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# The Existence and Uniqueness of the Solution of Partial Differential Equation by Using the Non-linear Transform Function

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**Abstract** : The development of Burger equation through the transform function is studied and we prove the existence and uniqueness of solution , also we give some applications .

## 1- **Introduction** :

The application of Adomian's decomposition method to partial differential equations, when the exact solution is not reached , demands the use of truncated series , but the solution's may have small convergence radius and the truncated series may be inaccurate in many regions ,in order to enlarge the convergence domain of the truncated series , pade approximations to the Adomain's series solution have been tested and applied to partial and ordinary differential equations, in [**1**] . Turbulence generated by Burgers' model equation yielded good approximation to real turbulence , the energy spectrum fell off like k to the -2 power for times greater than 6 seconds , the energy spectrum of turbulence generated by the model equation was found to follow the k to the -2 power law , in [**2**] . In [**3**] they are studied Burgers' equation and vector Burgers' equation with initial and boundary conditions , they are considered the Burgers' equation in the quarter plane x>0 , t>0 with Riemann type of initial and boundary conditions and use the HOPf-cole transformation to linearize the problems and explicitly solve them . Time and space splitting techniques are applied to the Burgers' equation and the modified Burgers' equation , and then the quintic B-spline collocation procedure is employed to approximate the resulting system , some numerical examples are studied to demonstrate the accuracy and efficiency of the proposed method , comparisons with both analytical solutions and some published numerical results are done in computational section , in [**4**] . The researcher is examination the numerical solution to Burgers' equation on a finite spatial with various boundary condition , he first conduct experiments to confirm the numerical solution observed by other researchers for Neumann boundary condition , also he considered the case where the non – homogeneous Robin boundary conditions approach the non- homogenous Neumann conditions , in [**5**] . The Projective Ordinary Differential assisted projective integration method based on the equation – free framework is presented , the method is essentially based on the slow manifold governing of given system , they have applied tow variants which are the "on – line" and "of – line " methods for solving the non- dimensional viscous Burgers' equation for the on – line method , they have computed the slow manifold by extracting the POD modes and used them on the fly along the projective integration process without assuming know ledge of the underlying slow manifold . In [**7**] they proposed differential quadratic method for calculating the numerical solution of nonlinear on – dimensional Burger – Huxley equation with appropriate initial and boundary conditions . In [8] they are constructed hump solution of Burgers' equation in terms of the self – similar solution of the heat equation following Kloosterziel approach , these self similar solution involve Hermite polynomials , the solution obtained here are compared with miller and Bernoff (large time) asymptotic solution . In [**9**] they are devoted to the numerical studied of various finite difference approximations to the stochastic Burgers' equation , of particular interest in the one- dimensional case is the situation where the driving noise is white both in space and in time , they are demonstrated that in this case , different finite difference schemes converge to different limiting processes as the mesh size tends to zero . In [**10**] , they are conducted a numerical studied if the one dimension viscous Burgers' equation and several Reduced Order Models over a range of parameter values , this studied is motivated by the need for robust reduced order model that can be used both for design and control , thus the model should first , allow for selection of optimal parameter value in a trade space and second , identify impacts from changes of parameter values that occur during development , production and sustainment of the designs . Burgers'- Huxley equations and their reduced form are of vital importance in modeling the interaction between reaction mechanisms , convection effects and diffusion transports , they applied the reduced form of differential transform method , in [**11**] . In [**12**] , first they are used the classification of one-dimensional subalgebras of Lie point symmetries admitted by Burgers' Equation and the corresponding reduced differential equations to construct a large class of new exact solutions, second, by using the Riccati transformation method , we obtain some new solutions of the Burgers equation namely , exponential, rational and periodic solutions. In **[13]** , they are presented the Burgers' equation in its viscous and non-viscous version, after submitting, as a motivation, some applications of this paradigmatic equations, they continue with the mathematical analysis of them . For Riemann data consisting of a single decreasing jump , they found that the Leray regularization captures the correct shock solution of the in viscid Burgers equation , for Riemann data consisting of a single increasing jump , the leray regularization captures an unphysical shock , this behavior can be remedied by consisting the behavior of the leray regularization with initial data consisting of an arbitrary mollification of the Riemann data , in [**14**] . Numerical solutions for the Burgers' equation based on the Galerkin's method using cubic B- splines as both weight and interpolation functions are set up , it is shown that this method is capable of solving Burgers' equation accurately for values of viscosity ranging form very small to large . in [**15**] . A new method for the solution of Burgers' equation is described, the marker method relies on the definition of a convective field associated with the underlying partial differential equation; the information about the approximate solution is associated with the response of an ensemble of markers to this convective field , some key aspects of the method, such as the selection of the shape function and the initial loading, are discussed in some details , the marker method is applicable to a general class of nonlinear dispersive partial differential equations, in [**16**]. In this work , we prove the existence and uniqueness of the development of Burgers' equation .

## 2- ***Statement of the problem :***

*Consider the second – order partial differential equation definitions of the following form :*

$$h(x)\,\frac{\partial u(x,t)}{\partial t} + C\,\frac{\partial u(x,t)}{\partial x} = k\,\frac{\partial^2 u(x,t)}{\partial x^2} \qquad (1.2)$$

$$C \ \ is\ constsnt \quad , \quad x \in R \ \ ,t > 0$$

$u(x,0) = f(x) \quad , \quad x \in R$

*where h(x)and f(x) are continuous functions define in* $\Omega$ , $\Omega=\{(x,t),$ $x \in R, t > 0\}$ *also we suppose the following condition is valid .*

*For given* $x \in R$ *If there exist* $\tau$ *and* $\xi_1 < \xi_2$ *such that*

$$u_0(\xi_j) = \frac{x-\xi_j}{\tau} \quad ,j = 1,2 \qquad (2.2)$$

$then\ u_k(x,t) \sim u_0(\xi_1), where\ \ t > \tau \ \ and\ u_k(x,t) \sim u_0(\xi_2)$ ,

$where\ \ t < \tau, as\ \varepsilon \to 0$ .

*We can change the equation by use the new variables*

$$u(x,t\ ) = -k^2\,\frac{V_x(x,t)}{V(x,t)} \qquad (3.2)$$

*where V(x , t) is continuous and differentiable function with respect to x and t . from (3.2) and (1.2) we get :*

$$u_x(x,t) = -k^2\,\frac{VV_{xx} - V_xV_x}{V^2} = -k^2\,\frac{V_{xx}}{V} + k^2\,\frac{V_x^2}{V^2}$$

$$u_{xx}(x,t) = -k^2\frac{VV_{xxx} - V_{xx}V_x}{V^2} + k^2\frac{V^2\,2V_x\,V_{xx} - V_x^2\,2V}{V^4}$$

$$= -k^2\frac{V_{xxx}}{V} + k^2\,\frac{V_{xx}V_x}{k^2} + 2k^2\,\frac{V_xV_{xx}}{V^2} - 2k^2\,\frac{V_x^2}{V^3}$$

$$= -k^2\,\frac{V_{xxx}}{V} + 3k^2\,\frac{V_{xx}V_x}{V^2} - 2k^2\,\frac{V_x^2}{V^3}$$

$$u_t(x,t) = -k^2\,\frac{V\,V_{xt} - V_x\,V_t}{V^2} = -h(x)k^2\,\frac{V_{xt}}{V} + h(x)\,k^2\,\frac{V_x\,V_t}{V^2}$$

*Compensate derivatives of x and t in the equation (1.2) , we get :*

 

$$[-h(x)k^2\frac{V_{xt}}{V}+h(x)k^2\frac{V_xV_t}{V^2}]+C\left[-k^2\frac{V_{xx}}{V}+k^2\frac{V_x^2}{V^2}\right]$$
$$=k\left[-k^2\frac{V_{xxx}}{V}+3k^2\frac{V_{xx}V_x}{k^2}-2k^2\frac{V_x^2}{V^3}\right]$$
$$-k^2C\frac{V_{xx}}{V}+k^2C\frac{V_x^2}{V^2}+2k^3\frac{V_x^2}{V^3}-3k^3\frac{V_{xx}V_x}{V^2}-h(x)k^2\frac{V_{xt}}{V}+$$
$$h(x)\,k^2\frac{V_xV_t}{V^2}+k^3\frac{V_{xxx}}{V}=0$$
$$\left[-k^2C\frac{V_{xx}}{V}-3k^3\frac{V_{xx}V_x}{V^2}\right]+\left[k^2C\frac{V_x^2}{V^2}+2k^3\frac{V_x^2}{k^3}\right]-$$
$$\left[h(x)k^2\frac{V_{xt}}{V}-h(x)k^2\frac{V_xV_t}{V^2}-k^3\frac{V_{xxx}}{V}\right]=0$$
$$\frac{1}{V}\left[-C\frac{\partial V(x,t)}{\partial x}-h(x)\frac{\partial V(x,t)}{\partial t}+k\frac{\partial^2V(x,t)}{\partial x^2}\right]$$
$$-\left[h(x)\frac{\partial V(x,t)}{\partial t}-3kV\frac{\partial V(x,t)}{\partial x}+C\frac{\partial V(x,t)}{\partial x}+2k\frac{V_x}{V}\right]=0$$

*If we suppose u =u(x , t) , then we solving the equation (1.2), is :*

$$f(x)=-k^2\frac{V_x(x,t)}{V(x,t)}=-k^2\frac{dv}{dx}\frac{1}{v}$$
$$\frac{f(x)}{dv}=-k^2\frac{1}{vdx}\rightarrow\int\frac{dv}{f(x)}=\frac{1}{-k^2}\int vdx$$
$$lnv_1=\frac{1}{-k^2}\int_0^x f(s)ds\ \rightarrow e^{lnv_1}=e^{\frac{-1}{k^2}\int_0^x f(s)ds}$$
$$V_1(x,0)=g(x)=Ce^{-F(x)}\qquad where\quad F(x)=\frac{1}{k^2}\int_0^x f(s)ds\,,$$

*and*

$$V(x,t)=\frac{1}{k\sqrt{\pi t}}\int_R e^{\frac{-1}{k^2}\int_0^\eta f(s)ds-\frac{1}{k^2t}(x-\eta)}\ h(\eta)d\eta\,,$$
$$=\frac{1}{k\sqrt{\pi t}}\int_R e^{\frac{-1}{k^2}Z(x,t,\eta)}\,h(\eta)d\eta\ ,$$

*where*

$$Z(x,t,\eta)=\frac{(x-\eta)}{t}+\int_0^\eta f(s)ds\ ,$$

*and*

$$u(x,t)=\frac{\int_R\frac{(x-\eta)}{t}e^{\frac{-1}{k^2}Z(x,t,\eta)}\ h(\eta)\,d\eta}{C}\tag{4.2}$$

*which agrees with what was found by the first approach for solving the Development Burger's equation. from (4.2) we get*

$$u(x,t)\sim\frac{(x-\eta)}{t}=u_0(\eta)$$

*Can rewrite the solution to be asymptotically*

$$u=u_0(\eta)$$
$$x=\eta+tu_0(\eta)\ \rightarrow\eta=x-tu_0\ \rightarrow\ \eta=x-tu$$

*Since* $u=u_0(\eta)$ , *then we get :*

$$u=u_0(x-tu)\tag{5.2}$$

*The equation (5.2) is exactly the solution of the problem*

$$h(x)\frac{\partial u(x,t)}{\partial t}+C\frac{\partial u(x,t)}{\partial x}=0\tag{6.2}$$

$$c \text{ is constsnt} \quad , \quad x \in R \quad , t > 0$$
$$u(x,0) = f(x) \quad , \quad x \in R \quad .$$

*We found the equation (6.2) by the method of characteristics, the solution of equation (5.2) is smooth for t, if the function $u_0(x)$ is smooth function and differentiate for x , then we have*

$$u_x = u_0'(\eta)(1 - tu_x) \ ,$$
$$u_x = \frac{u_0'(\eta)}{1 + u_0'(\eta)t} \ ,$$

*where*

$$1 + u_0'(\eta)t \neq 0 \quad ,$$

*If suppose $u'(x) < 0$ for every x , then $u_x = \infty$, if $t = \frac{-1}{u_0'(\eta)}$ is the first instant $T_O$ when $u_x = \infty$, known as gradient catastrophe, corresponds to a $S_o$ where $u'_0(x)$ has a minimum*

$$T_0 = \frac{-1}{u_0'(\eta_0)} \quad , \qquad u_0''(s_0) = 0$$

## 3- *Study some application about equation (1.2):*

*1.3 : Let consider the instant of gradient catastrophe for the problem*

$$h(x)\frac{\partial u(x,t)}{\partial t} + C\frac{\partial u(x,t)}{\partial x} = 0$$
$$C \text{ is constsnt} \quad , \quad x \in R \quad , t > 0$$
$$u(x,0) = e^{2kx}, \quad x \in R$$

*Solution: The solution of the problem*

$$u(x,t) = e^{2k(x - u(x,t)t)}$$

*For the function* $u_0(x) = e^{2kx}$

$$u_0' = 2ke^{2kx} \qquad \text{now at} \quad x = 0$$
$$\min u_0' = \max\,(2ke^{2kx})$$
$$\min u_0'(x) = 2k\,(\min e^{2kx}) = 2k$$

*Because* $e^{2kx} = e^{2k(0)} = e^0 = 1$

*Then* $T_0 = k \to 0 \text{ as } k \to 0$ *Note that*

$$\lim_{k \to 0}(e^{2kx}) = 1 + H(x)$$
$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

*The graphs of the functions* $e^{2kx}$

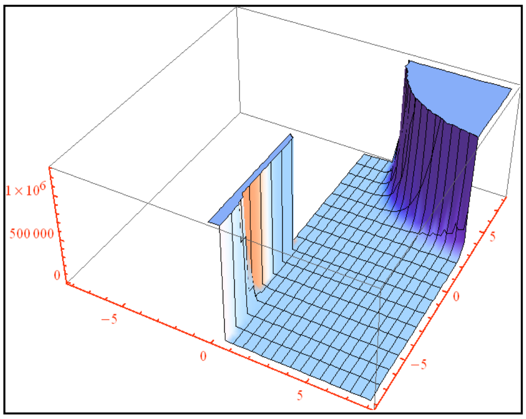


*Figure (1.3)*

*Plot3D[$e^{2kx}$, {e, -8 , 8 } , { kx , -8 , 8 } ]*

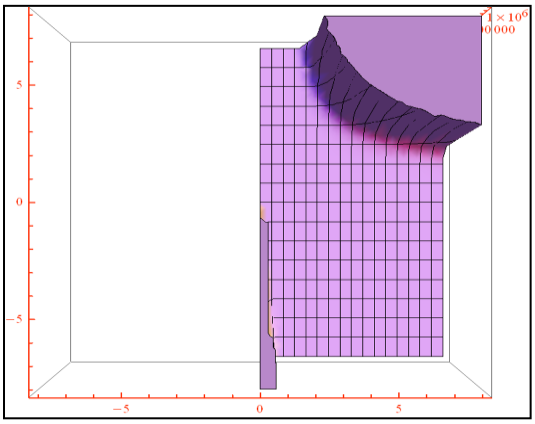


*Figure (2.3)*

*Plot3D[$e^{2kx}$, {e, -8 , 8 } , { kx , -8 , 8 } ]*

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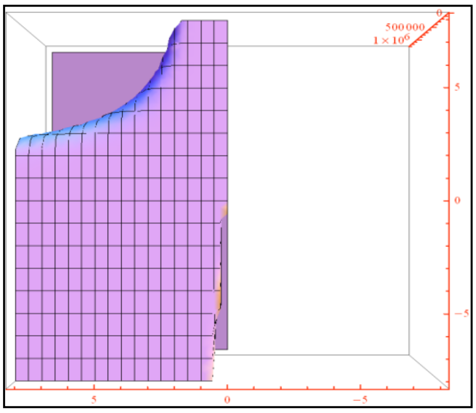

*Figure (3.3)*
*Plot3D[$e^{2kx}$, {e, -8 , 8 } , { kx , -8 , 8 } ]*

## 4- Procedure of solving the problem :

*Consider the following problem :*

$$h(x)\frac{\partial u(x,t)}{\partial t}+C\frac{\partial u(x,t)}{\partial x}=0 \qquad (1.4)$$

$$x\in R\ ,t>0\ \ ,C\ \ conistant$$

$$u(x,0)=f(x)=u_0(x)\ \ ,\ \ x\in R\ ,$$

*which is a limit case of Development Burgers' equation as* $\varepsilon\to 0$ .

*Definition 1.4 :*

*Assume* $u_0(x)\in L^1(R), A\ function\ u(x,t)\in L^2(R\times[0,\infty])is\ a\ weak\ solution\ of (1.2)\ \ iff$

$$\int_0^\infty\int_{-\infty}^\infty (h(x)u\beta_t+Cu\beta_x)dxdt+\int_{-\infty}^\infty u_0(x)\beta(x,0)dx=0$$

*for every test function* $\beta\in C_0^1\,(R\times[0,\infty])$ .

*And we need the following Proposition:*

***Proposition 1.4 [4]***

*Let u (x, t) be a smooth solution of the problem* $\frac{\partial u(x,t)}{\partial t}+a\frac{\partial u(x,t)}{\partial x}=0,\ x\in R\ ,t>0$ .

$u(x,0)=u_0(x)\ ,x\in R$ . *Then u (x ,t ) is a weak solution of the problem.*

***Proof:***

*Obviously* $u(x,t)\in L^1_{loc}(R\times[0,\infty])$ $.Let\ \rho(x,t)\in C_0^1(R\times[0,\infty])$ *and* $supp\rho\subseteq[-T,T]\times[0,T]$ *Multiplying* $u_t+au_x=0\ by\ \rho$ *integrating in* $[-T,T]\times[0,T]$ *and using* $\rho(\pm T,t)=\rho(x,T)=0$ *we obtain :*

$$0=\int_0^T\int_{-T}^T\left(\frac{\partial u(x,t)}{\partial t}+a\frac{\partial u(x,t)}{\partial x}\right)\rho dxdt$$

$$=\int_0^T\int_{-T}^T((u\rho)_t+a(u\rho)_x-u(\rho_t+a\rho_x)dxdt$$

 

$$= \int_0^T \int_{-T}^T (u\rho)_t dtdx - \int_0^T \int_{-T}^T u(\rho_t + a\rho_x) dxdt$$

$$= -\left( \int_{-T}^T u_0(x)\rho(x,0)dx + \int_0^T \int_{-T}^T u(\rho_t + a\rho_x) dxdt \quad \blacksquare\right.$$

***Theorem 1.4***

*For a given function $u \in W_{loc}^{2,1}(\Omega)$ the regularizations $J_\varepsilon u$ tend to u in $W^{2,1}$ (k) for every compact $k \subset \Omega$ , i ,e $\| J_\varepsilon u - u \| \, W^{2,1}(k) \to 0$ as $\varepsilon \to 0$ .* [4]

*From the theorem (1.4) and proposition (1.4), the following theorem is holds :*

***Theorem 2.4***

*Let $u \in C^1(R \times [0,\infty])$ be a smooth solution of the equation $h(x)u_t + Cu_x = 0$ and a weak solution of the problem (1.4). If $u_0(x)$ is continuous at a point $x_0$ , then $u(x_0 , 0) = u_0(x)$ .*

***Proof :***

*Let $\beta(x,t) \in C_0^1(R \times [0,\infty])$. As in proposition (1.4) we are led to*

$$\int_{-\infty}^{\infty} \left(u(x,0) - u_0(x)\right)\beta(x,0)dx = 0$$

*Suppose $u(x_0,0) > u_0(x_0)$ . By continuity there exist a neighborhood U such that*

$$u(x,0) > u_0(x) \ , \quad x \in U$$

*Take $\beta(x,t) \in C_0^1(R \times [0,\infty])$ such that*

$$supp\beta(x,0) = [a,b] \subset U \ , \beta(x) > 0 \ , x \in (a,b) \ ,$$
$$\beta(a) = \beta(b) = 0$$

*Then*

$$\int_{-\infty}^{\infty} \left(u(x,0) - u_0(x)\right) \beta(x,0)dx$$
$$= \int_{-\infty}^{\infty} \left(u(x,0) - u_0(x)\right)\beta(x,0)dx > 0$$

*Which is a contradiction , similarly $u(x_0 , 0) < u_0(x_0)$ is impossible . then $u(x_0 , 0) = u_0(x_0)$* ■

*Now let us consider the initial data*

$$u_0(x) = \begin{cases} u_l & x < 0 \\ u_\tau & x > 0 \end{cases} \qquad (2.4)$$

*where $u_l$ and $u_\tau$ are constant .*

*The two cases $u_l > u_\tau$ and $u_l < u_\tau$ are quite different with respect to the solvability of problem(1.4) . It can be proved that if $u_l > u_\tau$ then the weak solution is unique, while if $u_l < u_\tau$ then there exist infinitely many solutions.*

## *5- Studying the following cases for problem (1.4) :*

***<u>Case I</u>*** *: $u_l > u_\tau$ then we have the following problem*

$$h(x)\, u_t(x,t) + Cu_x(x,t) = ku_{xx}(x,t)$$
$$C \text{ is constsnt } , \quad x \in R \ , t > 0$$
$$u(x,0) = f(x) = u_0(x) \ , \quad x \in R$$

*If $u_l > u_\tau$ we are in a situation to apply the theorem* (1.4)

*Let x > 0 be fixed and*

 

$$s = \frac{u_l + u_\tau}{2}$$

*The instant $\tau$ of the theorem (2.4)and by using the condition (2.2) is determined by the slope k of the straight line through the points (x,0) and (0,s)*

$$k = -\frac{1}{\tau} = -\frac{s}{x} \quad ,$$

*then*

$$\tau = \frac{x}{s} \quad ,$$

*and*

$$u(x,t) \sim \begin{cases} u_l & x < st \\ u_\tau & x > st \end{cases} \qquad as\ \varepsilon \to 0$$

*The unique solution of (1.4) is known as a shock wave , while $s = (u_l + u_\tau)/2$ is a shock speed the speed at which the discontinuity of the solution travels .*

***Theorem 1.5***

*Let the function*

$$u(x,t) = \begin{cases} u_l & x < st \\ u_\tau & x > st \end{cases} \qquad (1.5)$$

*Is a weak solution of the problem (1.4) with initial data (2.4) , where*

$$u_l > u_\tau \quad , \quad s = \frac{u_l + u_\tau}{2}$$

***Proof** :*

*Let $\beta(x,t) \in C_0^1(R \times [0,\infty])$. Denote for simplicity*

$$A = \int_0^\infty \int_{-\infty}^{\infty} (h(x)u\beta_t + Cu\beta_x)dxdt$$

$$B = -\int_{-\infty}^{\infty} u(x,0)\ \beta(x,0)dx$$

*We have*

$$A = \int_0^\infty \int_{-\infty}^{st} (h(x)\beta_t u_l + C\beta_x u_l)dxdt + \int_0^\infty \int_{st}^{\infty} (h(x)\beta_t u_\tau + C\beta_x u_\tau)dxdt$$

$$A_1 = \int_0^\infty \int_{-\infty}^{st} (h(x)\beta_t u_l + C\beta_x u_l)dxdt$$

$$= u_l \int_0^\infty ( \int_{-\infty}^{st} h(x)\beta_t dx)dt + Cu_l \int_0^\infty ( \int_{-\infty}^{st} \beta_x dx)dt$$

*By*

$$\int_{-\infty}^{st} \beta_t(x,t)dx = \frac{d}{dt}\int_{-\infty}^{st} \beta(x,t)dx - \beta(st,t)s \quad ,$$

*and*

$$\int_{-\infty}^{st} \beta_x(x,t)dx = \beta(st,t) \quad ,$$

*then by using the Retail Integration of $A_1$ we get*

$$A_1 = \int_0^{\infty} \int_{-\infty}^{st} (h(x)\beta_t u_l + C\beta_x u_l) dx dt = 0$$

*Similarly*

$$A_2 = \int_0^{\infty} \int_{st}^{\infty} (h(x)\beta_t u_\tau + C\beta_x u_\tau) dx dt \quad ,$$

$$= u_\tau \int_0^{\infty} \Big( \int_{st}^{\infty} h(x)\beta_t dx \Big) dt + C u_\tau \int_0^{\infty} \Big( \int_{st}^{\infty} \beta_x dx \Big) dt \quad ,$$

*by*

$$\int_{st}^{\infty} \beta_x(x,t) dx = \beta(\infty, t) - \beta(st,t) = -\beta(st,t) \quad ,$$

*and*

$$\int_{st}^{\infty} \beta_t(x,t) dx = \frac{d}{dt} \int_{st}^{\infty} \beta(x,t) dx = -\beta(st,t) \quad ,$$

*then by using the Retail Integration of $A_2$ we get*

$$A_2 = \int_0^{\infty} \int_{st}^{\infty} (h(x)\beta_t u_\tau + C\beta_x u_\tau) dx dt = 0 \quad ,$$

*then*

$A = A_1 + A_2 = 0$ .

*On the other hand*

$$B = -\int_{-\infty}^{\infty} u(x,0)\, \beta(x,0) dx \quad ,$$

$$= -\int_{-\infty}^{0} u(x,0)\, \beta(x,0) dx - \int_{0}^{\infty} u(x,0)\, \beta(x,0) dx \quad ,$$

$$= -u_l \int_{-\infty}^{0} \beta(x,0) dx - u_\tau \int_{0}^{\infty} \beta(x,0) dx = 0 \quad ,$$

*then* $A = B$ ■

***<u>Case II</u>*** *: If $u_l < u_\tau$ then we have the following problem :*
*In this case there exist more than one weak solution, we show the following theorem is valid .*

***Theorem 2.5***

*Let the function*

$$u(x,t) = \begin{cases} u_l & x < u_l t \\ x & u_l t \le x \le u_\tau t \\ u_\tau & x > u_\tau t \end{cases}$$

*is a weak solution of the problem (1.4)with initial data (2.4) .*

***Proof :*** *Let* $\beta(x,t) \in \big(R \times [0,\infty)\big)$
*for simplicity we take $u_l = -1$ and $u_\tau = 1$ and denote*

$$F = \int_0^{\infty} \int_{-\infty}^{\infty} (h(x)\, u\beta_t + C u\beta_x) dxdt$$

$$M = -\int_{-\infty}^{\infty} u(x,0)\beta(x,0)dx = \int_{-\infty}^{0} \beta(x,0)dx - \int_0^{\infty} \beta(x,0)dx \ .$$

*Where* $u_l = -1$ *and* $u_\tau = 1$

*the function* $x$ *satisfies the equation* $h(x)\frac{\partial u(x,t)}{\partial t} + C\frac{\partial u(x,t)}{\partial x} = 0$ . *we have*

$$F = \int_0^{\infty} \int_{-\infty}^{-t} (-h(x)\beta_t(x,t) + C\beta_x(x,t))dxdt$$

$$+ \int_0^{\infty} \int_{-t}^{t} (h(x)\; x\; \beta_t(x,t) + C\; x\; \beta_x(x,t))dxdt$$

$$+ \int_0^{\infty} \int_t^{\infty} (h(x)\beta_t(x,t) + C\beta_x(x,t))dxdt = F_1 + F_2 + F_3 \ ,$$

*where*

$$F_1 = \int_0^{\infty} \int_{-\infty}^{-t} (-h(x)\beta_t + C\beta_x)dxdt = C\int_0^{\infty} \beta(-t,t)dt + \int_{-\infty}^{0} \beta(x,0)dx \ ,$$

$$F_2 = \int_0^{\infty} \int_{-t}^{t} (h(x)\; x\; \beta_t(x,t) + C\; x\; \beta_x(x,t))dxdt \ ,$$

$F_2$ *has a singularity at 0*

$$F_2 = \lim_{k \to 0} F_{2,k} \ ,$$

*where*

$$F_2 = \int_k^{\infty} \int_{-t}^{t} (h(x)\; x\; \beta_t(x,t) + C\; x\; \beta_x(x,t))dxdt \ ,$$

*we have*

$$\left(Cx\beta_x(x,t)\right) = C\; x\beta(x,t) + C\; \beta_x(x,t) \ ,$$

*that*

$$F_{2,k} = \int_k^{\infty} \int_{-t}^{t} (h(x)\; x\; \beta_t(x,t) + C\; x\; \beta_x(x,t))dxdt \ ,$$

$$F_3 = \int_0^{\infty} \int_t^{\infty} \left(h(x)\beta_t(x,t) + C\beta_x(x,t)\right)dxdt = C\int_0^{\infty} \beta(-t,t)dt - \int_0^{\infty} \beta(x,0)dx \ ,$$

*so* $F = F_1 + F_2 + F_3 = C\int_0^{\infty} \beta(-t,t)dt + \int_{-\infty}^{0} \beta(x,0)dx +$

$$\int_k^{\infty} \int_{-t}^{t} (h(x)\; x\; \beta_t(x,t) + C\; x\; \beta_x(x,t))dxdt +$$

$$C\int_0^{\infty} \beta(-t,t)dt - \int_0^{\infty} \beta(x,0)dx = -\int_0^{\infty} \beta(x,0)dx + \int_{-\infty}^{0} \beta(x,0)dx = M$$

*The proof is complete* ∎

***6- We have some applications about the case II :***

***1.6*** *: We can find the solution of the following Development Burger's equation with discontinuous initial condition:*

$$\begin{cases} h(x)\dfrac{\partial u(x,t)}{\partial t} + C\dfrac{\partial u(x,t)}{\partial x} = 0 \\ u(x,0) = u_0(x) = \begin{cases} -1 & x > 0 \\ 1 & x < 0 \end{cases} \end{cases}$$

*Solution: Let the following problem:*

$$h(x)\frac{\partial u(x,t)}{\partial t} + C\frac{\partial u(x,t)}{\partial x} = 0$$

$$u(x,0) = u_0(x) = \begin{cases} -1 & x < -k \ , \\ \dfrac{x}{k} & -k < x < k \ , \\ 1 & x > k \ . \end{cases}$$

$$h(x)\frac{\partial u}{\partial t} = C\frac{\partial u}{\partial x} \ ,$$

$$If \ \frac{\partial t}{h(x)} = \frac{\partial x}{C} \quad then \ \ x = \frac{Ct}{h(x)}$$

*Then we have the following problem :*

$$u(x,t) = u_0\left(\frac{xh(x)-ct}{kh(x)}\right) = \begin{cases} -1 & \dfrac{xh(x)-ct}{kh(x)} < -k \\ \dfrac{xh(x)-ct}{kh(x)} & -k < x\dfrac{xh(x)-ct}{kh(x)} < k \\ 1 & \dfrac{xh(x)-ct}{kh(x)} > k \end{cases}$$

*that means ,*

$$u(x,t) = \begin{cases} -1 & x < -(\frac{xh(x)}{t} - kc) \\ \frac{xh(x)}{t} - kc & -(\frac{xh(x)}{t} - kc) < x < (\frac{xh(x)}{t} - kc) \\ 1 & x > (\frac{xh(x)}{t} - kc) \end{cases}$$

*Sending* $k \to 0$ *we get the solution of the original problem*

$$u(x,t) = \begin{cases} -1 & x < -\dfrac{xh(x)}{t} \\ \dfrac{xh(x)}{t} & -\dfrac{xh(x)}{t} < x < \dfrac{xh(x)}{t} \\ 1 & x > \dfrac{xh(x)}{t} \end{cases}$$

*It is interesting to observe that the initial discontinuity is smoothed as illustrated in following figure. We can say Development Burger's equation favors non-decreasing initial values and dislikes other ones.*

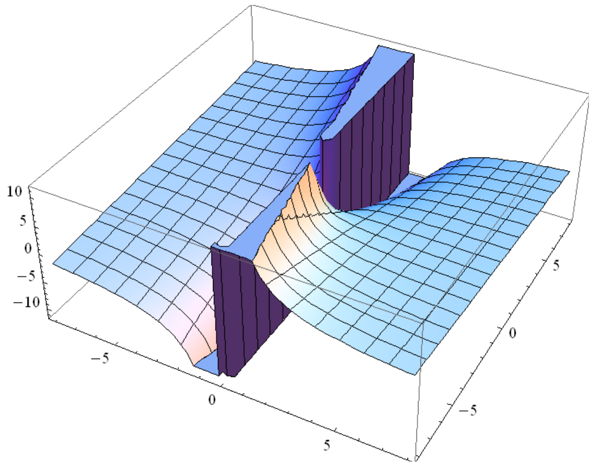


*Figure (1.6)*

$$\text{Manipulate}[\text{Plot3D}[\frac{xh(x)}{t}, \{t,-8,8\}, \{x,-8,8\}], \{\text{xh},-2,2\}]$$

***2.6** : We solve the initial value problem for u(x , t) , t > 0 in terms of t and a characteristic variable*

$$h(x)\frac{\partial u(x,t)}{\partial t} + C\frac{\partial u(x,t)}{\partial x} = 0 \qquad with$$

$$u(x,0) = u_0(x) = \begin{cases} 1 & , \quad x \le -1 \\ x^2 & , \quad -1 \le x \le 1 \\ 1 & , \quad x \ge 1 \end{cases}$$

$$h(x)\frac{\partial u}{\partial t} = C\frac{\partial u}{\partial x}$$

$$If \ \frac{\partial t}{h(x)} = \frac{\partial x}{C} = \frac{\partial u}{0} \quad \to so \ \frac{\partial u}{\partial t} = 0 \quad \to u = B(L)$$

$$and \ \frac{\partial x}{\partial t} = \frac{C}{h(x)} \quad then \quad x = \frac{Ct}{h(x)} + L$$

*Hence in terms of t and L*

$$u = B(L) \quad with \ \ x = \frac{Ct}{h(x)} + L$$

$then \ at \ t = 0 \quad \to \quad x = L \ ,$

*and* $\quad u = B(L) = u_0(x) = \begin{cases} 1 & for \quad |x| \ge 1 \ , \\ x^2 & for \quad |x| \le 1 \ , \end{cases}$

*so* $\quad u = \begin{cases} 1 & for \quad |L| \ge 1 \ , \\ L^2 & for \quad |L| \le 1 \ , \end{cases}$

*With* $\quad x = L + t \times \begin{cases} 1 & for \quad |L| \ge 1 \ , \\ |L| & for \quad |L| \le 1 \ , \end{cases}$

*We have ,*

$$0 = x_L = 1 + t \times \begin{cases} 0 & |L| > 1 \ , \\ 1 & 0 < L < 1 \ , \\ -1 & -1 < L < 0 \ . \end{cases}$$

*That is when* $t = 1 \quad for \ -1 < L < 0 \ \ at \quad x = L + t|L| = 0$

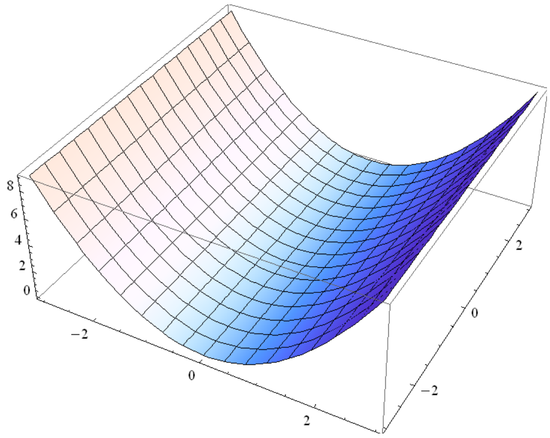


*Figure (2.6)*
Plot3D$[x^2, \{x, -3,3\}, \{y, -3,3\}]$

**References** :

[1] Mario Basto , Visiato Semiano and Francisco L.Calheiros , Numerical study of modified Adomian's method applied to Burgers' equation , Journal of Computational and Applied Mathematics , Volume 206 . Issue 2 , pp 927-949 , 2007 .
[2] Jeng , Dah – teng , A study of Burgers' model equation of turbulence , Minnesota univ Minneapolis of tech , 1965 .
[3] Joseph KT , Sachdev PL , Initial boundary value problems for scalar and vector Burgers' equations , Journal Article , Volume 106 , pp 481-505 , 2010 .
[4] Bulent Saka , A numerical study of the Burgers' equation , Journal of the Franklin Institute , Volume 345 , Issue 4 , pp328-348 , 2008 .
[5] Vinh Q . Nguyen , A numerical study of Burgers' equation with Robin boundary conditions , Masters of Science in Mathematics , Virginia Polytechnic Institute and State University , 2001 .
[6] Montri Maleewong , Sirod Sirisup , On – line and off – line POD assisted projective integral for non – linear problems : A case study with Burgers' equation , World Academy of Science , Volume 55 , 2011 .
[7] R.C. Mittal , Ram Jiwari , Numerical study of Burgers' – Huxley equation by differential quadrature method , Department of Mathematics , Indian Institute of technology Roorkee , 2009 .
[8] Ch. Srinivasa Rao , Engu Satyanarayana , Solutions of Burgers' equation , International Journal of Nonlinear Scince , Volume 9 , pp 290-295 , 2010 .
[9] Martin Hairer , Jochen Voss , Approximations to the stochastic Burgers' equation , 2010 .
[10] Christopher Jarvis , Reduced Order model study of Burgers' equation using proper orthogonal decomposition , Masters of Science in Mathematics , Virginia Polytechnic Institute and State University , 2012 .
[11] Alagacone Sri Ranga , Numerical study of Burgers' – Huxley equations via reduced differential transform method , Computational and Applied Mathematics , Volume 32 , Issue 1 , pp1-17 , 2013 .
[12] A. Ouhadan , T. Mekkaoui and E.H. El Kinani , Invariant and Some New Exact Solutions of Burgers Equation , International Journal of Nonlinear Science , Volume 11 , pp.51-58 , 2011.
[13] Mikel Landajuela , Burgers' equation , BCAM Internship , 2011 .
[14] H.S.Bhat , R.C.Fetecau , The Riemann problem for the leray – Burgers' equation , Journal of differential equations , pp 3957- 3979 , 2009 .
[15] A .A. Soliman , A Galerrkin solution for Burgers' equation using cubic B-spline finite elements , Department of mathematics , Faculty of education , AL – Arish , Egypt , 2011 .
[16] Jerome L. V. Lewandowski , Solution of Burgers' equation using the marker method , International journal of numerical analysis and modeling , Volume 3, Number 1, pp 80-93 , 2006 .
[17] Ioannis P. Stavroukis and stepan A. Tersian , Partial Differential Equations ,, world scientific, pp135 , 2003 .