, and $\varphi_{1,2}.\mathrm{int}_G$ is $\varphi_{1,2}$-interior operator for an final characterized L-topology on a group G of $(\psi_{1,2}.\mathrm{int}_{G_i})_{i\in I}$ we get that $(G, \varphi_{1,2}.\mathrm{int}_G)$ is also characterized L-topological group. To prove these results we need at first the following lemma.

Lemma 5.1 If $f : (G, \varphi_{1,2}.\mathrm{int}_G) \to (H, f(\varphi_{1,2}.\mathrm{int}_G))$ is surjective homomorphism mapping from the characterized L-topological groups $(G, \varphi_{1,2}.\mathrm{int}_G)$ to the group H equipped with the final characterized L-topology generated by $f(\varphi_{1,2}.\mathrm{int}_G)$ as a base with respect to f, then f is $\varphi_{1,2}\ \varphi_{1,2}$ L-open.

Proof. Immediate from Proposition 2.4. □

Now, we consider the case of I being a singleton.

Proposition 5.1 Let $(G, \varphi_{1,2}.\mathrm{int}_G)$ be a characterized L-topological group and let $f : G \to H$ be a homomorphism from a group G onto a group H. Then the final characterized L-space $(H, f(\varphi_{1,2}.\mathrm{int}_G))$ of $(G, \varphi_{1,2}.\mathrm{int}_G)$ with respect to f is characterized L-topological group.

Proof. Let at first $\gamma_H : (H \times H, f(\varphi_{1,2}.\mathrm{int}_G) \times f(\varphi_{1,2}.\mathrm{int}_G)) \to (H, f(\varphi_{1,2}.\mathrm{int}_G))$ and $\gamma_G : (G \times G,\ \varphi_{1,2}.\mathrm{int}_G \times \varphi_{1,2}.\mathrm{int}_G) \to (G,\ \varphi_{1,2}.\mathrm{int}_G)$ are the mappings defined by (2.8) and let

$\mu \in \beta_{f(\varphi_{1,2}.\mathrm{int}_G)}$, where $\beta_{f(\varphi_{1,2}.\mathrm{int}_G)}$ is the base of $(H, f(\varphi_{1,2}.\mathrm{int}_G))$ which is generated by $f(\varphi_{1,2}.\mathrm{int}_G)$, then $f^{-1}(\mu) \in \beta_{\varphi_{1,2}.\mathrm{int}_G}$. Since $(G, \varphi_{1,2}.\mathrm{int}_G)$ is characterized L-topological group, then γ_G is $\varphi_{1,2}$ $\varphi_{1,2}$ L-continuous and therefore from Proposition 2.3, we have $\gamma_G^{-1}(f^{-1}(\mu)) \in \beta_{\varphi_{1,2}.\mathrm{int}_G \times \varphi_{1,2}.\mathrm{int}_G}$. Because of Lemma 5.1, we have that the mapping $f : (G, \varphi_{1,2}.\mathrm{int}_G)) \rightarrow (H, f(\varphi_{1,2}.\mathrm{int}_G))$ is $\varphi_{1,2}$ $\varphi_{1,2}$ L-open and therefore Lemma 3.1 implies that the product mapping $f \times f : G \times G \rightarrow H \times H$ is also $\varphi_{1,2}$ $\varphi_{1,2}$ L-open. Since, $\gamma_H^{-1}(\mu) = (f \times f)(\gamma_G^{-1}(f^{-1}(\mu)))$, then we have $\gamma_H^{-1}(\mu) \in \beta_{f(\varphi_{1,2}.\mathrm{int}_G) \times f(\varphi_{1,2}.\mathrm{int}_G)}$. Therefore, because of Proposition 2.3, it follows that γ_H is $\varphi_{1,2}$ $\varphi_{1,2}$ L- continuous and consequently $(H, f(\varphi_{1,2}.\mathrm{int}_G))$ is characterized L-topological group. □

Generally, we consider the case of I is any class consistes of more than one element. Then we have the following result.

Proposition 5.2 Let $\left((G_i, \psi_{1,2}.\mathrm{int}_{G_i})\right)_{i \in I}$ be a family of characterized L-topological groups and for each $i \in I$, let $f_i : G_i \rightarrow G$ be a homomorphism from a group G onto a group G_i. If $(G, \varphi_{1,2}.\mathrm{int}_G)$ is the initial characterized L-space of the family $\left((G_i, \psi_{1,2}.\mathrm{int}_{G_i})\right)_{i \in I}$ with respect to the family $(f_i)_{i \in I}$, then $(G, \varphi_{1,2}.\mathrm{int}_G)$ is characterized L-topological group.

Proof. Let $\gamma_{G_i} : (G_i \times G_i, \psi_{1,2}.\mathrm{int}_{G_i} \times \psi_{1,2}.\mathrm{int}_{G_i}) \rightarrow (G_i, \psi_{1,2}.\mathrm{int}_{G_i})$ is a mapping defined by (2.8) and $\mu \in \beta_{\varphi_{1,2}.\mathrm{int}_G}$. Since $f_i : (G_i, \psi_{1,2}.\mathrm{int}_{G_i}) \rightarrow (G, \varphi_{1,2}.\mathrm{int}_G)$ is $\psi_{1,2}$ $\varphi_{1,2}$ L- continuous for all $i \in I$, then $f^{-1}(\mu) \in \beta_{\psi_{1,2}.\mathrm{int}_{G_i}}$ for all $i \in I$ and because of γ_{G_i} is $\psi_{1,2}$ $\psi_{1,2}$ L-continuous for all $i \in I$, then we have $\gamma_{G_i}^{-1}(f_i^{-1}(\mu)) \in \beta_{\psi_{1,2}.\mathrm{int}_{G_i} \times \psi_{1,2}.\mathrm{int}_{G_i}}$. Consider $\gamma_G : (G \times G, \varphi_{1,2}.\mathrm{int}_G \times \varphi_{1,2}.\mathrm{int}_G) \rightarrow (G, \varphi_{1,2}.\mathrm{int}_G)$ is a mapping defined by (2.8), then $\gamma_G^{-1}(\mu) = (f_i \times f_i)(\gamma_{G_i}^{-1}(f_i^{-1}(\mu)))$ and by a similar way to the proof of Proposition 5.1, we have the product mapping $f_i \times f_i$ is $\psi_{1,2}$ $\varphi_{1,2}$ L-open for all $i \in I$. Hence, $\gamma_G^{-1}(\mu) \in \beta_{\varphi_{1,2}.\mathrm{int}_G \times \varphi_{1,2}.\mathrm{int}_G}$ and therefore γ_G is $\varphi_{1,2}$ $\varphi_{1,2}$ L- continuous and consequently $(G, \varphi_{1,2}.\mathrm{int}_G)$ is characterized L-topological group. □

In the following proposition we show that the final lefts and then the final characterized L-topological groups uniquely exist in the concrete category **CRL-TopGrp**, that is, the characterized L-topological groups mentiond in Propositions 5.1 and 5.2 fulfills the conditions of the final lifts in the category **CRL-TopGrp**.

Proposition 5.3 Let $\left((G_i, \psi_{1,2}.\mathrm{int}_{G_i})\right)_{i \in I}$ be a family of characterized L-topological groups and for each $i \in I$, let $f_i : G_i \rightarrow G$ be an surjective homomorphism from the groups G_i into a group G. If $(G, \varphi_{1,2}.\mathrm{int}_G)$ is the final characterized L-space of the family $\left((G_i, \psi_{1,2}.\mathrm{int}_{G_i})\right)_{i \in I}$ with respect to the family of surjective homomorphism mappings $(f_i)_{i \in I}$, then the family $\left(f_i : (G_i, \psi_{1,2}.\mathrm{int}_{G_i}) \rightarrow (G, \varphi_{1,2}.\mathrm{int}_G)\right)_{i \in I}$ is a final lift of $\left(f_i : G_i \rightarrow G,\ \psi_{1,2}.\mathrm{int}_{G_i}\right)_{i \in I}$ in the category **CRL-TopGrp**.

Proof. The proof goes similarly by using Propositions 5.1 and 5.2 with the properties of the final lifts in the category as in case of Proposition 4.3. □

Because of Proposition 5.3, the characterized L-topological groups mentioned in Propositions 5.1 and 5.2 are coincide with the final characterized L-topological groups, that is, if $\big((G_i, \psi_{1,2}.\text{int}_{G_i})\big)_{i\in I}$ is a family of characterized L-topological groups and for each $i \in I$, the mapping $f_i : G_i \to G$ is an surjective homomorphism and $(G, \varphi_{1,2}.\text{int}_G)$ is the final characterized L-space of the family $\big((G_i, \psi_{1,2}.\text{int}_{G_i})\big)_{i\in I}$ with respect to the family of surjective homomorphism mappings $(f_i)_{i\in I}$, then $(G, \varphi_{1,2}.\text{int}_G)$ is final characterized L-topological groups. Hence, the category **CRL-TopGrp** is co-concrete category of the category **L-Top** of all topological spaces and the faithful functor $\mathscr{F}^*$: **L – Top → CRL – TopGrp** is isomorphism.

In the following we consider some special cases for the final characterized L-topological groups.

Characterized L-topological quotient groups. The characterized L-topological group is special final characterized L-topological group when the mapping $f : G \to H$ replaced by the canonical mapping $h : G \to G/N$, where N is normal subgroup the group G.
Let N be normal subgroup of the characterized L-topological group $(G, \varphi_{1,2}.\text{int}_G)$ and G/N is the corresponding quotient group. If $h : G \to G/N$ is the canonical homomorphism mapping defined by: $h(x) = x\,N$ for all $x \in G$, then $(G/N, h(\varphi_{1,2}.\text{int}_G))$ is final characterized L-topological group called characterized L-topological quotient group of the characterized L-topological group $(G, \varphi_{1,2}.\text{int}_G)$.

Proposition 5.4 Let $(G, \varphi_{1,2}.\text{int}_G)$ be a characterized L-topological group and N is a normal subgroup of G . If G/N is the corresponding quotient group, then the canonical surjective homomorphism $h : (G, \varphi_{1,2}.\text{int}_G) \to (G/N, h(\varphi_{1,2}.\text{int}_G))$ which is defined as $h(x) = x\,N$ for all $x \in G$ is $\varphi_{1,2}$ $\varphi_{1,2}$ L- open.
Proof. Follows directly from Lemma 5.1. □

In the following proposition we give the relation between characterized L-topological quotient groups and the characterized product L-topological groups.

Proposition 5.5 Let I be a class and for each $i \in I$, let $(G_i, \psi_{1,2}.\text{int}_{G_i})$ be a characterized L-topological group and N_i be a normal subgroup of G_i . If $G = \prod_{\text{i}\in\text{I}} G_\text{i}$ and $N = \prod_{\text{i}\in\text{I}} N_\text{i}$ are the related products of the least two families $(G_i)_{i\in I}$ and $(N_i)_{i\in I}$, respectively, then the isomorphism mapping $f : \big(G/N, h(\prod_{\text{i}\in I}\psi_{1,2}.\text{int}_{\text{G}_i})\big) \to \big(\prod_{i\in I}(G_i/N_i), (\prod_{\text{i}\in I} h_i(\psi_{1,2}.\text{int}_{\text{G}_i}))\big)$ is $\psi_{1,2}$ $\psi_{1,2}$ L- homeomorphism, where $h : (G, \prod_{\text{i}\in I}\psi_{1,2}.\text{int}_{\text{G}_i}) \to \big(G/N, h(\prod_{\text{i}\in I}\psi_{1,2}.\text{int}_{\text{G}_i})\big)$ and $h_i : (G_i, \psi_{1,2}.\text{int}_{\text{G}_i}) \to \big(G_i/N_i, h_i(\psi_{1,2}.\text{int}_{\text{G}_i})\big)$ are the related canonical surjective homomorphism's.

Proof. Because of the definition of characterized product L-topological groups and the characterized L-topological quotient groups we have that $\big(G/N, h(\prod_{\text{i}\in I}\psi_{1,2}.\text{int}_{\text{G}_i})\big)$ and $\big(\prod_{i\in I}(G_i/N_i), (\prod_{\text{i}\in I} h_i(\psi_{1,2}.\text{int}_{\text{G}_i}))\big)$ are characterized L-topological groups. Since h_i is $\psi_{1,2}$ $\psi_{1,2}$ L-continuous for all $i \in I$, then from Lemma 3.1 it follows that the product mapping $\prod_{\text{i}\in I} h_i : \big(G_i, h(\prod_{\text{i}\in I}\psi_{1,2}.\text{int}_{\text{G}_i})\big) \to \big(\prod_{i\in I}(G_i/N_i), (\prod_{\text{i}\in I} h_i(\psi_{1,2}.\text{int}_{\text{G}_i}))\big)$ is $\psi_{1,2}$ $\psi_{1,2}$ L- continuous. Hence,

$f(\mu) \in \beta_{\prod_{i \in I}(h_i(\psi_{1,2}.\text{int}_{G_i}))}$ implies $h^{-1}(f^{-1}(\mu)) = (\prod_{i \in I} h_i)^{-1}(\mu) \in \beta_{\prod_{i \in I}\psi_{1,2}.\text{int}_{G_i}}$. Because of Proposition 5.3, h is $\psi_{1,2}$ $\psi_{1,2}$ L- open and surjective mapping, therefore $f^{-1}(\mu) \in \beta_{h(\prod_{i \in I}\psi_{1,2}.\text{int}_{G_i})}$. Then, f is $\psi_{1,2}$ $\psi_{1,2}$ L-continuous isomorphism, that is, f is bijective $\psi_{1,2}$ $\psi_{1,2}$ L- continuous.

Now, let $\eta \in \beta_{h(\prod_{i \in I}\psi_{1,2}.\text{int}_{G_i})}$. Since h is $\psi_{1,2}$ $\psi_{1,2}$ L- continuous, then $h^{-1}(\eta) \in \beta_{\prod_{i \in I}\psi_{1,2}.\text{int}_{G_i}}$. Because of $\prod_{i \in I} h_i$ is the product of $\psi_{1,2}$ $\psi_{1,2}$ L- open mappings, then Lemma 3.1 implies that $\prod_{i \in I} h_i$ is $\psi_{1,2}$ $\psi_{1,2}$ L- open mapping. Therefore, $f(\eta) = (\prod_{i \in I} h_i)(h^{-1}(\eta)) \in \beta_{\prod_{i \in I}(h_i(\psi_{1,2}.\text{int}_{G_i}))}$, that is, f is $\psi_{1,2}$ $\psi_{1,2}$ L- open. Consequently, f is $\psi_{1,2}$ $\psi_{1,2}$ L- homeomorphism. □

6. Conclusion

In this paper, we introduced and studied the notions of final characterized L-spaces and initial and final characterized L-topological groups. The properties of such notions are deeply studied. By the notion of final characterized L-spaces, the notions of characterized quotient pre L-spaces and characterized sum L-spaces are introduced and studied. We show that all the final lefts and all the final characterized L-spaces are uniquely exist in the category **CRL-Sp** and hence **CRL-Sp** is topological category over the category **SET** of all sets. The characterized L-subspaces together with their related inclusion mappings and the characterized quotient pre L-spaces together with their related canonical surjection are the equalizers and co-equalizers, respectively in **CRL-Sp**. Moreover, we show that the initial and final lefts and then the initial and final characterized L-topological groups uniquely exist in the category **CRL-TopGrp.** Hence, the category **CRL-TopGrp** is topological category over the category **Grp** of all groups. By the notion of initial and final characterized L-topological groups, the notions of characterized L-subgroups, characterized product L-topological groups and characterized L-topological quotient groups are introduced and studied. However, we show that the category **CRL-TopGrp** is concrete and co-concrete category of the category **L-Top** of all topological L-spaces and that the faithful functors $\mathscr{F}: \textbf{CRL}-\textbf{TopGrp} \rightarrow \textbf{L}-\textbf{Top}$ and $\mathscr{F}^*: \textbf{L}-\textbf{Top} \rightarrow \textbf{CRL}-\textbf{TopGrp}$ are isomorphism's. Thus, the category **CRL-TopGrp** is algebraic and co-algebraic category over the category **L-Top** in sense of [7]. Many new special classes for the final characterized L-spaces, initial characterized L-topological groups, final characterized L-topological groups, characterized product L-topological groups and characterized L-topological quotient groups are listed in **Table (1).**

	Operations	Final Characterized L-spaces	Initial Characterized L-topol.Groups	Final Characterized L-topol. Groups	Characterized Product L-topol. groups	Characterized L-topol. Quotient groups
1	$\varphi_1 = \text{int}$ $\varphi_2 = 1_{L^X}$	Final L-top. space [18]	Initial L-topol. Group [6,8]	Final L- topol. Group [6,8]	Product L-topol. Group [6,8]	L- topol. Quotient group [6,8]
2	$\varphi_1 = \text{int}$ $\varphi_2 = \text{cl}$	Final θ L-space	Initial θ L-topol. Group	Final θ L- topol. Group	θ - product L- topol. Group	θ L – topol. Quotient group
3	$\varphi_1 = \text{int}$ $\varphi_2 = \text{int} \circ \text{cl}$	Final δ L-space	Initial δ L-topol. Group	Final δ L- topol. Group	δ - product L- topol. Group	δ L - topol. Quotient group
4	$\varphi_1 = \text{cl} \circ \text{int}$ $\varphi_2 = 1_{L^X}$	Final semi L-space	Initial semi L- topol. Group	Final semi L-topol. Group	Semi- product L- topol. Group	Semi L- topol. Quotient group
5	$\varphi_1 = \text{cl} \circ \text{int}$ $\varphi_2 = \text{cl}$	Final $(\theta.S)$ L-space	Initial $(\theta.S)$ L-topol. Group	Final $(\theta.S)$ L-topol. Group	$(\theta.S)$ - product L- topol. Group	$(\theta.S)$ L- topol. Quotient group
6	$\varphi_1 = \text{cl} \circ \text{int}$ $\varphi_2 = \text{int} \circ \text{cl}$	Final $(\delta.S)$ L-space	Initial $(\delta.S)$ L- topol. Group	Final $(\delta.S)$ L-topol. Group	$(\delta.S)$ - product L- topol. Group	$(\delta.S)$ L- topol. Quotient group
7	$\varphi_1 = \text{int} \circ \text{cl}$ $\varphi_2 = 1_{L^X}$	Final pre L-space	Initial pre L-topol. Group	Final pre L-topol. Group	Pre- product L- topol. Group	Pre L- topol. Quotient group
8	$\varphi_1 = \text{cl} \circ \text{int}$ $\varphi_2 = S.\text{cl}$	Final $(S.\theta)$ L-space	Initial $(S.\theta)$ L- topol. Group	Final $(S.\theta)$ L-topol. Group	$(S.\theta)$ - product L- topol. Group	$(S.\theta)$ L- topol. Quotient group
9	$\varphi_1 = \text{cl} \circ \text{int}$ $\varphi_2 = S.\text{int} \circ S.\text{cl}$	Final $(S.\delta)$ L-space	Initial $(S.\delta)$ L- topol. Group	Final $(S.\delta)$ L-topol. Group	$(S.\delta)$ - product L- topol. Group	$(S.\delta)$ L- topol. Quotient group
10	$\varphi_1 = \text{cl} \circ \text{int} \circ \text{cl}$ $\varphi_2 = 1_{L^X}$	Final β L-space	Initial β L-topol. Group	Final β L-topol. Group	β - product L- topol. Group	β L- topol. Quotient group
11	$\varphi_1 = \text{int} \circ \text{cl} \circ \text{int}$ $\varphi_2 = 1_{L^X}$	Final λ L-space	Initial λ L-topol. Group	Final λ L-topol. Group	λ - product L- topol. Group	λ L- topol. Quotient group
12	$\varphi_1 = S.\text{cl} \circ \text{int}$ $\varphi_2 = 1_{L^X}$	Final feebly L- space	Initial feebly L- topol. Group	Final feebly L-topol. Group	Feebly product L- topol. Group	Feebly L-topol. Quotient group

Table (1): Some special classes of final characterized L-spaces; initial characterized L-topological groups, final characterized L-topological groups characterized product L-topological groups and characterized L-topological quotient groups.

References

[1] Abd-Allah, A. S. (2014). Initial characterized L-spaces and characterized L- topological groups, Mathematical Theory and Modeling **4 (2)**, 86 - 106.
[2] Abd-Allah, A. S. (2002). General notions related to fuzzy filters, J. Fuzzy Math., **10 (2)**, 321 - 358.

[3] Abd-Allah, A. S. and El-Essawy, M. (2003). On characterizing notions of characterized spaces, J. Fuzzy Math., **11(4)**, 835 - 875.
[4] Abd-Allah, A. S. and El-Essawy, M. (2004). Closedness and compactness in characterized spaces, J. Fuzzy Math., **12 (3)**, 591 - 632.
[5] Abd El-Monsef, M. E., Zeyada, F. M., Mashour A. S. and El-Deeb, S. N. (1983). Operations on the power set P(X) of a topological space (X, T), Colloquium on topology,
Janos Bolyai Math. Soc. Eger, Hungry.
[6] Ahsanullah, T.M.G (1984). On fuzzy topological groups and semi groups, Ph.D. Thesis, Faculty of Science, Free Univ. of Brussels.
[7] Adȃmek , J. Herrlich, H. and Strecker, G. 1990. Abstract and Concrete Categories, John Wiley & Sons, Inc., New York et al.
[8] Bayoumi, F. (2005). On initial and final L-topological groups, Fuzzy Sets and Systems, 1**56**, 43 - 54.
[9] Chang, C. L. (1968). Fuzzy topological spaces, J. Math. Anal. Appl., **24**, 182 - 190.
[10] Eklund, P. and G$\ddot{a}$hler, W. Dordrecht et al. (1992). Fuzzy filter functors and convergence, in: Applications of Category Theory Fuzzy Subsets, Kluwer Academic Publishers, 109 - 136.
[11] Gähler , W. (1995). The general fuzzy filter approach to fuzzy topology, I, Fuzzy Sets and Systems, **76**, 205 - 224.
[12] Gähler , W. I (1977), II(1978). Grundstrukturen der Analysis, I und II, , Akademie Verlag Berlin, Birkhäuser Verlag Basel-Stuttgar.
[13] Gähler , W., Abd-Allah, A. S. and Kandil, A. (2000). On Extended fuzzy topologies, Fuzzy Sets and Systems, **109**, 149 - 172.
[14] Goguen, J. A. (1967). L-fuzzy sets, J. Math. Anal. Appl., **18**, 145 - 174.
[15] Kandil, A., Abd-Allah, A. S. and Nouh, A. A. (1999). Operations and its applications on L-fuzzy bitopological spaces, Part I, Fuzzy Sets and Systems, **106**, 255 - 274.
[16] Kasahara, S. (1979). Operation-compact spaces, Math. Japon., **24(1)**, 97 - 105.
[17] Lowen, R. (1976). Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl., **56**, 621 - 633.
[18] Lowen, R. (1977). Intial and final topologies and the fuzzy Tychonof Theorem, J. Math. Anal. Appl., **58**, 11 - 21.
[19] Richter, G. 1979. Kategorielle Algebra, Akademie Verlag Berlin.