A Simple Proof of the Generalization of the Binomial Theorem Using Differential Calculus

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Abstract
The binomial theorem is a simple and important mathematical result, and it is of substantial interest to statistical scientists in particular. Its proof and applications appear quite often in textbooks of probability and mathematical statistics. In this article, a new and very simple proof of multinomial theorem is presented. The new proof is based on a direct computation involving partial derivatives.
Key Words: Binomial distribution; Combinatorial Analysis; Mathematical Induction; Multinomial theorem; Partial derivatives.

1. Introduction

The binomial theorem is a simple and important mathematical result, and it is of substantial interest to statistical scientists in particular. Its proof and applications appear quite often in textbooks of probability and mathematical statistics. The standard proof of the binomial theorem involves a rather tricky argument using mathematical induction (Courant and John, 1999; Ross, 2006). Fulton (1952) provided a simpler proof of the binomial theorem, which also involved an induction argument. A very nice proof of the binomial theorem based on combinatorial considerations was obtained by Ross (2006). Using a result of the binomial distribution in probability, Rosalsky (2007) presented a very simple proof of the binomial theorem. Furthermore, Hwang (2009) presented a simple proof of the binomial theorem using partial differential calculus.

It is our point of view that the existing proofs of the binomial theorem can be distiguished into three main methodologies. The first is based on mathematical induction (Fulton, 1952; Courant and John, 1999; Ross, 2006). The second is based on either a counting or an elementary probability argument (Ross, 2006; Rosalsky, 2007). The third is based on partial differential calculus (Hwang, 2009).

This article presents a new proof of the generalization of the binomial theorem (multinomial theorem) based on a direct computation involving partial derivatives. The proof we give is substantially simpler than the proofs by mathematical induction and somewhat simpler than the proofs by counting or an elementary probability argument, through generalization. Further, an interesting feature of the new proof, which is also showed by that of Ross (2006) and Rosalsky (2007), is that the exact form of the binomial coefficients in the proof does not need to be known in advance but, rather, is actually produced by the proof.
The new proof for the generalization of the binomial theorem is very suitable for those students who have completed a course in calculus and are enrolled in an introductory course in probability and mathematical statistics.

2. Proof of the Generalization of the Binomial Theorem using Differential Calculus

We begin by stating the multinomial theorem and then present the new proof of it.
Multinomial Theorem: If n is a positive integer, then

$$(x_1 + x_2 + \ldots + x_m)^n = \sum_{n_1+n_2+\ldots+n_m=n} c(n; n_1, n_2, \ldots, n_m)\, x_1^{n_1}\, x_2^{n_2} \ldots x_n^{n_m}$$

where the notation $c(n; n_1, n_2, \ldots, n_m) = n!/(n_1!\,n_2!\ldots n_m!)$ is the multinomial coefficient.
Proof: From the definition above,

$$(x_1 + x_2 + \ldots + x_m)^n = \sum_{n_1+n_2+\ldots+n_m=n} c(n; n_1, n_2, \ldots, n_m)\, x_1^{n_1}\, x_2^{n_2} \ldots x_n^{n_m} \quad . \tag{1}$$

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For any n_i, $i = 1, 2, \ldots, m$, we calculate the partial derivatives of both sides, n_1 times with respect to x_1, n_2 times with respect to x_2, and so on as, n_m times with respect to x_m. The partial derivative of the left side of (1) is always equal to

$$\frac{\delta^n}{\delta x_1^{n_1}\,\delta x_2^{n_2}\ldots\delta x_n^{n_m}}(x_1 + x_2 + \ldots + x_m)^n = n!. \tag{2}$$

Now for all integers $n_1, n_2, \ldots, n_m$ in n,

$$\frac{\delta^n}{\delta x_1^{n_1}\,\delta x_2^{n_2}\ldots\delta x_n^{n_m}} c\left(n; n_1, n_2, \ldots, n_m\right) x_1^{n_1}\, x_2^{n_2}\ldots x_n^{n_m} = 0, \qquad \text{for } n_i \text{ not equal,}$$

where $i = 1, 2, \ldots, m$

and

$$\frac{\delta^n}{\delta x_1^{n_1}\,\delta x_2^{n_2}\ldots\delta x_n^{n_m}} c\left(n; n_1, n_2, \ldots, n_m\right) x_1^{n_1}\, x_2^{n_2}\ldots x_n^{n_m} = n_1!\,n_2!\ldots n_m!\,c\left(n; n_1, n_2, \ldots, n_m\right)$$

and hence

$$\frac{\delta^n}{\delta x_1^{n_1}\,\delta x_2^{n_2}\ldots\delta x_n^{n_m}} \sum_{n_1+n_2+\ldots+n_m} c\left(n; n_1, n_2, \ldots, n_m\right) x_1^{n_1}\, x_2^{n_2}\ldots x_n^{n_m} = n_1!\,n_2!\ldots n_m!\,c\left(n; n_1, n_2, \ldots, n_m\right) \tag{3}$$

It then follows from (2) and (3) that we have, for any n_i, $i = 1, 2, \ldots, m$,

$$n! = n_1!\,n_2!\ldots n_m!\,c\left(n; n_1, n_2, \ldots, n_m\right).$$

This completes the proof of the theorem.

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