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ON THE PRODUCT AND RATIO OF PARETO AND KUMARASWAMY RANDOM VARIABLES

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Abstract. The distributions of the product XY and the ratio X/Y are derived when X and Y are Pareto and the Kumaraswamy random variables distributed independently of each other.

1. Introduction

For given random variables X and Y, the distributions of the product XY and the ratio X/Y are of interest in many areas of the sciences, engineer- ing and medicine. Examples of XY include traditional portfolio selection models, relationship between attitudes and behavior, number of cancer cells in tumor biology and stream flow in hydrology. Examples of X/Y include Mendelian inheritance ratios in genetics, mass to energy ratios in nuclear physics, target to control precipitation in meteorology, inventory ratios in economics and safety factor in engineering. The distributions of XY and X/Y have been studied by several authors especially when X and Y are independent random variables and come from the same family. With respect to XY, see Sakamoto (1943) for uniform family, Harter (1951) and Wallgren (1980) for Students t family, Springer and Thompson (1970) for normal family, Stuart (1962) and Podolski (1972) for gamma family, Steece (1976), Bhargava and Khatri (1981) and Tang and Gupta (1984) for beta family, AbuSalih (1983) for power function family, and Malik and Trudel (1986) for exponential family (see also Rathie and Rohrer (1987) for a comprehensive review of known results). With respect to X/Y, see Marsaglia (1965) and Korhonen and Narula (1989) for normal family, Press (1969) for Student's t family, Basu and Lochner (1971) for Weibull family, Shcolnick (1985) for stable family, Hawkins and Han (1986) for non-central chi-squared family, Provost (1989) for gamma family, and Pham-Gia (2000) for beta family. There is relatively little work of this kind when X and Y belong to different families. In the applications mentioned above, it is quite possible that X and Y could arise from different but similar distributions (see below for examples).

In this paper, we study the exact distributions of XY and X/Y when X and Y are independent Pareto and Kumaraswamy random variables with pdfs

$$f_X(x) = \frac{\alpha k^\alpha}{x^{\alpha+1}}, \tag{1.1}$$

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and

$$f_Y(y) = aby^{a-1}(1-y^a)^{b-1} \tag{1.2}$$

respectively, for $k \le x < \infty, \alpha, k > 0, 0 < y < 1, \alpha > 0, \beta > 0, \lambda > 0$

The calculations of this paper involve several special functions, including the gamma function defined by

$$\Gamma(a,) = \int_0^\infty \exp(-t)t^{a-1}dt,$$

thand beta function defined by

$$Beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx.$$

2. Main Results

Theorem 2.1. *Suppose X and Y are distributed according to (1.1) and (1.2), respectively. The cdf of $Z = XY$ can be expressed as:*

$$F(z) = Pr(Z = XY) = 1 - b(\frac{k}{z})^\alpha B\Big(1 + \frac{\alpha}{a}, b\Big) \tag{2.1}$$

for $z > 0, \alpha > 2 - a, a > 0, b > 0$

Proof. The cdf corresponding to (1.1) is $1 - \left(\frac{k}{x}\right)^\alpha$. Thus, one can write the cdf of X/Y as

$$\begin{aligned}
\Pr(XY \le z) &= \int_0^\infty F_X(\frac{z}{y})f_Y(y)dy \\
&= 1 - ab\int_0^1 \left(\frac{ky}{z}\right)^\alpha y^{a-1}(1-y^a)^{b-1}dy \\
&= 1 - ab(\frac{k}{z})^\alpha \int_0^1 y^{a+\alpha-1}(1-y^a)^{b-1}dy \\
&= 1 - b(\frac{k}{z})^\alpha \int_0^1 t^{\frac{a+\alpha}{a}-1}(1-t)^{b-1}dt \\
&= 1 - b(\frac{k}{z})^\alpha B(1 + \frac{\alpha}{a}, b)
\end{aligned}$$

□

Theorem 2.2. *The pdf of $Z = XY$ can be expressed as:*

$$f(z) = \frac{b\alpha k^\alpha}{z^{\alpha+1}} B\Big(1 + \frac{\alpha}{a}, b\Big) \tag{2.2}$$

for $z > 0, \alpha > 0, a > 0, b > 0$.

Proof. It is straight forward to show the results of Corollary by taking the differentiation equation (2.1). □

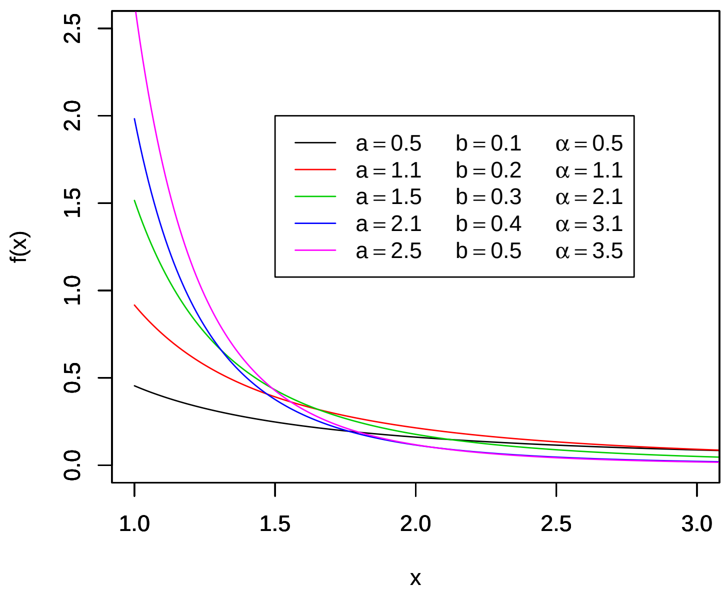

FIGURE 1. The pdf's of various values for Z=XY.

Theorem 2.3. *Suppose X and Y are distributed according to (1.1) and (1.2), respectively.The rth moment of $Z = XY$, say $E[Z^r]$, is*

$$E[Z^r] = \frac{b\alpha k^r}{(\alpha - r)} B\Big(1 + \frac{\alpha}{a}, b\Big) \tag{2.3}$$

Proof.

$$\begin{aligned} E[Z^r] &= \int_k^\infty z^r f(z)dx = \int_k^\infty \frac{b\alpha k^\alpha}{z^{\alpha-r+1}} B\Big(1 + \frac{\alpha}{a}, b\Big)dz \\ &= b\alpha k^\alpha B\Big(1 + \frac{\alpha}{a}, b\Big) \int_k^\infty \frac{dx}{z^{\alpha-r+1}} \\ &= b\alpha k^\alpha B\Big(1 + \frac{\alpha}{a}, b\Big) \Big[- \frac{1}{(\alpha - r)z^{\alpha-r}} \Big|_k^\infty \Big] \\ &= \frac{b\alpha k^r}{(\alpha - r)} B\Big(1 + \frac{\alpha}{a}, b\Big) \end{aligned}$$

□

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Theorem 2.4. *Suppose X and Y are distributed according to (1.1) and (1.2), respectively. The cdf of $Z = \frac{X}{Y}$ can be expressed as:*

$$F(Z) = 1 - b\Big(\frac{k}{z}\Big)^{\alpha} B\Big(\frac{a-\alpha}{a}, b\Big) \tag{2.4}$$

Proof. The cdf corresponding to (1.1) is $1 - \left(\frac{k}{x}\right)^{\alpha}$. Thus, one can write the cdf of X/Y as

$$\begin{aligned}
\Pr(\frac{X}{Y} \le z) &= \int_0^{\infty} F_X(zy) f_Y(y) dy \\
&= 1 - ab \int_0^1 \Big(\frac{k}{zy}\Big)^{\alpha} y^{a-1}(1-y^a)^{b-1} dy \\
&= 1 - ab\Big(\frac{k}{z}\Big)^{\alpha} \int_0^1 y^{a-\alpha-1}(1-y^a)^{b-1} dy \\
&= 1 - b\Big(\frac{k}{z}\Big)^{\alpha} \int_0^1 t^{\frac{a-\alpha}{a}-1}(1-t)^{b-1} dt \\
&= 1 - b\Big(\frac{k}{z}\Big)^{\alpha} B\Big(\frac{a-\alpha}{a}, b\Big)
\end{aligned}$$

□

Theorem 2.5. *The pdf of $Z = \frac{X}{Y}$ can be expressed as:*

$$f(z) = \frac{b\alpha k^{\alpha}}{z^{\alpha+1}} B\Big(\frac{a-\alpha}{a}, b\Big) \tag{2.5}$$

for $z > 0, \alpha > a, a > 0, b > 0$.

Proof. It is straight forward to show the results of Corollary by taking the differentiation equation (2.1). □

Theorem 2.6. *Suppose X and Y are distributed according to (1.1) and (1.2), respectively.The rth moment of $Z = \frac{X}{Y}$, say $E[Z^r]$, is*

$$E[Z^r] = \frac{b\alpha k^{r}}{(\alpha - r)} B\Big(\frac{a-\alpha}{a}, b\Big) \tag{2.6}$$

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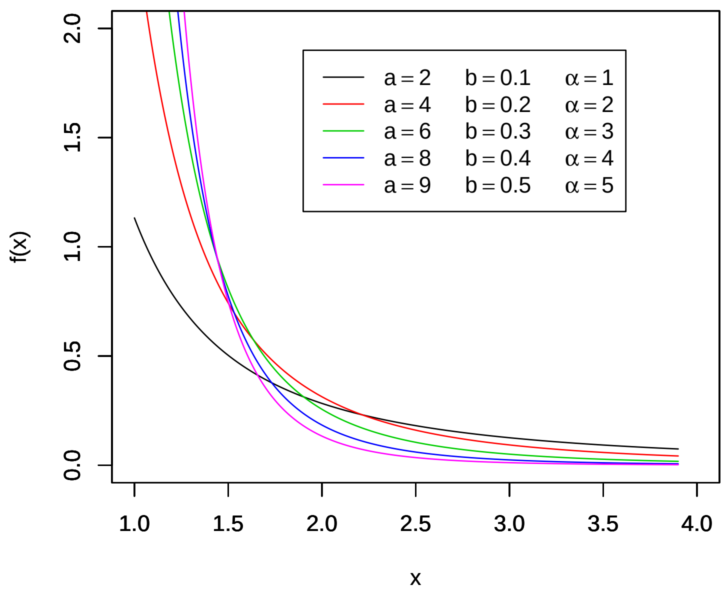

FIGURE 2. The pdf's of various values for Z=X/Y.

Proof.

$$\begin{aligned}
E[Z^r] &= \int_k^\infty z^r f(z)dx = \int_k^\infty \frac{b\alpha k^\alpha}{z^{\alpha+1}} B\Big(\frac{a-\alpha}{a}, b\Big) dz \\
&= b\alpha k^\alpha B\Big(\frac{a-\alpha}{a}, b\Big) \int_k^\infty \frac{dx}{z^{\alpha-r+1}} dz \\
&= b\alpha k^\alpha B\Big(\frac{a-\alpha}{a}, b\Big) \Bigg[- \frac{1}{(\alpha-r)z^{\alpha-r}} \Bigg|_k^\infty \Bigg] \\
&= \frac{b\alpha k^r}{(\alpha-r)} B\Big(\frac{a-\alpha}{a}, b\Big)
\end{aligned}$$

□

3. ORDER STATISTICS

Suppose $Z_1, Z_2, \ldots, Z_N$ is a random sample from (2.2). Let $Z_{1:N} < Z_{2:N} < \ldots < Z_{N:N}$ denote the corresponding order statistics. It is well known that the pdf and

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the cdf of the kth order statistic, say $H = Z_{k:N}$ are given by

$$f_H(h) = \frac{N!}{(k-1)!(N-k)!} f_Z(h)[F_Z(h)]^{k-1}[1-F_Z(h)]^{n-k} \tag{3.1}$$
$$= \frac{N!}{(k-1)!(N-k)!} \sum_{l=0}^{N-k} \binom{N-j}{l} (-1)^l F_Z^{k-1+l}(h) f_Z(h)$$

and

$$F_H(h) = \sum_{j=k}^{N} \binom{N}{j} [F_Z(h)]^j [1-F_Z(h)]^{N-j} \tag{3.2}$$
$$= \sum_{j=k}^{N} \sum_{l=0}^{N-j} \binom{N}{j} \binom{N-j}{l} (-1)^l F_Z^{j+l}(h)$$

respectively, for $k = 1, 2, \ldots, N$. It follows from (2.1) and (2.2) that

$$f_H(h) = \frac{N!}{(k-1)!(N-k)!} \sum_{l=0}^{N-k} \binom{N-j}{l} (-1)^l \Big[1 - b(\frac{k}{h})^\alpha B\Big(1+\frac{\alpha}{a}, b\Big)\Big]^{k-1+l} \tag{3.3}$$
$$\times \frac{b\alpha k^\alpha}{h^{\alpha+1}} B\Big(1+\frac{\alpha}{a}, b\Big)$$

$$F_H(h) = \sum_{j=k}^{N} \sum_{l=0}^{N-j} \binom{N}{j} \binom{N-j}{l} (-1)^l \Big[1 - b(\frac{k}{h})^\alpha B\Big(1+\frac{\alpha}{a}, b\Big)\Big]^{j+l} \tag{3.4}$$

$R_1, R_2, \ldots, R_N$ is a random sample from (2.5). Let $R_{1:N} < R_{2:N} < \ldots < R_{N:N}$ denote the corresponding order statistics.Let $H = R_{k:N}$, so the pdf and cdf of H follow from (2.4) and (2.5):

$$f_H(h) \frac{N!}{(k-1)!(N-k)!} \sum_{l=0}^{N-k} \binom{N-j}{l} (-1)^l \Big[1 - b\frac{k}{h} B\Big(1-\frac{\alpha}{a}, b\Big)\Big]^{k-1+l} \tag{3.5}$$
$$\times \frac{b\alpha k^\alpha}{h^{\alpha+1}} B\Big(1-\frac{\alpha}{a}, b\Big)$$

and

$$F_H(h) = \sum_{j=k}^{N} \sum_{l=0}^{N-j} \binom{N}{j} \binom{N-j}{l} (-1)^l \Big[1 - b\frac{k}{h} B\Big(1-\frac{\alpha}{a}, b\Big)\Big]^{j+l} \tag{3.6}$$

4. Application

An application of the result in Theorem 2.1 can be illustrated by deriving the percentage points z_q associated with $Z = XY$. These values can be obtained by numerically solving the equation

$$1 - b(\frac{k}{z})^\alpha B\Big(1+\frac{\alpha}{a}, b\Big) = z_q. \tag{4.1}$$

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k	a	b	p=0.01	p=0.05	p=0.1	p=0.90	p=0.95	p=0.99
1	1	0.1	.918	.957	1.010	9.091	18.182	90.908
1	2	0.2	.905	.943	.995	8.955	17.910	89.551
1	3	0.3	.902	.940	.992	8.928	17.855	89.277
1	4	0.4	.902	.940	.993	8.935	17.869	89.346
1	5	0.5	.905	.943	.995	8.955	17.911	89.554
2	1	0.1	1.837	1.914	2.020	18.182	36.364	181.820
2	2	0.2	1.809	1.885	1.990	17.911	35.821	179.110
2	3	0.3	1.804	1.879	1.984	17.855	35.709	178.550
2	4	0.4	1.805	1.881	1.986	17.870	35.739	178.700
2	5	0.5	1.809	1.885	1.990	17.911	35.821	179.110
3	1	0.1	2.755	2.871	3.030	27.272	54.544	272.720
3	2	0.2	2.714	2.828	2.985	26.865	53.730	268.650
3	3	0.3	2.705	2.819	2.976	26.783	53.565	267.830
3	4	0.4	2.708	2.822	2.978	26.804	53.608	268.040
3	5	0.5	2.714	2.828	2.985	26.867	53.733	268.670
4	1	0.1	3.673	3.828	4.040	36.363	72.726	363.630
4	2	0.2	3.618	3.771	3.980	35.820	71.639	358.200
4	3	0.3	3.607	3.759	3.968	35.711	71.422	357.110
4	4	0.4	3.610	3.762	3.971	35.739	71.478	357.390
4	5	0.5	3.618	3.771	3.980	35.822	71.644	358.220
5	1	0.1	4.591	4.785	5.050	45.454	90.908	454.540
5	2	0.2	4.523	4.713	4.975	44.776	89.551	447.760
5	3	0.3	4.509	4.699	4.960	44.636	89.273	446.360
5	4	0.4	4.513	4.703	4.964	44.675	89.350	446.750
5	5	0.5	4.523	4.713	4.975	44.777	89.555	447.770

TABLE 1. Percentage points of XY

Evidently, this involves computation of the gamma function, and routines for this are widely available. One can use the $GAMMA()$ function in MAPLE, the algebraic manipulation package. A MAPLE procedure for solving Equation (4.1) is:

```
alpha := 1;
k := 1;
a := 1;
b := .1;
f := 1-b*k^alpha*GAMMA(1+alpha/a)*GAMMA(b)/(z^alpha*GAMMA(1+alpha/a+b));
p1 := fsolve(f = 0.1e-1, z);
p2 := fsolve(f = 0.5e-1, z);
p3 := fsolve(f = .1, z);
p4 := fsolve(f = .90, z);
p5 := fsolve(f = .95, z);
p6 := fsolve(f = .99, z);
print(p1, p2, p3, p4, p5, p6)
```

Table 1 provides numerical values of z_q for $\alpha = 1$, $k = 1, 2, \dots 5$, $a = 1, 2, \dots, 5$; $b = 0.1, 0.2, \dots 0.5$

k	a	b	p=0.01	p=0.05	p=0.1	p=0.90	p=0.95	p=0.99
1	1.1	0.1	2.094	2.182	2.303	20.726	41.452	207.259
1	2.1	0.2	1.245	1.297	1.369	12.324	24.649	123.244
1	3.1	0.3	1.194	1.245	1.314	11.824	23.647	118.237
1	4.1	0.4	1.172	1.221	1.289	11.601	23.203	116.013
1	5.1	0.5	1.158	1.206	1.273	11.461	22.921	114.607
2	1.1	0.1	4.187	4.363	4.606	41.452	82.904	414.518
2	2.1	0.2	2.490	2.595	2.739	24.649	49.298	246.488
2	3.1	0.3	2.389	2.489	2.627	23.647	47.295	236.473
2	4.1	0.4	2.344	2.442	2.578	23.203	46.405	232.027
2	5.1	0.5	2.315	2.413	2.547	22.921	45.843	229.214
3	1.1	0.1	6.281	6.545	6.909	62.178	124.355	621.776
3	2.1	0.2	3.735	3.892	4.108	36.973	73.946	369.731
3	3.1	0.3	3.583	3.734	3.941	35.471	70.942	354.710
3	4.1	0.4	3.516	3.664	3.867	34.804	69.608	348.040
3	5.1	0.5	3.473	3.619	3.820	34.382	68.764	343.822
4	1.1	0.1	8.374	8.727	9.212	82.904	165.807	829.035
4	2.1	0.2	4.980	5.189	5.478	49.298	98.595	492.975
4	3.1	0.3	4.777	4.978	5.255	47.295	94.589	472.947
4	4.1	0.4	4.687	4.885	5.156	46.405	92.811	464.054
4	5.1	0.5	4.631	4.826	5.094	45.843	91.686	458.429
5	1.1	0.1	10.468	10.908	11.514	103.629	207.259	1036.294
5	2.1	0.2	6.224	6.487	6.847	61.622	123.244	616.219
5	3.1	0.3	5.972	6.223	6.569	59.118	118.237	591.184
5	4.1	0.4	5.859	6.106	6.445	58.007	116.013	580.067
5	5.1	0.5	5.788	6.032	6.367	57.304	114.607	573.036

TABLE 2. Percentage points of $\frac{X}{Y}$

An application of the result in Theorem 2.4 can be illustrated by deriving the percentage points z_q associated with $Z = \frac{X}{Y}$. These values can be obtained by numerically solving the equation

$$1 - b\Big(\frac{k}{z}\Big)^{\alpha} B\Big(\frac{a-\alpha}{a}, b\Big) = z_q. \tag{4.2}$$

A MAPLE procedure for solving Equation (4.2) is:

```
alpha := 1;
 k := 5; a := 5;
 b := .5;
 f := 1-b*k^alpha*GAMMA(1-alpha/a)*GAMMA(b)/(z^alpha*GAMMA(1-alpha/a+b));
 p1 := fsolve(f = 0.1e-1, z);
 p2 := fsolve(f = 0.5e-1, z);
 p3 := fsolve(f = .1, z);
 p4 := fsolve(f = .90, z);
 p5 := fsolve(f = .95, z);
 p6 := fsolve(f = .99, z);
 print(p1, p2, p3, p4, p5, p6)
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