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A Modified ODE Solver for Autonomous Initial Value Problems

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Abstract

In this work, modified version of a well-known variant of Euler method, known as the Improved Euler method, is proposed with a view to attain greater accuracy and efficiency. The attention is focused upon performance of the proposed method in autonomous initial value problems of ordinary differential equations. Order of accuracy of the proposed modified method is proved to be two using Taylor's expansion. Numerical experiments are performed using MS Excel 2010.

Keywords: ODE solver, numerical solution, initial value problem.

1. Introduction

Mathematical Modeling of several problems in areas such as Engineering, Science, Medicine, Economics, and etc. often leads to initial value problems of ordinary differential equations. There is no general analytical method to solve every differential equation as discussed in (Burden, Faires, & Reynolds, 1981) and (Chandio & Memon, 2010). A number of analytical methods exist for finding an exact solution to an ordinary differential equation. However, the limitation of analytical techniques to solve nonlinear differential equations has impeded the use of numerical techniques for obtaining an accurate approximate solution of the differential equation. Numerical methods for the solution of initial value problems in ordinary differential equations are efficient tools that have been gradually developed by various mathematicians since the late 18th century. Later; in 20th century, this subject made an enormous progress in the context of modern computers as evident in (Hassan, Abolarin, & Jimoh, 2006).

2. Existing ODE Solvers

Euler (1768) proposed the oldest and simplest numerical method to produce approximate solution for an initial value problem of an ordinary differential equation.

Given an initial value problem $\dot{x} = g(t\ ,\ x);\quad x(t_0) = x_0$, the formula for Euler's method is given as:

$$x_{i+1} = x_i + \Delta t\, g(t_i, x_i);\quad i = 0, 1,\ 2, \ldots \tag{1}$$

where $\Delta t = t_{i+1} - t_i$.

Euler's method is a first order accurate method. It requires a very small step size to yield sufficiently accurate numerical solutions. This makes it of seldom use in practice; much effort has been made to improve the efficiency of this method.

Runge (1895) was the first who proposed to improve Euler's method by increasing the number of function evaluations per step. He extended the Midpoint and Trapezoidal rules of numerical integration to formulate Modified Euler (ME) and Improved Euler (IE) methods, respectively.

$$ME: \quad x_{i+1} = x_i + \Delta t \; g\left(t_i + \frac{\Delta t}{2}, x_i + \frac{\Delta t}{2} g\left(t_i, x_i\right)\right) \tag{2}$$

$$IE: \quad x_{i+1} = x_i + \frac{\Delta t}{2}\left[g\left(t_i, x_i\right) + g\left(t_i + \Delta t,\; x_i + \Delta t \; g\left(t_i, x_i\right)\right)\right] \tag{3}$$

Both ME and IE methods are second order accurate involving two evaluations of the function g, per step, in contrast to the Euler method which involves a single function evaluation.

Abraham (2007) achieved an improvement on the ME Method for non-autonomous initial value problems. His method, referred to as the Improved Modified Euler (IME) method is given as:

$$x_{i+1} = x_i + \Delta t \, g\left(t_i + \frac{\Delta t}{2}, x_i + \frac{\Delta t}{2} g\left(t_i, x_i + \Delta t \; f\left(t_i, x_i\right)\right)\right) \tag{4}$$

The local truncation error for this method is $O\left(\Delta t^3\right)$, hence it is second order accurate.

Abraham (2008) improved the efficiency of ME method for autonomous initial value problems by proposing the following second order accurate Modified Improved Modified Euler (MIME) method:

$$x_{i+1} = x_i + \Delta t \, g\left(t_i + \frac{\Delta t}{2},\; x_i + \frac{\Delta t}{2} g\left(t_i,\; x_i + \frac{\Delta t}{2} g\left(t_i, x_i\right)\right)\right) \tag{5}$$

Akanbi (2010) developed an improvement on the ME method, known as Third Order Euler Method (TOEM), which is described as:

$$x_{i+1} = x_i + \Delta t \, g\left(t_i + \frac{\Delta t}{2},\; x_i + \frac{\Delta t}{2} g\left(x_i + \frac{\Delta t}{2},\; y_i + \frac{\Delta t}{3} g\left(t_i, x_i\right)\right)\right) \tag{6}$$

Afshar and Rohani (2009), Chandio & Memon (2010) and Qureshi *et al.* (2013 & 2014) independently developed second order accurate variants of the conventional ME method.

The ubiquity of differential equations in scientific and engineering fields has led to the continued research to improve existing numerical algorithms to yield greater accuracy with less computational effort.

3. The Modified ODE Solver

The IE method can be written as

$$\left.\begin{aligned} & x_{i+1} = x_i + \frac{1}{2}\left(k_1 + k_2\right) \\ & \text{where} \\ & k_1 = \Delta t \, g\left(t_i, x_i\right) \\ & k_2 = \Delta t \, g\left(t_i + \Delta t, x_i + \Delta t g\left(t_i, x_i\right)\right) \end{aligned}\right\} \tag{7}$$

Replacing the inner slope $g\left(t_i, x_i\right)$ in k_2 in (7) by the slope $g\left(t_i + \frac{\Delta t}{2}, x_i + \frac{\Delta t}{2} g\left(t_i, x_i\right)\right)$ at midpoint yields

$$\left.\begin{aligned} & x_{i+1} = x_i + \frac{1}{2}\left(k_1 + k_2\right) \\ & \text{where} \\ & k_1 = \Delta t\, g(t_i, x_i) = \Delta t\, g \\ & \tilde{k} = \Delta t\, g\left(t_i + \frac{\Delta t}{2}, x_i + \frac{k_1}{2}\right) \\ & k_2 = \Delta t\, g\left(t_i + \Delta t, x_i + \tilde{k}\right) \end{aligned}\right\} \quad (8)$$

which is the proposed Modified ODE solver.

3.1 Order of the Modified ODE Solver

The Taylor series expansion of $\tilde{k}$ yields

$$\tilde{k} = \Delta t\left\{g + \frac{\Delta t}{2} g_t + \frac{k_1}{2} g_y + \frac{\Delta t^2}{8} g_{tt} + \frac{\Delta t}{4} k_1 g_{ty} + \frac{k_1{}^2}{8} g_{yy} + O\left(\Delta t^3\right)\right\}$$

$$= \Delta t\, g + \frac{\Delta t^2}{2} g_t + \frac{\Delta t^2}{2} g g_y + \frac{\Delta t^3}{8} g_{tt} + \frac{\Delta t^3}{4} g g_{ty} + \frac{\Delta t^3}{8} g^2 g_{yy} + O\left(\Delta t^4\right)$$

Similarly, the Taylor series expansion of k_2 yields

$$k_2 = \Delta t\left\{g + \Delta t g_t + \tilde{k} g_y + \frac{\Delta t^2}{2} g_{tt} + \Delta t \tilde{k} g_{ty} + \frac{\tilde{k}^2}{2} g_{yy} + O\left(\Delta t^3\right)\right\}$$

$$= \Delta t\, g + \Delta t^2 g_t + \Delta t\left\{\Delta t\, g + \frac{\Delta t^2}{2} g_t + \frac{\Delta t^2}{2} g g_y + O\left(\Delta t^3\right)\right\} g_y$$

$$+ \frac{\Delta t^3}{2} g_{tt} + \Delta t^2\left\{\Delta t\, g + O\left(\Delta t^2\right)\right\} g_{ty} + \frac{\Delta t}{2}\left\{\Delta t\, g + O\left(\Delta t^2\right)\right\}^2 g_{yy} + O\left(\Delta t^4\right)$$

$$= \Delta t\, g + \Delta t^2\left(g_t + g g_y\right) + \frac{\Delta t^3}{2}\left(g_{tt} + 2 g g_{ty} + g^2 g_{yy} + g_t g_y + g g_y{}^2\right) + O\left(\Delta t^4\right)$$

$$= \Delta t\, g + \Delta t^2 g' + \frac{\Delta t^3}{2} g'' + O\left(\Delta t^4\right)$$

Hence, the Taylor series expansion of the proposed method (8) is

$$x_{i+1} = x_i + \Delta t\, g + \frac{\Delta t^2}{2} g' + \frac{\Delta t^3}{4} g'' + O\left(\Delta t^4\right)$$

$$= x_i + \Delta t\, g\left(t_i, x_i\right) + \frac{\Delta t^2}{2} g'\left(t_i, x_i\right) + \frac{\Delta t^3}{4} g''\left(t_i, x_i\right) + O\left(\Delta t^4\right),$$

And, that of the exact solution $x(t_{i+1})$ is

$$x(t_{i+1}) = x(t_i) + \Delta t\, x'(t_i) + \frac{\Delta t^2}{2} x''(t_i) + \frac{\Delta t^3}{6} x'''(t_i) + O(\Delta t^4)$$

$$= x(t_i) + \Delta t\, g(t_i, x(t_i)) + \frac{\Delta t^2}{2} g'(t_i, x(t_i)) + \frac{\Delta t^3}{6} g''(t_i, x(t_i)) + O(\Delta t^4)$$

Assuming $x(t_i) = x_i$, the local truncation error τ of the proposed method is

$$\tau = x_{i+1} - x(t_{i+1}) = \frac{\Delta t^3}{12} g'' + O(\Delta t^4) = O(\Delta t^3)$$

Hence, the proposed method is second order accurate.

4. Numerical Experiments

In this section; first and second order initial value problems are chosen to analyze performance of the proposed modified ODE solver with respect to absolute errors taking step size of 0.1 over the interval [0, 4]; and the number of function evaluations involved for a given error tolerance at $t = 4$.

Problem 01 Consider the IVP: $\dot{x} = x(x-2)$; $x(0) = 1$ which possesses the analytical solution: $x(t) = \dfrac{2}{1+e^{2t}}$.

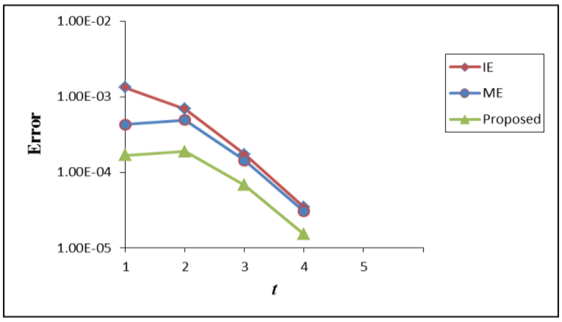

Figure 1. Error comparison at $\Delta t = 0.1$

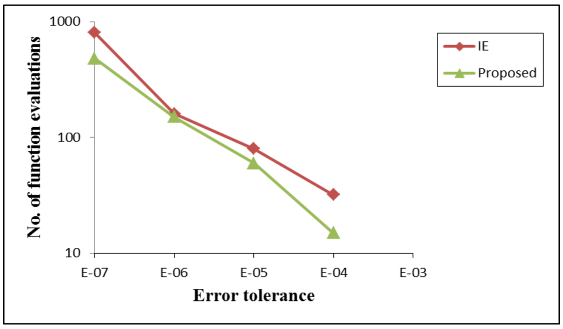

Figure 2. Function evaluations vs. error tolerance at $t = 4$

Problem 02 $\ddot{x}+6\dot{x}+9x=0; x(0)=2,\ \dot{x}(0)=-3$ Exact solution: $x(t)=(2+3t)e^{-3t}$.

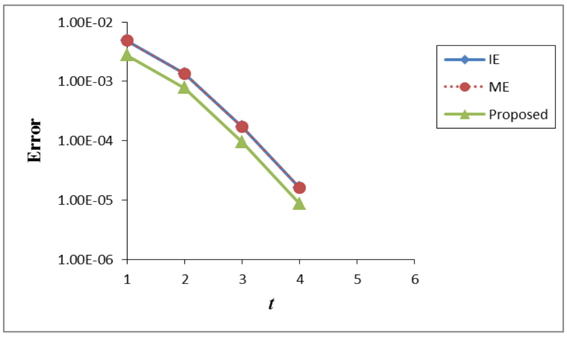

Figure 3. Error comparison at $\Delta t = 0.1$

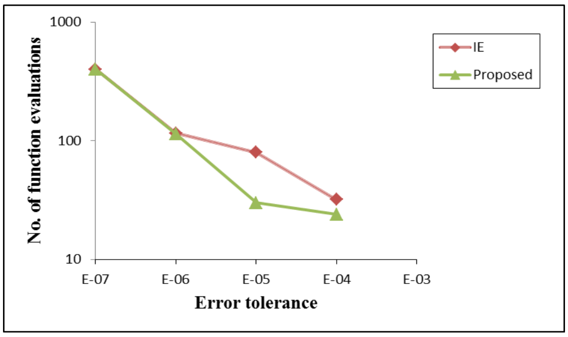

Figure 4. Function evaluations vs. error tolerance at $t=4$

Problem 03 Consider the IVP: $\ddot{x}=3\sqrt{x}\ ; x(0)=1, \dot{x}(0)=2$ with exact solution: $x(t)=(0.5t+1)^4$.

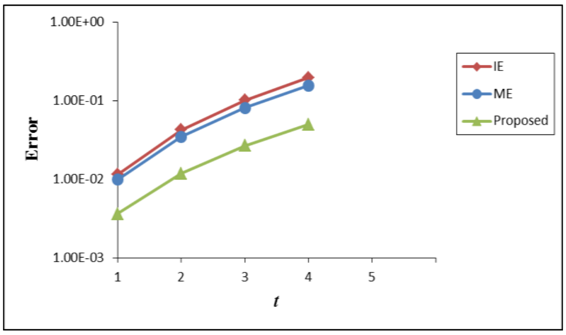

Figure 5. Error comparison at $\Delta t = 0.1$

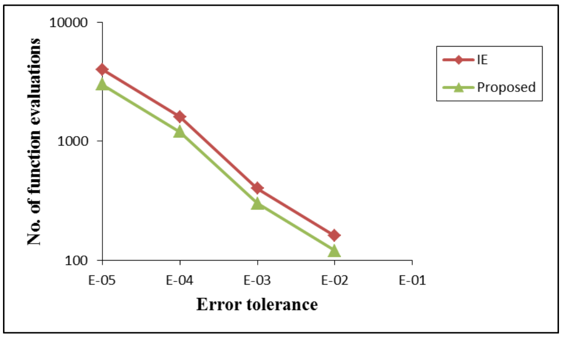

Figure 6. Function evaluations vs. error tolerance at $t=4$

5. Discussion & Conclusion

In this paper a modified version of IE method is proposed which is found to be second order accurate. The proposed method is tested on three autonomous initial value problems. Figures 1, 3 and 5 show that the proposed scheme yields smaller errors (absolute) than both IE and ME methods at sufficiently large step size of 0.1. Moreover, it is observed from figures 2, 4 and 6 that for the specified error tolerance (in terms of absolute error) at $t=4$, the proposed scheme takes into account the number of function evaluations less than or equal to that taken by the IE method; thus depicting the faster convergence of the proposed scheme.

6. Future Work

Future work will be devoted to testing the performance of proposed method on non-autonomous initial value problems along with a comprehensive analysis of the scheme with respect to the critical issues including convergence, consistency, stability and error bounds.

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