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On Best One-Sided Approximation By Interpolation Polynomials In Space $L_{p.w}(X)$

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Abstract

The aim of this article is to obtain the order of convergence of weighted space by interpolation polynomials on [-π, π]. Our order of convergence is given in terms of error of the best one-sided approximation or in terms averaged modulus. However if f is a smooth function, then we can given the order in terms of $E_n(f^{(m)})_{p,w}$.

Keyword : One-sided approximation, Averaged modulus, Interpolation polynomials

1. Introduction

We shall consider the functions defined on $\mathbb{R}$ which are 2 π-periodic on every variable. With $\mathbb{T}_n$ we denote the set of all trigonometric polynomials of degree n on every variable. Set X=[-π,π]. We denote the set of 2 π-periodic bounded measurable functions with usual sup-norm by L_∞ such that

(1.1)……. $L_\infty(X) = \{f : \|f\|_\infty = \sup\{|f(x)|, \forall\, x \in X\} < \infty\}$.

The space $L_p(X)$, ($1 \le p < \infty$) is equipped with the following norm ($f \in L_p(X)$)

(1.2)……. $\|f\|_p = \left(\int_X |f(x)|^p\, dx\right)^{\frac{1}{p}} < \infty.$

Further, for $\delta > 0$, locally global norm of a function f is defined by

(1.3)……. $\|f\|_{\delta,p} = \left(\int_X \, sup\left\{|f(y)|^p; y \in [x-\frac{\delta}{2}, x+\frac{\delta}{2}]\right\} dy\right)^{\frac{1}{p}}$.

Now , let W be the set of all weight functions on X. Consider $L_{p,w}(X)$, ($1 \le p < \infty$)the space of all functions f on X which is given the following norm ($f \in L_{p,w}(X)$)

(1.4)……. $\|f\|_{p,w} = \left(\int_X \left|\frac{f(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}} < \infty.$

The degree of best approximation of a function $f \in L_p(X)$ with trigonometric polynomials from $\mathbb{T}_n$ on X given by

(1.5)……. $E_n(f)_p = inf\{\|f - T_n\|_p \ , \ T_n \in \mathbb{T}_n\},$

the degree of best approximation of a function $f \in L_{\delta,p}(X)$ with trigonometric polynomials from $\mathbb{T}_n$ on X is given by

(1.6)……. $E_n(f)_{\delta,p} = inf\{\|f - T_n\|_{\delta,p} \ , \ T_n \in \mathbb{T}_n\}$

and the degree of best approximation of a function $f \in L_{p,w}(X)$ with trigonometric polynomials from $\mathbb{T}_n$ on X is given by

(1.7)……. $E_n(f)_{p,w} = inf\{\|f - T_n\|_{p,w} \ , \ T_n \in \mathbb{T}_n\}.$

The degree of best one-sided approximation of a function $f \in L_p(X)$, $f \in L_{p,w}(X)$ and $f \in L_{\delta,p,w}(X)$ with trigonometric polynomials from $\mathbb{T}_n$ on X are respectively given by

(1.8)……. $\tilde{E}_n(f)_p = inf\{\|p_n - q_n\|_p \ , \ p_n, q_n \in \mathbb{T}_n \ and \ q_n(x) \leq f(x) \leq p_n(x), \forall\, x \in X\}$

(1.9)……. $\tilde{E}_n(f)_{p,w} = inf\{\|p_n - q_n\|_{p,w} \ , \ p_n, q_n \in \mathbb{T}_n \ and \ q_n(x) \leq f(x) \leq p_n(x), \forall\, x \in X\}$

(1.10)...…. $\tilde{E}_n(f)_{\delta,p,w} = inf\{\|p_n - q_n\|_{\delta,p,w} \ , p_n, q_n \in \mathbb{T}_n \ and \ q_n(x) \leq f(x) \leq p_n(x), \forall x \in X\}$

For characterization of the structural properties for a given function $f \in L_p(X)$ or $f \in L_{p,w}(X)$, we shall use the following modulus.

The k^th average modulus of smoothness for $f \in L_p(X)$ and $f \in L_{p,w}(X)$ are respectively given by

(1.11)……. $\tau_k(f,\delta)_p = \|\omega_k(f,.,\delta\|_p$, where

$\omega_k(f,\delta)_p = sup_{0<h<\delta}\left\{\left\|\Delta_h^k f(.)\right\|_p\right\}, \delta > 0$, the kth ordinary modulus of continuity for $f \in L_p(X)$ and

(1.12)……. $\tau_k(f,\delta)_{p,w} = \|\omega_k(f,.,\delta\|_{p,w}$, where

$\omega_k(f,\delta)_{p,w} = sup_{0<h<\delta}\left\{\left\|\Delta_h^k f(.)\right\|_{p,w}\right\}, \delta > 0$ such that

$\Delta_h^k f(x) = \sum_{i=0}^{k}(-1)^{i+k}\binom{k}{i} f(x+ih)\ , x, h \in X.$

The kth locally modulus of smoothness for $f \in L_\infty(X)$ is defined by

$$\omega_k(f,x,\delta)_\infty = sup\left\{\left|\Delta_h^k f(t)\right|; t, t+kh \in [x-\frac{kh}{2}, x+\frac{kh}{2}]\right\}.$$

The unique trigonometric polynomial from $\mathbb{T}_n$ interpolating a given function $f \in L_{p,w}(X)$ at a points $\{x_j\}_0^n$ is denoted by $I_n(f)$.

If t,u $\in\mathbb{R}$, then we denoted by

$D_n = \frac{\sin(\frac{n+1}{2})u}{2sin\frac{u}{2}}$ the Dirichlet kernel . Interpolating polynomial $I_n(f)$ has representation

(1.13)……. $I_n(f) = \frac{2}{2n+1}\sum_{j\in N} f(xj)\ D_n\ (x-x_j)$, which has the following properties

i. $I_n(f, x_{kn}) = f(x_{kn}),\ 0 \leq k < n-1.$

ii. $I_n^{(j)}(f, x_{kn}) = f^{(j)}(x_{kn})$, j= $m_1, m_2, \ldots m_q$, where $0< m_1< m_2<\ldots< m_q$ are distinct integer and $x_{nk}= 2k\pi/n$ [4].

Recently, similar results have been proved for mean convergence of interpolation by trigonometric polynomial in Xu (1991). For interpolation we do not really need continuity of the underling function *f*. The interpolation is well defined already for bounded measurable function *f* on X. To get $L_{p,w}$-approximation of the Langrange interpolation it is sufficient to assume Riemann integrablity of *f* , which can be found already (Zygmand 1958).

The purpose of this note is to obtain the order of approximation of the Lagrange interpolation and more generally interpolation in $L_{p,w}$-norm for unbounded function *f* .

Since the interpolating polynomials are based on the point values of *f*, it is unrealistic to expect that the order be given by either $E_n(f)_{p,w}$ or $(f,\delta)_{p,w}$.

Our order of approximation is given in terms of degree of best one-sided approximation. However if *f* is a smooth function, then we can give the order in terms of $E_n(f^{(m)})_{p,w}$.

1. Auxiliary Lemmas :

Lemma (2.1) (Hristov 1989) **:**

If $T_n \in \mathbb{T}_n$, (1≤ p<∞), then

$$\|T_n\|_p \le \|T_n\|_{\delta,p} \le c(p)\|T_n\|_p.$$

Lemma (2.2) (Hristov 1989) **:**

For every $f \in L_\infty(X)$, (1≤ p<∞) we have

$$\|f\|_p \le \|f\|_{\delta,p} \le \|f\|_\infty = \|f\|_{\delta,\infty}.$$

Lemma (2.3) (Jassim, et al. 2010) **:**

Let f, g be two functions define on the same domain, (1≤ p<∞). Then

$$\tau_k(f,\tfrac{1}{n})_p \le \tau_k(f-g,\tfrac{1}{n})_p + \tau_k(g,\tfrac{1}{n})_p.$$

Lemma (2.4) (Hristov 1989) **:**

For every $f \in L_\infty(X)$, (1≤ p<∞) we have

$$\|I_n(f)\|_{\delta,p} \le c(p)\|f\|_{\delta,p}.$$

Lemma (2.5) (Jurgen et al. 1994) **:**

Let $T_n \in \mathbb{T}_n$, (1≤ p<∞). Then

$$\|T_n\|_p \le c(p)\left(\frac{1}{n}\sum_{k=0}^{n-1}\left|T_n^{(m)}(x_{kn})\right|^p\right)^{\frac{1}{p}}.$$

Lemma (2.6) (Jurgen et al. 1994) **:**

Let $T_n \in \mathbb{T}_n$, (1≤ p<∞). Then

$$\left(\frac{1}{n}\sum_{k=0}^{n-1}|T_n(x_{kn})|^p\right)^{\frac{1}{p}} \le c(p)\|T_n\|_p.$$

Lemma (2.7) (Jassim, et al. 2011) **:**

If f is a bounded measurable function on [a,b], then

$$\int_a^b f(x)dx \approx \frac{b-a}{n}\sum_{i=1}^{n} f(x_i), \quad \text{where } x_i = a + \frac{(b-a)(2i-1)}{2n}.$$

Lemma (2.8) (Sendov, et al. 1988)**:**

If $f \in L_p(X)$, (1≤ p<∞) ,then

$$\tilde{E}_n(f)_p \le \frac{2\pi}{n+1}E_n(f')_p .$$

Lemma (2.9) (Popov, et al. 1984**) :**

If $T_n \in \mathbb{T}_n$, r is positive integer, then

$$\left\|T_n^{(r)}\right\|_p \le c(p)n^{-r}\omega_r(f,\delta)_p.$$

Lemma (2.10) (Popov, et al. 1984) **:**

For 2π-periodic bounded Riemann integrable functions, we have

$\|f - I_n(f)\| = O(1)\tau_1(f, \frac{1}{n})$, (1≤ p<∞), where $O(1)$ is bounded function.

Lemma (2.11) (Sendov, et al. 1988) **:**

Let $f \in L_p(X)$, (1≤ p<∞) .Then

i. $E_n(f)_p \leq c_k\omega_k(f,\delta)_p \leq \tau_k(f,\delta)_p$.
ii. $\tilde{E}_n(f)_p \leq c_k\tau_k(f)_p$.

Lemma (2.12) **(**Jassim 1991) **:**

Let $f \in L_{p,w}(X)$, (1≤ p<∞). Then

$$\tilde{E}_n(f)_{p,w} \leq c(p)E_n(f)_{\delta,p,w} \leq c(p)\tilde{E}_n(f)_{p,w}.$$

2. **Formulation of the main results:**

The object of our paper is to find the degree of best one-sided approximation in $L_{p,w}(X)$ space b interpolating $I_n(f)$ in terms of average modulus and modulus of continuity for $f \in L_{p,w}(X)$.

Theorem (3.1) :

If $f \in L_{p,w}(X)$; (1≤ p<∞), then

$$\tilde{E}_n(f)_{p,w} \leq c(p)\|f - I_n(f)\|_{\delta,p,w} \leq c(p)\tilde{E}_n(f)_{p,w} .$$

Theorem (3.2) :

Let $n \geq 1$, (1≤ p<∞) and$f^{(m)} \in f \in L_{p,w}(X)$. Then

$$\|f - I_n(f)\|_{p,w} \leq c(p)\, n^{-m}\, \tilde{E}_n(f)_{p,w}.$$

Theorem (3.3) :

Let $n \geq 1$, $f \in L_{p,w}(X)$, (1≤ p<∞). Then

$$\|f - I_n(f)\|_{p,w} \leq c(p)\, [\tilde{E}_n(f)_{p,w} + \omega_r(f,\delta)_{p,w}].$$

Theorem (3.4) :

Let $f \in L_{p,w}(X)$, (1≤ p<∞). Then

$$\|f - I_n(f)\|_{\delta,p,w} \leq c_k\tau_k(f,\delta)_{p,w} ,$$

where c is constant depending only on p.

We need the following lemmas to prove our theorems.

Lemma (A) :

Let $f \in L_{p,w}(X)$, (1≤ p<∞). Then

$$\|f\|_{p,w} \leq \|f\|_{\delta,p,w} .$$

Proof : From (1.3) and (1.4) we get

$$\|f\|_{p,w} = \left(\int_X \left|\frac{f(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}} \le sup\left(\int_X \left|\frac{f(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}}$$
$$\le sup\left(\int_X sup\left\{\left|\frac{f(x)}{w(x)}\right|^p ; y \in \left[x-\frac{\delta}{2}, x+\frac{\delta}{2}\right]\right\} dx\right)^{\frac{1}{p}} = \|f\|_{\delta,p,w}.$$

Lemma (B) :

Let $f \in L_{p,w}(X)$, ($1 \le p < \infty$). Then

$$\tau_k(f,\delta)_{p,w} \le c(p)\|f\|_{p,w}.$$

Proof :

By using (1.12) and definition of modulus of continuity we get

$$\tau_k(f,\delta)_{p,w} = \|\omega_k(f,.,\delta\|_{p,w} = \left(\int_X sup\left\{\left|\frac{\Delta_h^k f(t)}{w(t)}\right|^p dt\right\}\right)^{\frac{1}{p}}$$
$$\le \sum_{k=0}^{n}\left(\int_X sup\left\{\left|\frac{\Delta_h^k f(t)}{w(t)}\right|^p dt\right\}\right)^{\frac{1}{p}} \le c(p)\left(\int_X \left\{\left|\frac{f(x)}{w(x)}\right|^p dt\right\}\right)^{\frac{1}{p}} = c\,(p)\|f\|_{p,w}.$$

Proof of theorem (3.1) :

We shall to prove $\|f - I_n(f)\|_{\delta,p,w} \le c(p)\tilde{E}_n(f)_{p,w}$ ………(1)

From (2.1), (2.2), Interpolation conditions and lemma(A) we get

$$\|I_n(f)\|_{\delta,p,w} = \left\|\frac{I_n(f)}{w}\right\|_{\delta,p} \le c(p)\left\|\frac{I_n(f)}{w}\right\|_p = c(p)\left\|\frac{f}{w}\right\|_p \le c(p)\left\|\frac{f}{w}\right\|_{\delta,p}.$$

Thus $\|I_n(f)\|_{\delta,p,w} \le c(p)\left\|\frac{f}{w}\right\|_{\delta,p} = c(p)\|f\|_{\delta,p,w}$ ……..(2)

In order to obtain inequality (1), we consider $p_n \in \mathbb{T}_n$, which $E_n(f)_{\delta,p,w} = \|f - p_n\|_{\delta,p,w}$.

$$\|f - I_n(f)\|_{\delta,p,w} \le \|f - p_n\|_{\delta,p,w} + \|p_n - I_n(f)\|_{\delta,p,w}$$
$$= \|f - p_n\|_{\delta,p,w} + \|\mathrm{I_n}(p_n - f)\|_{\delta,p,w}$$

By using (2) and (2.12), we get

$$\|f - I_n(f)\|_{\delta,p,w} \le E_n(f)_{\delta,p,w} + c_1(p)\|f - p_n\|_{\delta,p,w}$$
$$= c(p)E_n(f)_{\delta,p,w} \le c(p)\,\tilde{E}_n(f)_{p,w}.$$

We need to prove $\tilde{E}_n(f)_{p,w} \le c(p)\,\|f - I_n(f)\|_{\delta,p,w}$.

Let $p_n, q_n \in \mathbb{T}_n$, such that , $q_n(x) \le f(x) \le p_n(x)$ $\forall x \in X$ and $\tilde{E}_n(f)_{p,w} = \|p_n - q_n\|_{p,w}$.

Thus $\tilde{E}_n(f)_{p,w} \le c(p)E_n(f)_{\delta,p,w} \le c(p)\,\|f - I_n(f)\|_{\delta,p,w}$.

Proof of theorem (3.2) :

Since $I_n(f)$ preserves trigonometric polynomials in$\mathbb{T}_n$, then

$\|f-I_n(f)\|_{p,w} \le \|f-T_n\|_{p,w} + \|T_n - I_n(f)\|_{p,w}$ where $T_n \in \mathbb{T}_n$ is best trigonometric polynomial approximation to f. Let p_n *and* q_n be the polynomials in$\mathbb{T}_n$ such that

$$\tilde{E}_n(f^{(m)})_{p,w} = \|p_n - q_n\|_{p,w}\,;\ q_n(x) \le f^{(m)}(x) \le p_n(x)\ \ \forall\, x \in X$$

From (2.5), (2.6), (2.7) and Minkowaski's inequality, we get

$$\|T_n - I_n(f)\|_{p,w} = \|I_n(T_n - f)\|_{p,w} = \left\| I_n(\frac{T_n}{w} - \frac{f}{w}) \right\|_p \le c(p)\left(\frac{1}{n}\sum_{k=0}^{n-1}\left|\frac{T_n^{(m)}(x_{kn})}{w(x_{kn})} - \frac{f^{(m)}(x_{kn})}{w(x_{kn})}\right|^p\right)^{\frac{1}{p}}$$

$$\approx c(p)\left(\int_X \left|\frac{T_n^{(m)}(x)}{w(x)} - \frac{f^{(m)}(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}}$$

$$\le c(p)\left\{\left(\int_X \left|\frac{T_n^{(m)}(x)}{w(x)} - \frac{q_n(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}} + \left(\int_X \left|\frac{q_n(x)}{w(x)} - \frac{f^{(m)}(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}}\right\}$$

$$\le c(p)\left\{\left(\int_X \left|\frac{T_n^{(m)}(x)}{w(x)} - \frac{q_n(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}} + \left(\int_X \left|\frac{p_n(x)}{w(x)} - \frac{q_n(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}}\right\}$$

$$\le c(p)\left\{\{\left(\int_X \left|\frac{T_n^{(m)}(x)}{w(x)} - \frac{f^{(m)}(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}} + \left(\int_X \left|\frac{f^{(m)}(x)}{w(x)} - \frac{q_n(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}}\}\right.$$

$$\left. + \left(\int_X \left|\frac{p_n(x)}{w(x)} - \frac{q_n(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}}\right\}$$

$$\le c(p)\left\{\left\|f^{(m)} - T^{(m)}\right\|_{p,w} + \|p_n - q_n\|_{p,w} + \|p_n - q_n\|_{p,w}\right\}$$

$$= c(p)\{E_n(f^{(m)})_{p,w} + 2\,\tilde{E}_n(f)_{p,w}\}$$

Therefore

$$\|f - I_n(f)\|_{p,w} \le E_n(f)_{p,w} + c(p)[E_n(f^{(m)})_{p,w} + 2\,\tilde{E}_n(f)_{p,w}]$$

By using (2.8), we get

$$\|f - I_n(f)\|_{p,w} \le c(p)\frac{1}{n^m}[\,\tilde{E}_n(f)_{p,w} + E_n\left(f^{(m)}\right)_{p,w}]$$

$$\le c(p)n^{-m}\,\tilde{E}_n(f)_{p,w}\ .$$

Proof of theorem (3.3) :

Let $T_n \in \mathbb{T}_n$ be the best trigonometric polynomial approximation to a function $f \in L_{p,w}(X)$. Then

$$\|f - I_n(f)\|_{p,w} \le$$

$$\|f - T_n\|_{p,w} + \|T_n - I_n(T_n)\|_{p,w} + \|I_n(T_n) - I_n(f)\|_{p,w}$$

$$= E_n(f)_n + \|T_n - I_n(T_n)\|_{p,w} + \|I_n(T_n - f)\|_{p,w}$$

From (2.5), (2.7) and (2.9) we get

$$\|T_n - I_n(T_n)\|_{p,w} \le \|T_n\|_{p,w} = \left\|\frac{T_n}{w}\right\|_p \le c_1(p)\left(\frac{1}{n}\sum_{k=1}^{n-1}\left|\frac{T_n^{(r)}(x_k)}{w(x_k)}\right|^p\right)^{\frac{1}{p}} \approx c_1(p)\left(\int_X \left|\frac{T_n^{(r)}(x)}{w(x)}\right|^p dx\right)^{\frac{1}{p}}$$

$$= c_1(p)\|T_n^{(r)}\|_{p,w} = c_1(p)\left\|\frac{T_n^{(r)}}{w}\right\|_p$$

$$\le c(p)n^{-r}\omega_r(\frac{f}{w},\delta)_p = c(p)n^{-r}\omega_r(f,\delta)_{p,w}.$$

Now, from (2.4),(2.5) and (2.9) we get

$$\|I_n(T_n - f)\|_{p,w} = \left\|I_n\left(\frac{T_n}{w} - \frac{f}{w}\right)\right\|_p \le c(p)\left\|\frac{T_n}{w} - \frac{f}{w}\right\|_p = c(p)\|T_n - f\|_{p,w}$$

$$\le c(p)\left(\|T_n - q_n\|_{p,w} + \|q_n - f\|_{p,w}\right)$$

$$\le c(p)\left(\|T_n - f\|_{p,w} + \|f - q_n\|_{p,w} + \|q_n - f\|_{p,w}\right)$$

$$\le c(p)\left(\|T_n - f\|_{p,w} + \|p_n - q_n\|_{p,w} + \|p_n - q_n\|_{p,w}\right)$$

$$= c(p)(2\tilde{E}_n(f)_{p,w} + E_n(f)_{p,w}).$$

By using (2.11)(i),we get

$$E_n(f)_{p,w} = E_n(\tfrac{f}{w})_p \le c(p)\omega_r\left(\tfrac{f}{w},\delta\right)_p = c(p)\omega_r(f,\delta)_{p,w}\ .$$

Therefore $\|f - I_n(f)\|_{p,w} \le c(p)[\tilde{E}_n(f)_{p,w} + \omega_r(f,\delta)_{p,w}]$.

Proof of theorem (3.4) :

Consider p_n, q_n are the best one-sided approximation of a function f is space (X), such that $\tilde{E}_n(f)_{p,w} = \|p_n - q_n\|_{p,w}$

From (2.3), (2.4), (A), (2.10) and (B)

Now, $\|f - I_n(f)\|_{\delta,p,w} \le \|f - p_n\|_{\delta,p,w} + \|p_n - I_n(p_n)\|_{\delta,p,w} + \|I_n(p_n) - I_n(f)\|_{\delta,p,w}$

$$= \left\|\frac{f}{w} - \frac{p_n}{w}\right\|_{\delta,p} + \left\|\frac{p_n}{w} - \frac{I_n(p_n)}{w}\right\|_{\delta,p} + \left\|I_n\left(\frac{p_n}{w} - \frac{f}{w}\right)\right\|_{\delta,p}$$

$$\le \left\|\frac{p_n}{w} - \frac{q_n}{w}\right\|_{\delta,p} + \left\|\frac{p_n}{w} - I_n(\frac{p_n}{w})\right\|_{\delta,p} + c_1(p)\left\|\frac{p_n}{w} - \frac{q_n}{w}\right\|_p$$

$$= c_2(p)\left\|\frac{p_n}{w} - \frac{q_n}{w}\right\|_p + \left\|\frac{p_n}{w} - I_n(\frac{p_n}{w})\right\|_{\delta,p} + c_1(p)\left\|\frac{p_n}{w} - \frac{q_n}{w}\right\|_p$$

$$= c_3\tilde{E}_n(f)_{p,w} + \left\|\frac{p_n}{w} - I_n(\frac{p_n}{w})\right\|_{\delta,p}$$

$$\le c_3(p)\tilde{E}_n(f)_{p,w} + c_4(p)\left(O(1)\tau_1\left(\frac{p_n}{w},\delta\right)_p\right)$$

$$\le c_3(p)\tilde{E}_n(f)_{p,w} + c_5(p)\tau_1\left(\frac{p_n}{w},\delta\right)_p$$

$$\leq c_3(p)\tilde{E}_n(f)_{p,w} + c_5(p)(\tau_1\left(\frac{f}{w},\delta\right)_p + \tau_1\left(\frac{f-p_n}{w},\delta\right)_p$$

$$\leq c_3(p)\tilde{E}_n(f)_{p,w} + c_6(p)(\tau_1\left(\frac{f}{w},\delta\right)_p + \left\|\frac{f}{w}-\frac{p_n}{w}\right\|_p) \leq c_7(p)\tilde{E}_n(f)_{p,w} + c_6(p)\tau_1\left(\frac{f}{w},\delta\right)_p$$

$$\leq c_k(p)\tau_1(f,\delta)_{p,w}.$$

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