Orlicz Space of Difference Analytic Sequences

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Abstract

In this Paper, we introduce difference analytic sequence spaces defined by Orlicz function and study some topological properties.

Keywords: analytic sequence, Orlicz sequence space, difference sequence space .

1. Introduction

A complex sequence, whose k[th] term is denoted by (x_k) . A sequence $x = (x_k)$ is said to be analytic, if $sup_k |x_k|^{1/k} < \infty$ .The vector space of all analytic sequence will be denoted by Λ .A sequence is entire sequence if $\lim_{k\to\infty} |x_k|^{1/k} = 0$. The vector space of all entire sequence will be denoted by Γ.

The notion of difference sequence space was introduced by Kizmaz [2],who studied the difference sequence spaces $L_\infty(\Delta), C(\Delta)$ and $C_0(\Delta)$. Kizmaz [2] defined the following difference sequence spaces,

$Z(\Delta) = \{x = (x_k): \Delta_x \in Z\}$, where

$\Delta_x = (\Delta_x)_{k=1}^{\infty} = (x_k - x_{k+1})_{k=1}^{\infty}$.

An Orlicz function $M: [0,\infty) \to [0,\infty)$ is a continuous ,non decreasing and convex function such that $M(0) = 0, M(x) > 0, for\ x > 0\ and\ M(x) \to \infty\ as\ x \to \infty$.

Lindenstrass and Tzafriri [4] used the idea of Orlicz function to defined the following sequence space

$\ell_M = \left\{x \in w: \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty\right\}$, where

$w = \{all\ complex\ sequence\ \}$, which is called an Orlicz sequence space. Also ℓ_M is a Banach space with the nom

$\|x\| = \inf\left\{\rho > 0: \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1\right\}$, and , they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p $(1 \le p < \infty)$.

2. Definition and Preliminaries

2.1 Definition

Let M be an Orlicz function . The space consisting of all those sequences x in w such that

$\left(sup_k \left(M \frac{|x_k|^{\frac{1}{k}}}{\rho}\right)\right) < \infty$, for some arbitrarily fixed $\rho > 0$ .Is denoted by Λ_M and is known as a

sequence of analytic sequences defined by a sequence of Orlicz function.

2.2 Definition [1]

A sequence space E is said to be solid or normal if $(\alpha_k x_k) \in E$ where $(x_k) \in E$ and for all sequences of scalars (α_k) with $|\alpha_k| \le 1$.

2.3 Definition [3]

Let V be a vector space over scalars k . A semi norm v on V is a real – valued function on V so that:

1. $v(x) \geq 0, for\ all\ x \in V$

2. $v(\alpha x) = |\alpha| v(x), for\ all\ \alpha \in k, x \in V$

3. $v(x+y) \leq v(x) + v(y), for\ all\ x, y \in V.$

2.4 Definition

Let (X, q) be a semi normed space over the field of complex numbers with the semi norm q. We denote $\Lambda(X)$ as the space of all analytic sequences defined over X.

We define the following sequence spaces :

$$\Lambda_M(\Delta, p, q) = \left\{ x \in \Lambda(X) : \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M\left(q\left(\frac{|\Delta x_k|^{1/k}}{\rho} \right) \right) \right]^{p_k} \right\} < \infty.$$

2.5 Definition [5]

Let X be a linear metric space .A function $p: X \to R$ is called paranorm, if

1. $p(x) \geq 0, for\ all\ x \in X,$

2. $p(-x) = p(x), for\ all\ x \in X,$

3. $p(x+y) \leq p(x) + p(y), for\ all\ x, y \in X,$

4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda\ as\ n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0\ \ as\ n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0\ \ as\ n \to \infty.$

2.6 Note

The following inequality will be used throughout the paper

Let $p = (p_k)$ be a sequence of positive real numbers with $0 \leq p_k \leq \sup p_k = G,\ K = \max(1, 2^{G-1})$ then

$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\}$

3. Main Results

3.1 Theorem

Let M be an Orlicz function ,then $\Lambda_M(\Delta, p, q)$ is linear space.

Proof:

Let $x = (x_k), y = (y_k) \in \Lambda_M(\Delta, p, q)$ and $\alpha, \beta \in C$, then we have

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M\left(q\left(\frac{|\Delta x_k|^{1/k}}{\rho_1} \right) \right) \right]^{p_k} < \infty \text{ , for some } \rho_1 > 0$$

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M\left(q\left(\frac{|\Delta y_k|^{1/k}}{\rho_2} \right) \right) \right]^{p_k} < \infty \text{ , for some } \rho_2 > 0,$$

Since M is Orlicz function and q semi norm and Δ is linear , then we get

$\sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\Delta(\alpha x_k + \beta y_k)|^{1/k}}{\rho_3} \right) \right) \right]^{p_k} < \infty$, for some $\rho_3 > 0$

$\leq \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\alpha|^{1/k} |\Delta x_k|^{1/k}}{\rho_3} + \frac{|\beta|^{1/k} |\Delta y_k|^{1/k}}{\rho_3} \right) \right) \right]^{p_k}$,where $\rho_3 = max\{|\alpha|^{1/k}\rho_1, |\beta|^{1/k}\rho_2\}$

$\leq K \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho_1} + \frac{|\Delta y_k|^{1/k}}{\rho_2} \right) \right) \right]^{p_k}$

$\leq K \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho_1} \right) \right) + M \left(q \left(\frac{|\Delta y_k|^{1/k}}{\rho_2} \right) \right) \right]^{p_k}$

$\leq K \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho_1} \right) \right) \right]^{p_k} + K \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\Delta y_k|^{1/k}}{\rho_2} \right) \right) \right]^{p_k} < \infty.$

3.2 Theorem

Let M', M'' be two an Orlicz functions, then $\Lambda_{M'}(\Delta, p, q) \cap \Lambda_{M''}(\Delta, p, q) \subseteq \Lambda_{M'+M''}(\Delta, p, q)$.

Proof:

Let $x = (x_k) \in \Lambda_{M'}(\Delta, p, q) \cap \Lambda_{M''}(\Delta, p, q)$, then $\exists\ \rho_1, \rho_2 > 0$ such that

$\sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M' \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho_1} \right) \right) \right]^{p_k} < \infty$

$\sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M'' \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho_1} \right) \right) \right]^{p_k} < \infty$, since $\rho > 0, such\ that\ \rho = \min(2\rho_1, 2\rho_2)$, then we have

$\sup_n \frac{1}{n} \sum_{k=1}^{n} \left[(M' + M'') \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho_1} \right) \right) \right]^{p_k}$

$\leq \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M' \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho_1} \right) \right) \right]^{p_k} + \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M'' \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho_1} \right) \right) \right]^{p_k} < \infty.$

Hence $(x_k) \in \Lambda_{M'+M''}(\Delta, p, q)$.

3.3 Theorem

The sequence space $\Lambda_M(\Delta, p, q)$ is solid .

Proof

Let $x = (x_k) \in \Lambda_M(\Delta, p, q)$, then

$\sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho} \right) \right) \right]^{p_k} < \infty.$

Let (α_k) sequence of scalars such that $|\alpha_k| \leq 1, \forall\ k \in \mathbb{N}$, then , we have

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\alpha_k \Delta x_k|^{1/k}}{\rho} \right) \right) \right]^{p_k} \leq \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\alpha_k|^{\frac{1}{k}} |\Delta x_k|^{1/k}}{\rho} \right) \right) \right]^{p_k}$$

$$\leq \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho} \right) \right) \right]^{p_k} < \infty.$$

3.4 Theorem

$\Lambda \subset \Lambda_M(\Delta, p, q)$ with the hypothesis that

$$\frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho} \right) \right) \right]^{p_k} \leq |x_k|^{1/k}$$

Proof

Let $(x_k) \in \Lambda$, then we have

$\sup |x_k|^{1/k} < \infty$. But

$\sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho} \right) \right) \right]^{p_k} \leq \sup |x_k|^{1/k}$, by our assumption, implies that

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho} \right) \right) \right]^{p_k} < \infty.$$

Then $(x_k) \in \Lambda_M(\Delta, p, q)$ and $\Lambda \subset \Lambda_M(\Delta, p, q)$.

3.5 Theorem

Let $0 \leq p_k \leq r_k$ and $\left\{\frac{r_k}{p_k}\right\}$ be bounded . Then $\Lambda_M(\Delta, r, q) \subseteq \Lambda_M(\Delta, p, q)$

Proof

Let $x = (x_k) \in \Lambda_M(\Delta, r, q)$, then

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho} \right) \right) \right]^{r_k} < \infty.$$

Let $t_k = \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho} \right) \right) \right]^{q_k}$ and $\lambda_k = \frac{p_k}{r_k}$, since $p_k \leq r_k$, we have $0 \leq \lambda_k \leq 1$. Take $0 < \lambda < \lambda_k$.

Define $u_k = \begin{cases} t_k & if \;\; t_k \geq 1 \\ 0 & if \;\; t_k < 1 \end{cases}$ and $r_k = \begin{cases} 0 & if \;\; t_k \geq 1 \\ t_k & if \;\; t_k < 1 \end{cases}$

$t_k = u_k + r_k$, $t_k^{\lambda_k} = u_k^{\lambda_k} + r_k^{\lambda_k}$. It follows that $u_k^{\lambda_k} \leq u_k \leq t_k$, $u_k^{\lambda_k} \leq r_k^{\lambda}$.

Since $t_k^{\lambda_k} = u_k^{\lambda_k} + r_k^{\lambda_k}$, then $t_k^{\lambda_k} \leq t_k + r_k^{\lambda}$. Thus

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho} \right) \right)^{r_k} \right]^{\lambda_k} \leq \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho} \right) \right) \right]^{r_k}$$

$\Rightarrow$

$$\sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho} \right) \right)^{r_k} \right]^{p_k/r_k} \leq \sup_n \frac{1}{n} \sum_{k=1}^{n} \left[M \left(q \left(\frac{|\Delta x_k|^{1/k}}{\rho} \right) \right) \right]^{r_k}$$

$\Rightarrow$

$$\sup_n \frac{1}{n} \sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta x_k|^{1/k}}{\rho}\right)\right)\right]^{p_k} \leq \sup_n \frac{1}{n} \sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta x_k|^{1/k}}{\rho}\right)\right)\right]^{r_k}.$$

But $\sup_n \frac{1}{n} \sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta x_k|^{\frac{1}{k}}}{\rho}\right)\right)\right]^{r_k} < \infty$.

Therefore $\sup_n \frac{1}{n} \sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta x_k|^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} < \infty$.

Hence $x = (x_k) \in \Lambda_M(\Delta, p, q)$. Thus, we have $\Lambda_M(\Delta, r, q) \subseteq \Lambda_M(\Delta, p, q)$.

3.6 Theorem

Let $0 < \inf p_k \leq p_k \leq 1$. Then $\Lambda_M(\Delta, p, q) \subset \Lambda_M(\Delta, q)$

Proof

Let $x = (x_k) \in \Lambda_M(\Delta, p, q)$. Then

$\sup_n \frac{1}{n} \sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta x_k|^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} < \infty$, since $0 < \inf p_k \leq p_k \leq 1$,

$$\sup_n \frac{1}{n} \sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta x_k|^{\frac{1}{k}}}{\rho}\right)\right)\right] \leq \sup_n \frac{1}{n} \sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta x_k|^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} < \infty.$$

Thus it follows that $x = (x_k) \in \Lambda_M(\Delta, q)$. Thus $\Lambda_M(\Delta, p, q) \subset \Lambda_M(\Delta, q)$.

3.7 Theorem

Let $1 \leq p_k \leq \sup p_k < \infty$. Then $\Lambda_M(\Delta, q) \subset \Lambda_M(\Delta, p, q)$.

Proof

Let $p_k \geq 1$ for each k and $\sup p_k < \infty$, let $x = (x_k) \in \Lambda_M(\Delta, q)$, $\Rightarrow$

$\sup_n \frac{1}{n} \sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta x_k|^{\frac{1}{k}}}{\rho}\right)\right)\right] < \infty$, since $1 \leq p_k \leq \sup p_k < \infty$, we have

$$\sup_n \frac{1}{n} \sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta x_k|^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \leq \sup_n \frac{1}{n} \sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta x_k|^{\frac{1}{k}}}{\rho}\right)\right)\right] \Rightarrow$$

$$\sup_n \frac{1}{n} \sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta x_k|^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} < \infty .$$

$\Rightarrow x = (x_k) \in \Lambda_M(\Delta, p, q)$. Therefore $\Lambda_M(\Delta, q) \subset \Lambda_M(\Delta, p, q)$.

3.8 Theorem

Let M be a sequence of Orlicz function .Then $\Lambda_M(\Delta, p, q)$ is a paranormed space with paranorm defined by

$$g(x) = \inf\left\{\rho^{\frac{p_m}{H}} : \sup \sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta x_k|^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} < \infty, \rho > 0\right\}, \text{ where } H = \max(1, \sup_k p_k).$$

Proof

Clearly $g(x) \geq 0, g(x) = g(-x)$ and $g(\theta) = 0, where\theta$ is the zero sequence of X.

Let $(x_k), (y_k) \in \Lambda_M(\Delta, p, q), \rho_1, \rho_2 > 0$, such that

$$\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta x_k|^{\frac{1}{k}}}{\rho_1}\right)\right)\right]^{p_k} < \infty \text{, and } \sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta y_k|^{\frac{1}{k}}}{\rho_2}\right)\right)\right]^{p_k} < \infty.$$

Let $\rho_1 + \rho_2 = \rho$. Then by using Minkowski inequality, we have

$$\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta(x_k+y_k)|^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k} \leq$$

$$\left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta x_k|^{\frac{1}{k}}}{\rho_1}\right)\right)\right]^{p_k} + \left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta y_k|^{\frac{1}{k}}}{\rho_2}\right)\right)\right]^{p_k} < \infty$$

.

$$\text{Hence } g(x+y) = \inf\left\{(\rho_1+\rho_2)^{\frac{p_m}{H}} : \left(\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta(x_k+y_k)|^{\frac{1}{k}}}{\rho_1+\rho_2}\right)\right)\right]^{p_k}\right)^{\frac{1}{t}}\right\}$$

$$\leq$$

$$\inf\left\{(\rho_1)^{\frac{p_m}{H}} : \left(\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta x_k)|^{\frac{1}{k}}}{\rho_1}\right)\right)\right]^{p_k}\right)^{\frac{1}{t}}\right\} +$$

$$\inf\left\{(\rho_2)^{\frac{p_m}{H}} : \left(\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{|\Delta y_k)|^{\frac{1}{k}}}{\rho_2}\right)\right)\right]^{p_k}\right)^{\frac{1}{t}}\right\}$$

.

Thus , we have $g(x+y) \leq g(x) + g(y)$. Hence satisfies the triangle inequality .

Now, let $\lambda_n \to \lambda \text{ as } n \to \infty \text{ and } g(x_n - x) \to 0 \text{ as } n \to \infty, \text{then } g(\lambda_n x_n - \lambda_x) \to 0 \text{ as } n \to \infty$. Since

$$g(\lambda_x) = \inf\left\{(\rho)^{\frac{p_m}{H}} : \left(\sup_n \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{|\lambda\Delta x_k)|^{\frac{1}{k}}}{\rho}\right)\right)\right]^{p_k}\right)^{\frac{1}{t}}\right\} < \infty.$$

Hence $\Lambda_M(\Delta, p, q)$ is a paranormed space.

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