A New (Proposed) Formula for Interpolation and Comparison with Existing Formula of Interpolation

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Abstract

The word "interpolation" originates from the Latin verb interpolare, a contraction of "inter," meaning "between," and "polare," meaning "to polish." That is to say, to smooth in between given pieces of information. A number of different methods have been developed to construct useful interpolation formulas for evenly and unevenly spaced points. The aim of this paper is to develop a central difference interpolation formula which is derived from Gauss's Backward Formula and another formula in which we retreat the subscripts in Gauss's Forward Formula by one unit and replacing u by $u+1$. Also, we make the comparisons of the developed interpolation formula with the existing interpolation formulas based on differences. Results show that the new formula is very efficient and posses good accuracy for evaluating functional values between given data.

Keywords: Interpolation, Central Difference, Gauss's Formula.

1. Introduction

The word "interpolation" originates from the Latin verb interpolare, a contraction of "inter," meaning "between," and "polare," meaning "to polish." That is to say, to smooth in between given pieces of information. It seems that the word was introduced in the English literature for the first time around 1612 and was then used in the sense of "to alter or enlarge [texts] by insertion of new matter", see J. Simpson and E. Weiner, Eds. (1989). The original Latin word appears to have been used first in a mathematical sense by Wallis in his 1655 book on infinitesimal arithmetic, (see J. Bauschinger, 1900-1904); J. Wallis, 1972). Sir Edmund Whittaker, a professor of Numerical Mathematics at the University of Edinburgh from 1913 to 1923, observed "the most common form of interpolation occurs when we seek data from a table which does not have the exact values we want." The general problem of interpolation consists, then, in representing a function, known or unknown, in a form chosen in advance, with the aid of given values which this function takes for definite values of the independent variable, see James B. Scarborough (1966). A number of different methods have been developed to construct useful interpolation formulas for evenly and unevenly spaced points. Newton's divided difference formula (e.g. Kendall

E. Atkinson, 1989; S. D. Conte, Carl de Boor, 1980) and Lagrange's formula (e.g. R. L. Burden, J. D. Faires, 2001; Endre Süli and David Mayers, 2003; John H. Mathews, Kurtis D. Fink, 2004) are the most popular interpolation formulas for polynomial interpolation to any arbitrary degree with finite number of points.

Lagrange interpolation is a well known, classical technique for interpolation. Using this; one can generate a single polynomial expression which passes through every point given. This requires no additional information about the points. This can be really bad in some cases, as for large numbers of points we get very high degree polynomials which tend to oscillate violently, especially if the points are not so close together. It can be rewritten in two more computationally attractive forms: a modified Lagrange form (see Berrut, J. P. and Trefethen, L. N., 2004) and a bary centric form (e.g. Berrut, J. P. and Trefethen, L. N., 2004; Nicholas J. Higham, 2004). Newton's formula for constructing the interpolation polynomial makes the use of divided differences through Newton's divided difference table for unevenly spaced data, (see Kendall E. Atkinson, 1989). Based on this formula, there exists many number of interpolation formulas using differences through difference table, for evenly spaced data. The best formula is chosen by speed of convergence, but each formula converges faster than other under certain situations, no other formula is preferable in all cases. For example, if the interpolated value is closer to the center of the table then we go for any one of central difference formulas, (Gauss's, Stirling's and Bessel's etc) depending on the value of argument position from the center of the table. However, Newton interpolation formula is easier for hand computation but Lagrange interpolation formula is easier when it comes to computer programming. In this paper, we develop a new central difference interpolation formula which is derived from Gauss's Backward Formula and another formula in which we retreat the subscripts in Gauss's Forward Formula by one unit and replacing u by $u+1$. Also, we make the comparison of the new (proposed) interpolation formula with the existing interpolation formulas based on differences.

2. Literature Review

In his 1909 book on interpolation, T. N. Thiele characterized the subject as "the art of reading between the lines in a [numerical] table." Examples of fields in which this problem arises naturally and inevitably are astronomy and, related to this, calendar computation. Because man has been interested in these since day one, it should not surprise us that it is in these fields that the first interpolation methods were conceived. Throughout history, interpolation has been used in one form or another for just about every purpose under the sun. Speaking of the sun, some of the first surviving evidence of the use of interpolation came from ancient Babylon and Greece. In antiquity, astronomy was all about time keeping and making predictions concerning astronomical events. This served important practical needs: farmers, e.g., would base their planting strategies on these predictions. To this end, it was of great importance to keep up lists—so-called ephemerides—of the positions of the sun, moon, and the known planets for regular time intervals. Obviously, these lists would contain gaps, due to either atmospherical conditions hampering observation or the fact that celestial bodies may not be visible during certain periods. From his study of ephemerides found on ancient astronomical cuneiform tablets originating from Uruk and Babylon in the Seleucid period (the last three centuries BC), the historian-mathematician Neugebauer (see O. Neugebauer, 1955; O. Neugebauer, 1975) concluded that interpolation was used in order to fill these gaps.

Around 300 BC, they were using not only linear, but also more complex forms of interpolation to predict the positions of the sun, moon, and the planets they knew of. Farmers, timing the planting of their crops, were the primary users of these predictions. Also in Greece sometime around 150 BC, Toomer (1978) believes that

Hipparchus of Rhodes used linear interpolation to construct a “chord function”, which is similar to a sinusoidal function, to compute positions of celestial bodies. Farther east, Chinese evidence of interpolation dates back to around 600 AD. Liu Zhuo used the equivalent of second order Gregory-Newton interpolation to construct an “Imperial Standard Calendar” see Martzloff (1997) and Yˇan and Shírán (1987). In 625 AD, Indian astronomer and mathematician Brahmagupta introduced a method for second order interpolation of the sine function and, later on, a method for interpolation of unequal-interval data (see R. C. Gupta, 1969). The general interpolation formula for equidistant data was first written down in 1670 by Gregory (1939) can be found in a letter by him to Collins. Particular cases of it, however, had been published several decades earlier by Briggs, the man who brought to fruition the work of Napier on logarithms. In the introductory chapters to his major works (e.g. H. Briggs, 1624; H. Briggs, 1633), he described the precise rules by which he carried out his computations, including interpolations, in constructing the tables contained therein.

It is justified to say that “there is no single person who did so much for this field, as for so many others, as Newton”, (See H. H. Goldstine, 1977). His enthusiasm becomes clear in a letter he wrote to Oldenburg (1960), where he first describes a method by which certain functions may be expressed in series of powers of and then goes on to say. The contributions of Newton to the subject are contained in: (1) a letter to Smith in 1675 (see I. Newton, 1959); (2) a manuscript entitled *Methodus Differentialis* (see I. Newton, 1981), published in 1711, although earlier versions were probably written in the middle 1670s; (3) a manuscript entitled *Regula Differentiarum*, written in 1676, but first discovered and published in the 20th century (e.g. D. C. Fraser, 1927; D. C. Fraser, 1927); and (4) Lemma V in Book III of his celebrated *Principia* (see I. Newton, 1960), which appeared in 1687. The latter was published first and contains two formulae. The first deals with equal-interval data, which Newton seems to have discovered independently of Gregory. The second formula deals with the more general case of arbitrary-interval data.

The presentation of the two interpolation formulae in the *Principia* is heavily condensed and contains no proofs. Newton’s *Methodus Differentialis* contains a more elaborate treatment, including proofs and several alternative formulae. Three of those formulae for equal-interval data were discussed a few years later by Stirling (1719). These are the Gregory–Newton formula and two central-difference formulae, the first of which is now known as the Newton-Stirling formula. It is interesting to note that Brahmagupta’s formula is, in fact, the Newton-Stirling formula for the case when the third and higher order differences are zero. A very elegant alternative representation of Newton’s general formula that does not require the computation of finite or divided differences was published in 1779 by Waring. It is nowadays usually attributed to Lagrange who, in apparent ignorance of Waring’s paper, published it 16 years later (see J. L. Lagrange, 1877). The formula may also be obtained from a closely related representation of Newton’s formula due to Euler (1783). According to Joffe (1917), it was Gauss who first noticed the logical connection and proved the equivalence of the formulae by Newton, Euler, and Waring–Lagrange, as appears from his posthumous works (see C. F. Gauss, 1866), although Gauss did not refer to his predecessors.

In 1812, Gauss delivered a lecture on interpolation, the substance of which was recorded by his then student, Encke (1830), who first published it not until almost two decades later. Apart from other formulae, he also derived the one which is now known as the Newton-Gauss formula. In the course of the 19th century, two more formulae closely related to Newton-Gauss formula were developed. The first appeared in a paper by Bessel (1824) on computing the motion of the moon and was published by him because, in his own words, he could "not recollect having seen it anywhere." The formula is, however, equivalent to one of Newton's in his *Methodus Differentialis*, which is the second central-difference formula discussed by Stirling (1719) and has, therefore, been called the Newton–Bessel formula. The second formula, which has frequently been used by statisticians and actuaries, was developed by Everett (1900), (1901) around 1900 and the elegance of this formula lies in the fact that, in contrast with the earlier mentioned formulae, it involves only the even-order differences of the two table entries between which to interpolate. Alternatively, we could expand the even-order differences so as to end up with only odd-order differences. The resulting formula appears to have been described first by Steffensen (1950) and is, therefore, sometimes referred to as such F. B. Hildebrand, 1974; M. K. Samarin, 1992, although he himself calls it Everett's second interpolation formula. It was noted later by Joffe (1917) and Lidstone (1922) that the formulae of Bessel and Everett had alternatively been proven by Laplace by means of his method of generating functions (see P. S. de Laplace, 1820; P. S. de Laplace, 1894).

By the beginning of the 20th century, the problem of interpolation by finite or divided differences had been studied by astronomers, mathematicians, statisticians, and actuaries. Many of them introduced their own system of notation and terminology, leading to confusion and researchers reformulating existing results. The point was discussed by Joffe (1917), who also made an attempt to standardize yet another system. It is, however, Sheppard's (1899) notation for central and mean differences that has survived in later publications. Most of the now well-known variants of Newton's original formulae had been worked out. This is not to say, however, that there are no more advanced developments to report on. Quite to the contrary. Already in 1821, Cauchy (1821) studied interpolation by means of a ratio of two polynomials and showed that the solution to this problem is unique, the Waring–Lagrange formula being the special case for the second polynomial equal to one. It was Cauchy also who, in 1840, found an expression for the error caused by truncating finite-difference interpolation series (see A. Cauchy, 1841). The absolute value of this so-called Cauchy remainder term can be minimized by choosing the abscissae as the zeroes of the polynomials introduced later by Tchebychef (1874). See, e.g., Davis (1963); Hildebrand (1974), or Schwarz (1989) for more details. Generalizations for solving the problem of multivariate interpolation in the case of fairly arbitrary point configurations began to appear in the second half of the 19th century, in the works of Borchardt and Kronecker (e.g. C. W. Borchardt, 1860; L. Kronecker, 1865; M. Gasca and T. Sauer, 2000).

A generalization of a different nature was published in 1878 by Hermite, who studied and solved the problem of finding a polynomial of which also the first few derivatives assume pre-specified values at given points, where the order of the highest derivative may differ from point to point. Birkhoff (1906) studied the even more general problem: given any set of points, find a polynomial function that satisfies pre-specified criteria concerning its value and/or the value of any of its derivatives for each individual point. Birkhoff interpolation, also known as *lacunary interpolation*, initially received little attention, until Schoenberg (1966) revived interest in the subject. Hermite and Birkhoff type of interpolation problems—and their multivariate versions, not necessarily on Cartesian grids—have received much attention in the past decades.

3. New (proposed) and Existing Interpolation Formulas

We consider the following difference table [Table 1] in which the central ordinate is taken for convenience as $y = y_0$ corresponding to $x = x_0$.

The Gauss's Central-Difference Formulas are given below (see, James B. Scarborough, 1966):

Gauss's Forward Formula:

$$y = y_0 + u\Delta y_0 + u(u-1)\frac{\Delta^2 y_{-1}}{2!} + u(u^2-1)\frac{\Delta^3 y_{-1}}{3!} + u(u^2-1)(u-2)\frac{\Delta^4 y_{-2}}{4!} + u(u^2-1)(u^2-2^2)\frac{\Delta^5 y_{-2}}{5!} + \cdots \quad (1)$$

Gauss's Backward Formula:

$$y = y_0 + u\Delta y_{-1} + u(u+1)\frac{\Delta^2 y_{-1}}{2!} + u(u^2-1)\frac{\Delta^3 y_{-2}}{3!} + u(u^2-1)(u+2)\frac{\Delta^4 y_{-2}}{4!} + u(u^2-1)(u^2-2^2)\frac{\Delta^5 y_{-3}}{5!} + \cdots \quad (2)$$

Stirling's Interpolation Formula: Taking the mean of the Gauss's Forward Formula and Gauss's Backward Formula *i.e.* by adding them and dividing the sums throughout by 2, we get Stirling's Interpolation Formula as (see, James B. Scarborough, 1966):

$$y = y_0 + u\frac{(\Delta y_{-1} + \Delta y_0)}{2} + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\frac{(\Delta^3 y_{-2} + \Delta^2 y_{-1})}{2} + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1)(u^2-2^2)}{5!}\frac{(\Delta^5 y_{-3} + \Delta^5 y_{-2})}{2} + \cdots \quad (3)$$

Table 1: Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$				
		Δy_2					
x_3	y_3						

Bessel's Interpolation Formula: For the derivation of Bessel's Formula**,** we need a Third Gauss's Formula, to derive the Third Gauss's Formula**,** we advance the subscripts in Gauss's Backward Formula by one unit and replacing u by $u-1$ then we obtain,

$$y = y_1 + (u-1)\Delta y_0 + u(u-1)\frac{\Delta^2 y_0}{2!} + u(u-1)(u-2)\frac{\Delta^3 y_{-1}}{3!} + u(u^2-1)(u-2)\frac{\Delta^4 y_{-1}}{4!} + u(u^2-1)(u-2)(u-3)\frac{\Delta^5 y_{-2}}{5!} + \cdots \quad (4)$$

Now taking mean of the Gauss's Forward Formula and Third Gauss's Formula we obtain the Bessel's Formula as (see, James B. Scarborough, 1966):

$$y = \frac{(y_0+y_1)}{2} + \left(u-\frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\frac{(\Delta^2 y_{-1}+\Delta^2 y_0)}{2} + \frac{u\left(u-\frac{1}{2}\right)(u-1)}{3!}\Delta^3 y_{-1} + \frac{u(u^2-1)(u-2)}{4!}\frac{(\Delta^4 y_{-2}+\Delta^4 y_{-1})}{2} + \frac{u\left(u-\frac{1}{2}\right)(u^2-1)(u-2)}{5!}\Delta^5 y_{-2} + \cdots \quad (5)$$

Everett's Formula: This is an extensively used interpolation formula and uses only even order differences, as shown in the following table:

x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		—		—		—	—
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$

Hence the formula has the form (see, S. S. Sastry, 1998),

$$y = E_0 y_0 + E_2\Delta^2 y_{-1} + E_4\Delta^4 y_{-2} + E_6\Delta^6 y_{-3} + \cdots + F_0 y_1 + F_2\Delta^2 y_0 + F_4\Delta^4 y_{-1} + F_6\Delta^6 y_{-2} + \cdots \quad (6)$$

where the coefficients $E_0, F_0, E_2, F_2, E_4, F_4, E_6, F_6, \cdots$ can be determined as:

$$\left.\begin{array}{ll} E_0 = 1-u = v, & F_0 = u \\ E_2 = \dfrac{v(v^2-1^2)}{3!}, & F_2 = \dfrac{u(u^2-1^2)}{3!} \\ E_4 = \dfrac{v(v^2-1^2)(v^2-2^2)}{5!}, & F_4 = \dfrac{u(u^2-1^2)(u^2-2^2)}{5!} \\ \vdots & \vdots \end{array}\right\}$$

New (Proposed) Interpolation Formula: To derive the new formula we retreat the subscripts in Gauss's Forward Formula by one unit and replacing u by $u+1$ then we obtain,

$$y = y_{-1} + (u+1)\Delta y_{-1} + u(u+1)\frac{\Delta^2 y_{-2}}{2!} + u(u+1)(u+2)\frac{\Delta^3 y_{-2}}{3!} + u(u^2-1)(u+2)\frac{\Delta^4 y_{-3}}{4!} + u(u^2-1)(u+2)(u+3)\frac{\Delta^5 y_{-3}}{5!} + \cdots \quad (7)$$

Now taking the mean of (2) and (7), we obtain the New (proposed) Interpolation Formula as:

$$y = \frac{(y_{-1}+y_0)}{2} + \left(u+\frac{1}{2}\right)\Delta y_{-1} + \frac{u(u+1)}{2!}\frac{(\Delta^2 y_{-2}+\Delta^2 y_{-1})}{2} + \frac{u\left(u+\frac{1}{2}\right)(u+1)}{3!}\Delta^3 y_{-2} + \frac{u(u^2-1)(u+2)}{4!}\frac{(\Delta^4 y_{-3}+\Delta^4 y_{-2})}{2} + \frac{u\left(u+\frac{1}{2}\right)(u^2-1)(u+2)}{5!}\Delta^5 y_{-3} + \cdots \quad (8)$$

4. Comparisons of the Formulas by Examples

In order to compare our proposed formula of interpolation with the existing formulas we consider different examples. They are discussing in below.

***Problem 1*:** In the following table, the values of y are consecutive terms of the polynomial $y = 1 + 2x + 3x^2$.

x	2	3	4	5	6	7	8
y	17	34	57	86	121	162	209

Now we find the values of y for $x = 4.5$. We form the difference table below:

x	y	Δ	Δ^2
2	17		
		17	
3	34		6
		23	
4	57		6
		29	
5	86		6
		35	
6	121		6
		41	
7	162		6
		47	
8	209		

For $x = 4.5$, here we take $x_0 = 5$ and since $h = 1$ we have, $u = \frac{x - x_0}{h} = \frac{4.5 - 5}{1} = -0.5$ and $1 - u = v = 1 - (-0.5) = 1.5$.

Now Gauss's Forward Formula gives, $y(4.5) = 86 + (-0.5)(35) + (-0.5)(-0.5-1)(6) = 70.75$

Gauss's Backward Formula gives, $y(4.5) = 86 + (-0.5)(29) + (-0.5)(-0.5+1)(6) = 70.75$

Stirling's Interpolation Formula gives, $y(4.5) = 86 + (-0.5)\left(\frac{29+35}{2}\right) + \frac{(-0.5)^2}{2!}(6) = 70.75$

Bessel's Interpolation Formula gives, $y(4.5) = \frac{(86+121)}{2} + \left(-0.5 - \frac{1}{2}\right)(35) + \frac{(-0.5)(-0.5-1)}{2!}\left(\frac{6+6}{3}\right) = 70.75$

Everett's Formula gives, $y(4.5) = (1.5)(86) + (0.3125)(6) + (-0.5)(121) + (0.0625)(6) = 70.75$

Proposed Formula gives, $y(4.5) = \frac{(57+86)}{2} + (2(-0.5)+1)\left(\frac{29}{2}\right) + \frac{(-0.5)(-0.5+1)}{2!}\frac{(6+6)}{2} = 70.75$

***Problem 2*:** The following table gives the values of e^x for certain equidistant values of x. We find the value of e^x when $x = 1.7489$.

x	1.72	1.73	1.74	1.75	1.76	1.77	1.78
e^x	5.5845	5.6406	5.6973	5.7546	5.81244	5.87085	5.92986

Here we take $x = 1.7489$, $x_0 = 1.75$ and since $h = 0.01$ we have, $u = \frac{x - x_0}{h} = \frac{1.7489 - 1.75}{0.01} = -0.11$ and

$1-u=v=1-(-0.11)=1.11$. The difference table is shown on Table 2.

Table 2: Difference Table of Problem 2

x	$y=e^x$	Δ	Δ^2	Δ^3	Δ^4
1.72	5.5845285				
		0.056125444			
1.73	5.6406539		0.00056407		
		0.056689514		0.000005669	
1.74	5.6973434		0.000569739		0.00000005697
		0.057259253		0.00000572597	
1.75	5.7546027		0.000575465		0.00000005755
		0.057834718		0.00000578352	
1.76	5.8124374		0.000581249		0.00000005813
		0.058415967		0.00000584165	
1.77	5.8708534		0.00058709		
		0.059003057			
1.78	5.9298564				

Now Gauss's Forward Formula gives

$$y(1.7489)=5.7546027+(-0.11)(0.057834718)+(-0.11)(-0.11-1)\left(\frac{0.000575465}{2!}\right)$$
$$+(-0.11)\left((-0.11)^2-1\right)\left(\frac{0.00000578352}{3!}\right)+(-0.11)\left((-0.11)^2-1\right)(-0.11-2)\left(\frac{0.00000005755}{4!}\right)=5.748276093$$

Gauss's Backward Formula gives

$$y(1.7489)=5.7546027+(-0.11)(0.057259253)+(-0.11)(-0.11+1)\left(\frac{0.000575465}{2!}\right)$$
$$+(-0.11)\left((-0.11)^2-1\right)\left(\frac{0.00000572597}{3!}\right)+(-0.11)\left((-0.11)^2-1\right)(-0.11+2)\left(\frac{0.00000005813}{4!}\right)=5.748276093$$

Stirling's Interpolation Formula gives

$$y(1.7489)=5.7546027+(-0.11)\left(\frac{0.057259253+0.057834718}{2}\right)+\frac{(-0.11)^2}{2!}(0.000575465)$$
$$+\frac{(-0.11)\left((-0.11)^2-1\right)}{3!}\left(\frac{0.00000572597+0.00000578352}{2}\right)+\frac{(-0.11)^2\left((-0.11)^2-1\right)}{4!}(0.0000000575)=5.748276106$$

Bessel's Interpolation Formula gives

$$y(1.7489)=\frac{(5.7546027+5.8124374)}{2}+\left(-0.11-\frac{1}{2}\right)(0.0578347188)+\frac{(-0.11)(-0.11-1)}{2!}\left(\frac{0.000575465+0.000581249}{2}\right)$$
$$+\frac{(-0.11)\left(-0.11-\frac{1}{2}\right)(-0.11-1)}{3!}(0.00000578332)+\frac{(-0.11)\left((-0.11)^2-1\right)(-0.11-2)}{4!}\left(\frac{0.0000000575+0.0000000583}{2}\right)$$
$$=5.748276093$$

Everett's Formula gives

$$y(4.5)=(1.11)(5.7546027)+(0.0429385)(0.000575465)+(-0.02971237)(0.00000005755)+(-0.11)(5.8124374)$$
$$+(0.0181115)(0.000581249)+(-0.01805671)(0.00000005813)=5.748276091$$

Proposed Formula gives

$$y(1.7489)=\frac{(5.6973434+5.7546027)}{2}+\left(-0.11+\frac{1}{2}\right)(0.0572592533)+\frac{(-0.11)(-0.11+1)}{2!}\left(\frac{0.000569739+0.000575465}{2}\right)$$
$$+\frac{(-0.11)\left(-0.11+\frac{1}{2}\right)(-0.11+1)}{3!}(0.00000572599)+\frac{(-0.11)\left((-0.11)^2-1\right)(-0.11+2)}{4!}\left(\frac{0.00000005697+0.0000000575}{2}\right)$$
$$=5.748276093$$

Problem 3: The following table gives the values of $\sqrt{|x|}$ for certain equidistant values of x. we find the value of $\sqrt{|x|}$ for $x=1.3$.

x	-10	-7	-4	-1	2	5	8
$\sqrt{\lvert x\rvert}$	3.162278	2.645751	2	1	1.414214	2.236068	2.828427

Here we take $x=1.3$, $x_0=-1$ and since $h=3$ we have, $u=\frac{x-x_0}{h}=\frac{1.3-(-1)}{3}=0.766666667$ and $1-u=v=1-(0.766666667)=0.233333333$. The difference table is shown on Table 3.

Now Gauss's Forward Formula gives

$$y(1.3)=1+(0.766666667)(0.4142136)+(0.766666667)(0.766666667-1)\left(\frac{1.4142136}{2!}\right)$$
$$+(0.766666667)\left((0.766666667)^2-1\right)\left(\frac{-1.0065727}{3!}\right)+(0.766666667)\left((0.766666667)^2-1\right)(0.766666667-2)\left(\frac{-2.775035}{4!}\right)$$
$$+(0.766666667)\left((0.766666667)^2-1\right)\left((0.766666667)^2-2^2\right)\left(\frac{3.1444715}{5!}\right)$$
$$+(0.766666667)\left((0.766666667)^2-1\right)\left((0.766666667)^2-2^2\right)(0.766666667-3)\left(\frac{7.9129925}{6!}\right)=1.200809507$$

Table 3: Difference Table for Problem 3

x	$y=\sqrt{\lvert x\rvert}$	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
-10	3.16227766						
		-0.5165263					
-7	2.645751311		-0.129225				
		-0.6457513		-0.2250237			

-4	2		-0.3542487		1.993486		
		-1		1.76846225		-4.7685209	
-1	1		1.4142136		-2.775035		7.9129925
		0.4142136		-1.0065727		3.1444715	
2	1.414213562		0.4076409		0.3694366		
		0.8218544		-0.6371361			
5	2.236067977		-0.2294953				
		0.5923591					
8	2.828427125						

Gauss's Backward Formula gives

$$y(1.3)=1+(0.766666667)(-1)+(0.766666667)(0.766666667+1)\left(\frac{1.4142136}{2!}\right)+(0.766666667)\left((0.766666667)^2-1\right)$$
$$*\left(\frac{1.76846225}{3!}\right)+(0.766666667)\left((0.766666667)^2-1\right)(0.766666667+2)\left(\frac{-2.775035}{4!}\right)$$
$$+(0.766666667)\left((0.766666667)^2-1\right)\left((0.766666667)^2-2^2\right)\left(\frac{-4.7685209}{5!}\right)+(0.766666667)\left((0.766666667)^2-1\right)$$
$$*\left((0.766666667)^2-2^2\right)(0.766666667-3)\left(\frac{7.9129925}{6!}\right)=1.200809507$$

Stirling's Interpolation Formula gives

$$y(1.3)=1+(0.766666667)\left(\frac{-1+0.4142136}{2}\right)+\frac{(0.766666667)^2}{2!}(1.4142136)$$
$$+\frac{(0.766666667)\left((0.766666667)^2-1\right)}{3!}\left(\frac{1.76846225+(-1.0065727)}{2}\right)+\frac{(0.766666667)^2\left((0.766666667)^2-1\right)}{4!}(-2.775035)$$
$$+\frac{(0.766666667)\left((0.766666667)^2-1\right)\left((0.766666667)^2-2^2\right)}{5!}\left(\frac{-4.7685209+3.1444715}{2}\right)$$
$$+\frac{(0.766666667)^2\left((0.766666667)^2-1\right)\left((0.766666667)^2-2^2\right)}{6!}(7.9129925)=1.200809507$$

Bessel's Interpolation Formula gives

$$y(1.3)=\frac{(1+1.414213562)}{2}+\left(0.766666667-\frac{1}{2}\right)(0.4142136)+\frac{(0.766666667)(0.766666667-1)}{2!}\left(\frac{1.4142136+0.4076409}{2}\right)$$
$$+\frac{(0.766666667)\left(0.766666667-\frac{1}{2}\right)(0.766666667-1)}{3!}(-1.0065727)+\frac{(0.766666667)\left((0.766666667)^2-1\right)(0.766666667-2)}{4!}$$
$$*\left(\frac{-2.775035+0.3694366}{2}\right)+\frac{(0.766666667)\left(0.766666667-\frac{1}{2}\right)\left((0.766666667)^2-1\right)(0.766666667-2)}{5!}(3.1444715)$$
$$=1.22727848$$

Everett's Formula gives

$$
\left.
\begin{array}{l}
E_0 = 1 - 0.766666667 = 0.233333333, \qquad F_0 = 0.766666667 \\
E_2 = \dfrac{(0.233333333)\left((0.233333333)^2 - 1^2\right)}{3!} = -0.036771605, \quad F_2 = \dfrac{(0.766666667)\left((0.766666667)^2 - 1^2\right)}{3!} = -0.05267284 \\
E_4 = \dfrac{(0.233333333)\left((0.233333333)^2 - 1^2\right)\left((0.233333333)^2 - 2^2\right)}{5!} = 0.007254221, \\
F_4 = \dfrac{(0.766666667)\left((0.766666667)^2 - 1^2\right)\left((0.766666667)^2 - 2^2\right)}{5!} = 0.008986572, \\
E_6 = \dfrac{(0.233333333)\left((0.233333333)^2 - 1^2\right)\left((0.233333333)^2 - 2^2\right)\left((0.233333333)^2 - 3^2\right)}{7!} = -0.001545072
\end{array}
\right\}
$$

$$
\begin{aligned}
y(1.3) &= (0.233333333)(1) + (-0.036771605)(1.4142136) + (0.007254221)(-2.775035) + (-0.001545072)(7.9129925) \\
&+ (0.766666667)(1.414213562) + (-0.05267284)(0.4076409) + (0.008986572)(0.3694366) = 1.215052336
\end{aligned}
$$

Proposed Formula gives

$$
\begin{aligned}
y(1.3) &= \frac{(1+2)}{2} + \left(0.766666667 + \frac{1}{2}\right)(-1) + \frac{(0.766666667)(0.766666667+1)}{2!}\left(\frac{-0.3542487 + 1.4142136}{2}\right) \\
&+ \frac{(0.766666667)\left(0.766666667 + \frac{1}{2}\right)(0.766666667+1)}{3!}(1.76846225) + \frac{(0.766666667)\left((0.766666667)^2 - 1\right)(0.766666667+2)}{4!} \\
&* \left(\frac{1.993486 + (-2.775035)}{2}\right) + \frac{(0.766666667)\left(0.766666667 + \frac{1}{2}\right)\left((0.766666667)^2 - 1\right)(0.766666667+2)}{5!}(-4.7685209) \\
&= 1.156167806
\end{aligned}
$$

***Problem 4*:** The following table gives the values of $\cos x$ for certain equidistant values of x. We find the value of $\cos x$ when $x = 33.5$.

x	30	31	32	33	34	35	36
$\cos x$	0.154251	0.914742	0.834223	-0.01328	-0.84857	-0.90369	-0.12796

Here we take $x = 33.5$, $x_0 = 33$ and since $h = 1$ we have $u = \dfrac{x - x_0}{h} = \dfrac{33.5 - 33}{1} = 0.5$ and $1 - u = v = 1 - 0.5 = 0.5$. The difference table is shown on Table 4.

Table 4: Difference Table for Problem 4

x	$y=\cos x$	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^5
30	0.1542514						
		0.7604909					
31	0.9147424		-0.8410099				
		-0.0805190		0.0740288			
32	0.8342234		-0.7669811		0.7051589		
		-0.8475001		0.7791877		-0.7163816	
33	-0.0132767		0.0122066		-0.0112227		0.0103181
		-0.8352935		0.7679650		-0.7060635	
34	-0.8485703		0.7801716		-0.7172862		
		-0.0551219		0.0506788			
35	-0.9036922		0.8308504				
		0.7757285					
36	-0.1279637						

Now Gauss's Forward Formula gives

$$y(33.5)=-0.0132767+(0.5)(-0.8352935)+(0.5)(0.5-1)\left(\frac{0.0122066}{2!}\right)+(0.5)\left((0.5)^2-1\right)\left(\frac{0.7679650}{3!}\right)$$
$$+(0.5)\left((0.5)^2-1\right)(0.5-2)\left(\frac{-0.0112227}{4!}\right)+(0.5)\left((0.5)^2-1\right)\left((0.5)^2-2^2\right)\left(\frac{-0.7060635}{5!}\right)$$
$$+(0.5)\left((0.5)^2-1\right)\left((0.5)^2-4\right)(0.5-3)\left(\frac{0.0103181}{6!}\right)=-0.48903474$$

Gauss's Backward Formula gives

$$y(33.5)=-0.0132767+(0.5)(-0.8475001)+(0.5)(0.5+1)\left(\frac{0.0122066}{2!}\right)+(0.5)\left((0.5)^2-1\right)\left(\frac{0.7791877}{3!}\right)$$
$$+(0.5)\left((0.5)^2-1\right)(0.5+2)\left(\frac{-0.0112227}{4!}\right)+(0.5)\left((0.5)^2-1\right)\left((0.5)^2-2^2\right)\left(\frac{-0.7163816}{5!}\right)$$
$$+(0.5)\left((0.5)^2-1\right)\left((0.5)^2-4\right)(0.5-3)\left(\frac{0.0103181}{6!}\right)=-0.48903474$$

Stirling's Interpolation Formula gives

$$y(33.5) = -0.0132767 + (0.5)\left(\frac{-0.8475001 + -0.8352935}{2}\right) + \frac{(0.5)^2}{2!}(0.0122066)$$
$$+\frac{(0.5)((0.5)^2-1)}{3!}\left(\frac{0.7791877+0.7679650}{2}\right)+\frac{(0.5)^2((0.5)^2-1)}{4!}(-0.0112227)$$
$$+\frac{(0.5)((0.5)^2-1)((0.5)^2-2^2)}{5!}\left(\frac{-0.7163816+(-0.7060635)}{2}\right)+\frac{(0.5)^2((0.5)^2-1)((0.5)^2-2^2)}{6!}(0.0103181)$$
$$= -0.4890347$$

Bessel's Interpolation Formula gives

$$y(33.5) = \frac{(-0.0132767 + -0.8485703)}{2} + \left(0.5 - \frac{1}{2}\right)(-0.8352935) + \frac{(0.5)(0.5-1)}{2!}\left(\frac{0.0122066 + 0.7801716}{2}\right)$$
$$+\frac{(0.5)\left(0.5-\frac{1}{2}\right)(0.5-1)}{3!}(0.7679650)+\frac{(0.5)((0.5)^2-1)(0.5-2)}{4!}\left(\frac{-0.0112227+(-0.7172862)}{2}\right)$$
$$+\frac{(0.5)\left(0.5-\frac{1}{2}\right)((0.5)^2-1)(0.5-2)}{5!}(-0.7060635) = -0.4889844$$

Everett's Formula gives

$$y(33.5) = (0.5)(-0.0132767) + (-0.0625)(0.0122066) + (0.01171875)(-0.0112227) + (0.5)(-0.8485703)$$
$$+(-0.0625)(0.7801716) + (0.01171875)(-0.7172862) = -0.4889844$$

Proposed Formula gives

$$y(33.5) = \frac{(0.8342234 + (-0.0132767))}{2} + \left(0.5 + \frac{1}{2}\right)(-0.8475001)$$
$$+\frac{(0.5)(0.5+1)}{2!}\left(\frac{-0.7669811+0.0122066}{2}\right)+\frac{(0.5)\left(0.5+\frac{1}{2}\right)(0.5+1)}{3!}(0.7791877)$$
$$+\frac{(0.5)((0.5)^2-1)(0.5+2)}{4!}\left(\frac{0.7051589+(-0.0112227)}{2}\right)+\frac{(0.5)\left(0.5+\frac{1}{2}\right)((0.5)^2-1)(0.5+2)}{5!}(-0.7163816)$$
$$= -0.4891053$$

Table 5 shows the results of different interpolation methods of different example use in this study. Results shows that the new (proposed) formula is very efficient and posses good accuracy for evaluating functional values between given data.

Table 5: Results of different interpolation methods of different example use in this study

Problem No.	Gauss's Forward	Gauss's Backward	Stirling's	Bessel's	Everett's	New (Proposed)	True Value
Problem 1	70.75	70.75	70.75	70.75	70.75	70.75	70.75
Problem 2	5.748276093	5.748276093	5.748276106	5.748276093	5.748276091	5.748276093	5.748276093
Problem 3	1.200809507	1.200809507	1.200809507	1.22727848	1.215052336	1.156167806	1.140175425
Problem 4	-0.48903474	-0.48903474	-0.48903470	-0.4889844	-0.4889844	-0.4891053	-0.491034724

5. Conclusion

First, we propose a new interpolation given in equation (8) which is based on central difference and is derived from Gauss's Backward Formula and another formula in which we advance the subscripts in Gauss's Forward Formula by one unit and replacing u by $u+1$. Also comparisons of existing interpolation formulas (Gauss's, Stirling, Bessel's, etc.,) with the new formula by using different problems show that the new (proposed) formula is very efficient and posses good accuracy for evaluating functional values between given data.

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