Some Chaotic Properties of Ikeda Map

Iftichar M.T.AL-Shara'a[1] Wafaa H. Al-Hilli[2]*
Mathematics Department ,University of Babylon ,Hilla , Babylon, Iraq
* E-mail of the corresponding author: wafaahadi23@yahoo.com

Abstract

We study the dynamical system of Ikeda map on three dimension, we find some the general properties, and we show some chaotic properties of it. We prove the Lypaunov exponent of Ikeda map is positive and Ikeda map has sensitivity dependence to initial condition. Finally we use the Matlab program to draw the sensitivity of Ikeda map.

1. Introduction

The Ikeda map occurs in the modeling of optical recording media (crystals) the numerical results obtained to date show that under certain parameter values the Ikeda map exhibits highly complicated dynamical behavior [2] for chaotic systems, which exhibit sensitive dependence on initial conditions this problem is especially important [3]. When we iterate the map using a computer the inevitable rounding errors cause that the trajectory generated by a computer and the true one diverge exponentially and after a certain number of iterations become uncorrelated. In this work we study the Ikeda map which has form $I_{a,b}\begin{pmatrix} x \\ y \\ t \end{pmatrix} = \begin{pmatrix} 1 + a\,x\cos t - a\,y\sin t \\ a\,x\sin t + a\,y\cos t \\ bt \end{pmatrix}$.In the literature there are some authors study Ikeda map ,they study the bifurcation of it.

2. Preliminaries

Let F: X→X be a map and let x_0 be a point in the domain of F, then $F(x_0)$= the first iterate of x_0 for F and $F(F(x_0))$= the second iterate of x_0 for F. More generally, if n any positive integer, and x_n is the nth iterate of x_0 for F, then $F^{n+1}(x_0)$ is the (n+1)th iterate of x_0 for F .Let V be a subset of R^3. Let GL (3,R) be the set of all 3×3 matrices, $M = \begin{bmatrix} a & b & c \\ d & e & f \\ r & s & t \end{bmatrix}$ where a, b, c, d, e, f , r, s, t ∈ R such that det(M)=±1 The map F can always be represented in the form $F(v) = \begin{bmatrix} f_1(v) \\ f_2(v) \\ f_3(v) \end{bmatrix}$, for all v in V when f_1 , f_2 , f_3 are real valued coordinate map of F. In our work, Any $P = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$ for which , $f_1(P) = P_1$, $f_2(P) = P_2$, $f_3(P) = P_3$ is called a fixed point of the three dynamical system. Let $\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$ be a fixed point of F, then $\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$ is attracting fixed point. if and only if there is a disk centered of $\begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}$ such that $F^n\begin{bmatrix} x \\ y \\ t \end{bmatrix} \to \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ as n→∞ , for every $\begin{bmatrix} x \\ y \\ t \end{bmatrix}$ in the disk. x_0 is an eventually fixed point of F if there is a positive integer m such that $F^m(X_0)$ is a fixed point of F. assume that the first partials of the coordinate maps f_1 , f_2 and f_3 of F exit at v_0, the differential of F at v_0 is the nonlinear map $DF(v_0)$

defined on R^3 by: $DF(v_0)=\begin{bmatrix} \frac{\partial f_1(v_0)}{\partial x} & \frac{\partial f_1(v_0)}{\partial y} & \frac{\partial f_1(v_0)}{\partial t} \\ \frac{\partial f_2(v_0)}{\partial x} & \frac{\partial f_2(v_0)}{\partial y} & \frac{\partial f_2(v_0)}{\partial t} \\ \frac{\partial f_3(v_0)}{\partial x} & \frac{\partial f_3(v_0)}{\partial y} & \frac{\partial f_3(v_0)}{\partial t} \end{bmatrix}$, for all v_0 in R^3. The determinant of $DF(v_0)$ is called the Jacobian of F at v_0 and is denoted by J=det $DF(V_0)$. if $0 \le |\det DF(v_0)| < 1$, then F is said to be area-contracting at v_0, and if $|\det DF(v_0)| > 1$, then F is said to be area-expanding at v_0 A map F is called a diffeomorphism provided it is: one-to-one, onto, C^∞
and its inverse F^{-1} is C^∞.Suppose that A is a 3×3 matrix. The real number λ is an eigenvalue of A provided that there is a nonzero v in R^3 such that Av=λv ,in this case v is an eigenvector of A . ifλ_1, λ_2, λ_3 are eigenvalues of any matrix satisfying $|\lambda i| \neq 1$,i=1,2,3 then we call the matrix M, a hyperbolic matrix.

3. The General Properties of Ikeda Map

In this section, we introduce some general properties of Ikeda map $I_{a,b}$. if a≠1 and b≠1 then $I_{a,b}$ has unique fixed point, we show that by proposition below:-

Proposition (3.1):- If a≠1 and b≠1 then Ikeda map $I_{a,b}$ has unique fixed point

Proof:- by the definition of fixed point, we get:-

$\begin{bmatrix} 1+ax\cos t - ay\sin t \\ ax\sin t + ay\cos t \\ bt \end{bmatrix} = \begin{bmatrix} x \\ y \\ t \end{bmatrix}$, bt=t by hypothesis b≠1 so t=0 then t=0 so $ax\sin(t) + ay\cos(t) = y$ then ay=y, since a≠1 we get y=0

$1 + ax\cos t - ay\sin t = x$. Hence $x = \frac{1}{1-a}$ then $\begin{bmatrix} \frac{1}{1-a} \\ 0 \\ 0 \end{bmatrix}$ is the fixed point.☼

Proposition (3.2):-

If a=1, b=1 and $t \neq n\pi$ then Ikeda map has infinite fixed points.

If a≠0,1 , b=1 , t≠ nπ and t $\neq \cos^{-1}(\frac{a^2+1}{2a})$ then Ikeda map has infinite fixed points.

Proof:-

1. By the definition of fixed point, we get:- $\begin{bmatrix} 1 + x\cos t - y\sin t \\ x\sin t + y\cos t \\ t \end{bmatrix} = \begin{bmatrix} x \\ y \\ t \end{bmatrix}$, then t=t , so $x\sin t + y\cos t = y$

then $x\sin t = y(1-\cos t)$ since t≠nπ therefore $x = \frac{y(1-\cos t)}{\sin t}$ so $1 + \left(\frac{y(1-\cos t)}{\sin t}\right)\cos t - y\sin t = \frac{y(1-\cos t)}{\sin t}$,

then $\frac{\sin t + y\cos t(1-\cos t) - y\sin^2 t - y + y\cos t}{\sin t} = 0$, then

$sint + 2ycost - 2y = 0$ therefore $y=\frac{sint}{2(1-cost)}$ then $x = \frac{sint\,(1-cost)}{2(1-cost)} \cdot \frac{1}{sint} = \frac{1}{2}$, then the fixed point of Ikeda map is $\begin{pmatrix} \frac{1}{2} \\ \frac{sint}{2(1-cost)} \\ t \end{pmatrix}$, $\forall t\in R$.✿

2. By the definition of fixed point, we get:- $\begin{bmatrix} 1 + a\,x\,cost - a\,y\,sint \\ a\,x\,sint + ay\,cost \\ t \end{bmatrix} = \begin{bmatrix} x \\ y \\ t \end{bmatrix}$, then t=t , so

a x sint +a y cost=y then a x sint= y-a y cost since a≠0, and t≠ nπ therefore $x = \frac{y\,(1-a\,cost)}{a\,sint}$, so

$1+\frac{y\,cost\,(1-a\,cost)}{sint} - a\,y\,sint = \frac{y(1-cost)}{sint}$ then a snit +a y cost + a^2 y cos^2t – a^2 y sin^2t- y + a y cost=0 since $t\neq cos^{-1}(\frac{a^2+1}{2a}$) therefore y= y=(a sint) / (1+a^2-2 a cost) and x=(1-a cost)/(1+a^2-2 a cost) then Ikeda map has infinite fixed points✿

Proposition(3.3) :-The Jacobien of Ikeda map $I_{a,b}$ is a^2b .

Proof: the differential matrix of Ikeda is

$$DI_{a,b}(v_0)=\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial t} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial t} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial t} \end{bmatrix}$$

so

$$DI_{a,b}(v_0)=\begin{bmatrix} a\cos t & -a\sin t & -ax\sin t - ay\cos t \\ a\sin t & a\cos t & ax\cos t - ay\sin t \\ 0 & 0 & b \end{bmatrix}$$

Then

$$J = \det DI_{a,b}\,(v_0) = b\det\begin{bmatrix} a\,cost & -a\,sint \\ a\,sint & a\,cost \end{bmatrix} = ba^2$$

In some conditions the Ikeda map is area contracting at each point in R^3 but in other conditions it is expanding area.

Proposition (3.4):-

Let $I_{a,b}$ be the Ikeda map :-

(1) If |a|<1 and |b|<1 then $I_{a,b}$ is area contraction at v ; $\forall v\in R^3$ **.**

(2) If |b|>1,b≠0 and $|a|^2>\frac{1}{|b|}$ then $I_{a,b}$ is area expending at v ;$\forall$ $v\in R^3$ **.**

(3) If |a|>1,a≠0 and $|b|>\frac{1}{|a|2}$then $I_{a,b}$ is area expending at v ;$\forall$ $v\in R^3$

Proof:-

1. If |a|<1 and |b|<1 then $|ba^2|$ <1 so the absolute Jacobian of Ikeda map is least than 1 so from definition of area contracting.

2. If b≠0 since | J|=|det(D $I_{a,b}\begin{bmatrix} x \\ y \\ t \end{bmatrix}$| =|$a^2$b|=|a|^2|b| by hypothesis |b|>1 and so J>$\frac{1}{|b|}$. |b| > 1. Then $I_{a,b}$ is area expending at v∈R^3

3. Similarity proof (2).☼

Now, we study the conditions which Ikeda map is onto.

Proposition (3.5):-

If a≠0 and b≠0 then $I_{a,b}$ *is onto*

Proof:-

Case(1):- If t≠($\frac{1}{2}+n)\pi$, Let $\begin{bmatrix} v \\ w \\ s \end{bmatrix}$ any element in R^3 such that $\begin{bmatrix} v \\ w \\ s \end{bmatrix} = \begin{bmatrix} 1+ax\cos t - ay\sin t \\ ax\sin t + ay\cos t \\ bt \end{bmatrix}$.

Then $v = 1 + axcost - aysint$, since a≠0 and t≠($\frac{1}{2}+n)\pi$ then x= $\frac{v-1+aysint}{acost}$ so $w = axsint + aycost$ then w=a ($\frac{v-1+aysint}{acost}$)sin(t)+a ycos(t) therefore w=v tan(t) - tan(t) + (a y sin(t) tan(t)) + a y cos(t) then w-v tan(t) + tan(t) = y (a tan(t) sin(t) + (a cos(t)) therefore Y=$\frac{w-vtan(t)+tan(t)}{(aytan(t)sin(t)+acos(t))}$ then $bt = s$ since b ≠0 therefor $t = \frac{s}{b}$ then there exist

$$\begin{pmatrix} \dfrac{v-1+\left(\dfrac{w-vtan\left(\frac{s}{b}\right)+tan\left(\frac{s}{b}\right)}{tan\left(\frac{s}{b}\right)sin\left(\frac{s}{b}\right)+cos\left(\frac{s}{b}\right)}\right)sin\left(\frac{s}{b}\right.}{cos(t)} \\ \dfrac{w-vtan\left(\frac{s}{b}\right)+tan\left(t\left(\frac{s}{b}\right)\right)}{\left(a\,tan\left(\frac{s}{b}\right)sin\left(\frac{s}{b}\right)+acos\left(\frac{s}{b}\right)\right)} \\ \dfrac{s}{b} \end{pmatrix} \in R^3 \text{ such that } I_{a,b}\begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} v \\ w \\ s \end{bmatrix} \text{ then } I_{a,b} \text{ is onto}$$

Case (2):- If t= (n+$\frac{1}{2}$)π Let $\begin{bmatrix} v \\ w \\ s \end{bmatrix}$ any element in R^3 such that $\begin{bmatrix} v \\ w \\ s \end{bmatrix} = \begin{bmatrix} 1-ay \\ ax \\ bt \end{bmatrix}$, $s = bt$ then $t = \frac{s}{b}$ so

w= x a since a≠0 then x= $\frac{w}{a}$ so v=1 –a y then y = $\frac{1-v}{a}$ such that there exit $\begin{pmatrix} \frac{w}{a} \\ \frac{1-v}{a} \\ \frac{s}{b} \end{pmatrix} \in R^3$then $I_{a,b}$ is onto.

Remark:-

1. If a=0, b≠0 and $t \neq (\frac{1}{2}+n)\pi$ then $I_{0,b}$ is not onto.
2. If a≠0 , b=0 and $t \neq (\frac{1}{2}+n)\pi$ then $I_{a,b}$ is not onto.
3. If a=0 , b=0 and $t \neq (\frac{1}{2}+n)\pi$ then $I_{a,b}$ is not onto.

Remark:-If a=0 then $I_{a,b}$ is not one to one ,so it is not diffeomorphism.

Now, we study the conditions which Ikeda map is one to one.

Proposition (3.6):-ifa≠0 and b≠0 then Ikeda map is one to one.

Proof:- $1+ax\cos t - ay\sin t = 0$, $ax\sin t + ay\cos t = 0$, $bt = 0$ then t=0 so y=0 since a≠0 therefore x= $\frac{-1}{a}$.then the kernel of $I_{a,b}$ is unique set.

Proposition (3.7):-If a≠0 and b≠0 Ikeda map is diffeomorphism

Proof: . By proposition (3.5) and(3.6) Ikeda map is onto and one to one Note that

$\frac{\partial f_1(x,y,t)}{\partial x} = a\cos t$, $\frac{\partial^2 f_2(x,y,t)}{\partial x^2} = 0$ .... $\frac{\partial^n f_n(x,y,t)}{\partial x^n} = 0$ For all $n \in N$ and $n \geq 2$.

$\frac{\partial f_2(x,y,t)}{\partial y} = a\cos t$, $\frac{\partial^2 f_2(x,y,t)}{\partial y^2} = 0$, ... $\frac{\partial^n f_n(x,y,t)}{\partial y^n} = 0$ For all $n \in N$ and $n \geq 2$. $\frac{\partial f_3(x,y,t)}{\partial t} = 1$, $\frac{\partial^2 f_3(x,y,t)}{\partial t^2} = 0$, ... $\frac{\partial^n f(x,y,t)}{\partial t^n} = 0$ for all $n \in N$ and $n \geq 2$.Then the partial derivatives are exist and continuous then $I_{a,b}$ is C^{∞}, by the above the Ikeda map has inverse map. so by the definition $I_{a,b}$ is diffeomorphism map.☼

Proposition (3.8):- I f a≠1 and b≠1 then The eigenvalues of Ikeda map is $\lambda_{1,2}$=a, λ_3=b at the fixed point.

Proof :- so by proposition(3.1)Ikeda map has unique fixed point $v_0 = \begin{bmatrix} \frac{1}{1-a} \\ 0 \\ 0 \end{bmatrix}$

$$\det(a-\lambda I)v_0 = \det\begin{bmatrix} a-\lambda & 0 & 0 \\ 0 & a-\lambda & 0 \\ 0 & 0 & b-\lambda \end{bmatrix} = (a-\lambda)(a-\lambda)(b-\lambda) = 0$$ Then $\lambda_{1,2}$=a, λ_3=b.☼

Proposition (3.9):-Let $I_{a,b}$ be Ikeda map and a≠0 , b≠0 then

1. If |a|<1 and |b|<1 then the fixed point of Ikeda map is attracting fixed point.
2. If |a|>1 and |b|>1 then the fixed point of Ikeda map is repelling fixed point .
3. If |a|>1 and |b|<1 then the fixed point of Ikeda map is saddle fixed map.
4. If |a|<1 and |b|>1 then the fixed point of Ikeda map is saddle fixed map.

Proof:-

By proposition (3.4) and definition it's satisfying (1, 2, 3, 4)☼

Proposition (3.10):-1. $\forall \begin{bmatrix} x \\ y \\ t \end{bmatrix} \in R^3$, $DI_{a,b}\begin{bmatrix} x \\ y \\ t \end{bmatrix}$ is a hyperbolic matrix if and only if $|ba^2| = 1$.

Proof:-⇒) Let $DI_{a,b}\begin{bmatrix} x \\ y \\ t \end{bmatrix}$ be a hyperbolic matrix then in view of definition $DI_{a,b}\begin{bmatrix} x \\ y \\ t \end{bmatrix} \in GL(3,R)$ then

$\det\left(DI_{a,b}\begin{bmatrix} x \\ y \\ t \end{bmatrix}\right) = ba^2 = \mp 1$ .hence $|ba^2| = 1$.

⇐)) Let $|ba^2| = 1$ then $\det\left(DI_{a,b}\begin{bmatrix} x \\ y \\ t \end{bmatrix}\right) = ba^2 = \mp 1$, $DI_{a,b}\begin{bmatrix} x \\ y \\ t \end{bmatrix} \in GL(3,R)$ and by the relation between roots and coefficients. $|ba^2| = |\mp 1| = 1$ so if $|b| \neq 1$ and $|a| \neq 1$then $|a|^2 = \frac{1}{|b|}$such that $|b| \neq 0$ and by proposition(3.4)$\lambda_{1,2}$= a ,$\lambda 3$=bare three real numbers and since: $|\lambda_{1,2}| = |a| \neq 1$and $|\lambda_3| = |b| \neq 1$ and since R is totally order set so either $|a| > 1$ or $|b| < 1$ .If $|a| > 1$ then $|b| = \frac{1}{|a|^2} < 1$.and if $|b| < 1$ then $|a|^2 = \frac{1}{|b|} > 1$.☼

Proposition (3.11):- If a≠1 and b≠1 the eventually fixed points set of Ikeda map is the set of fixed points.

Proof:-By proposition (3.1), we have only one fixed points and clearly they is eventually fixed point $\begin{bmatrix} \frac{1}{1-a} \\ 0 \\ 0 \end{bmatrix}$.

Suppose that there exists an eventually fixed point $\begin{bmatrix} x \\ y \\ t \end{bmatrix}$ for $I_{a,b}\begin{bmatrix} x \\ y \\ t \end{bmatrix}$ and $\begin{bmatrix} x \\ y \\ t \end{bmatrix} \neq \begin{bmatrix} \frac{1}{1-a} \\ 0 \\ 0 \end{bmatrix}$ then by definition of eventually fixed point. Then exists a positive integer number n such that $I^n_{a,b}\begin{bmatrix} x \\ y \\ t \end{bmatrix}$ is a fixed point of $I_{a,b}$ so then is $n \in Z^{+}$ such that $I^{n}_{a,b}\begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{1-a} \\ 0 \\ 0 \end{bmatrix}$.we get If n=1, then $I_{a,b}\begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} 1+ax\cos t - ay\sin t \\ ax\sin t + ay\cos t \\ bt \end{bmatrix} = \begin{bmatrix} \frac{1}{1-a} \\ 0 \\ 0 \end{bmatrix}$ So bt=0

then t=0 , we get $axsin(0) + aycos(0) = 0$ this imply , y=0.hence 1+axcos(0)=$\frac{1}{1-a}$ then x=$\frac{1}{1-a}$ we generalize

this relation by induction $I^{n}_{a,b}\begin{pmatrix} x \\ y \\ t \end{pmatrix} = \begin{pmatrix} \frac{1}{1-a} \\ 0 \\ 0 \end{pmatrix}$. the set of all eventually fixed point of Ikeda map is $\left\{\begin{bmatrix} \frac{1}{1-a} \\ 0 \\ 0 \end{bmatrix}\right\}$.

Proposition (3.12):- If a≠1 and b≠1 then The Ikeda map has periodic points of period two

Proof:- Suppose that there exists a periodic point of period 2 for $I_{a,b}$ then for different we have

$I^{2}_{a,b}\begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} x \\ y \\ t \end{bmatrix}$ so $\begin{pmatrix} 1 + a(1 + axcost - aysint)cost - a(axsint + aycost)sint \\ a(1 + axcost - aysint)sint + a(axsint + aycost)cost \\ b^{2}t \end{pmatrix} = \begin{pmatrix} x \\ y \\ t \end{pmatrix}$, since b≠1, bt=t then

t=0 so a(1+axcos(0)-a y sin(0) + a (a x sin(0) + a y cos(0)) cos(0) = y therefore y = 0 so 1+ a (1+ a x cos(0) –a (0) sin(0)) cos(0) - $a\,(a\,x\sin(0) + a\,(0)\cos(0))\cos(0) = x$ then $1 + a + ax = x$, $1 + a = x - ax$ since a≠1

therefore x=$\frac{1+a}{1-a}$ then $I^{2}\begin{pmatrix} x \\ y \\ t \end{pmatrix} = \begin{pmatrix} \frac{1+a}{1-a} \\ 0 \\ 0 \end{pmatrix}$. Hence $\begin{pmatrix} \frac{1+a}{1-a} \\ 0 \\ 0 \end{pmatrix}$ is the periodic points of periodic 2

Proposition (3.13):-
Ikeda map has periodic points of period 3
Proof:-

Suppose that there exists a periodic point of three for $I_{a,b}$ then we have

$$\begin{pmatrix} 1 + a(1 + a(1 + axcost - aysint))cost - a((axsint + aycost)sint))cost - a(a(1 + axcost - aysint))sint + a(axsint + aycost)cost)sint \\ a(1 + a(1 + axcost - aysint)cost) - (a(axsint + aycost)sint)sint + a(a(1 + axcost - aysint)sint) + a(axsint + aycost)cost)cost \\ b^{3}t \end{pmatrix} = \begin{pmatrix} x \\ y \\ t \end{pmatrix}$$

since b≠1 , $b^{3}t$=t then t=0 so

a(1 + a(1 + a x cos(0) a y sin (0)) cos(0)) (a(a x sin(0) + a y cos(0) sin(0) + a(a(1 + a x cos(0) a y sin(0)) sin (0) + a (a x sin(0) + a y cos(0)) cos(0) = y then a^{2} y = y therefore y= 0 so 1 + a(1+ a(1+a x cos(0) – a (0) sin(0)) cos(0) – a((a x sin(0) + a(0) cos(0) sin(0)) cos(0) – a (a(1+a x cos(0) –a(0) sin(0)) sin(0) + a(a x sin(0) + a(0) cos(0)) cos(0)) sin(0) = x then 1 + a + a^{2} +a^{2} x = x therefor 1 + a + a^{2} = x – a^{2} x then x=$\frac{1+a+a^{2}}{1-a^{2}}$ therefore

$\begin{pmatrix} \frac{1+a+a^2}{1-a^2} \\ 0 \\ 0 \end{pmatrix}$ is the periodic points of periodic 3.

4. Sensitivity Dependence on Initial Condition of Ikeda Map:-

The f: X→X is said to be sensitive dependence on initial conditions if there exists ε > 0 such that for any $x_0 \in X$ and any open set $U \subset X$ containing x_0 there exists $y_0 \in U$ and $n \in z^+$ such that $d(f^n(x_0), f^n(y_0)) > \varepsilon$ That is $\exists \varepsilon>0, \forall x, \forall \delta > 0, \exists\, y \in B\,\delta(x), \exists\, n: d(f^n(x_0), f^n(y_0)) \geq \varepsilon$. A dynamical system has sensitive dependence on initial conditions on subset $x' \subseteq X$ if there is $\in>0$ such that for every $x \in x'$ and $\delta>0$ there are $y \in Y$ and $n \in N$ for which $d(x,y) < \delta$ and $d(f^n(x), f^n(y)) > \in$ .Although there is no universal agreement on definition of chaos, its generally agreed that a chaotic dynamical system should exhibit sensitive dependence on initial conditions chaotic. Al-Shara'a and Al-Yaseen [1] defined an order on R^n as: let $x=(x_1,x_2,\ldots\ldots,x_n)$ and $y=(y_1,y_2,\ldots\ldots,y_n) \in R^n$ we write $x \prec y$ if and only if $x_i<y_i$ $,\forall i=1,\ldots,n$

Proposition (3.14):-If |a|>1 or |b|>1 then $I_{a,b}$ is sensitive dependence on initial conditions

Proof:- let $x = \begin{bmatrix} x \\ y \\ t \end{bmatrix}$ be a point in R^3, since the sine and cose maps are bounded then $I_{a,b}(X) = \begin{bmatrix} 1+ax-ay \\ ax+ay \\ bt \end{bmatrix}$.

Case (1):- If $|x| \leq 1$ then by hypothesis and by definition of the order a bove.

$$I_{a,b}(X) \prec \begin{bmatrix} 1-ay \\ ay \\ bt \end{bmatrix} \text{ and } I^2_{a,b}(X) \prec \begin{bmatrix} 1-a^2y \\ a^2y \\ bt \end{bmatrix} \text{, that is , } I^n_{a,b}(X) \prec \begin{bmatrix} 1-a^ny \\ a^ny \\ bt \end{bmatrix}$$

thus if $|a|>1$ or |b|>1 then $n \to \infty$ .then $I^n_{a,b}(x) \to \infty$, let $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in R^3$ such that $d(x,y,t) \prec \delta$,

$$d(I_{a,b}(x), I_{a,b}(y), I_{a,b}(t)) = \sqrt{(1-ay)^2 + (ay)^2 + (bt)^2} ,$$

$$d\left(I^2{}_{a,b}(x), I^2{}_{a,b}(y), I^2{}_{a,b}(t)\right) = \sqrt{\left(1-a^2y\right)^2 + \left(a^2y\right)^2 + (bt)^2} = \sqrt{b^2t^2 + 1 - 2a^2y + a^4y + a^4y} =$$

$$\sqrt{b^2t^2 + 1 - (2a^2 - 2a^4)y}$$

and by induction $d\left(I^n{}_{a,b}(x), I^n{}_{a,b}(y), I^n{}_{a,b}(t)\right) = \sqrt{(1-ay)^{2^n} + (ay)^{2^n} + (bt)^n}$

If $|a^2b| > 1$ and $n\to\infty$, $d\left(I^n{}_{a,b}(x), I^n{}_{a,b}(y), I^n{}_{a,b}(t)\right)$ hence $I_{a,b}$ has sensitive dependence an initial condition.

Case (2):- If $|x|>1$ from of the Ikeda map, then the iterates of Ikeda map are diverage. Thus it has sensitive dependence on initial condition.☼

Then we study the sensitivity to initial condition of map by varying thepoint (x_i,y_i ,z_i) as follows (i.=1,2) control parameters (a,b) by using (Matlab) to analysis of view for sensitivity dependent on initial condition .this work show as in figure(1) Now consider the map we get sensitivity to initial condition on the initial

5. The Lyapunov Exponents of Ikeda Map

Let f:X→X be continuous differential map where X be any space then all x in X in direction V. the

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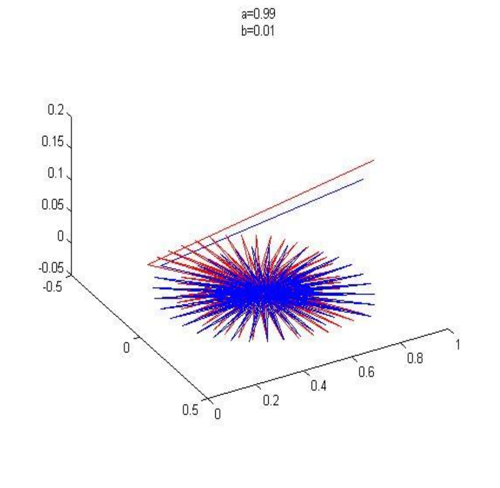
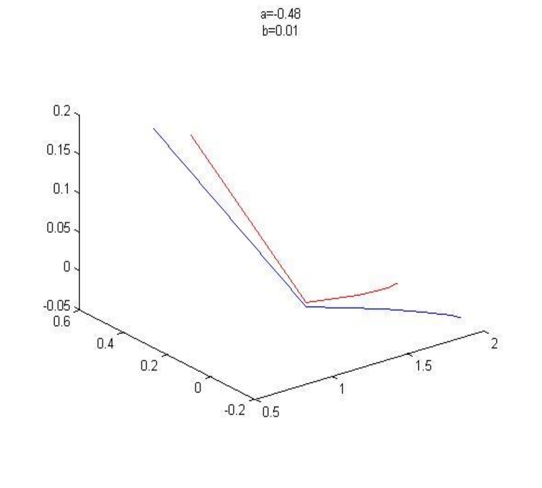
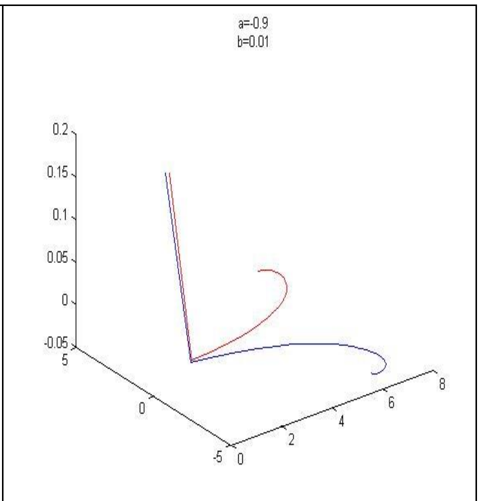
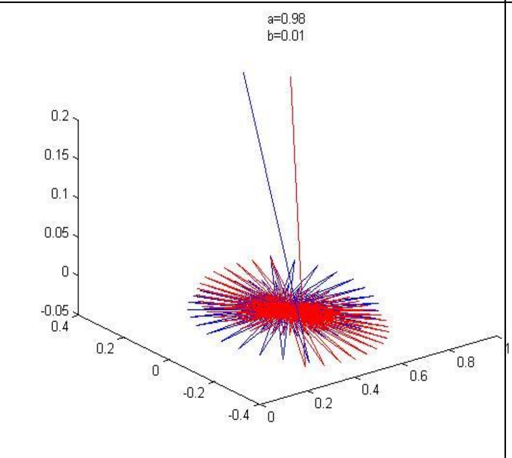
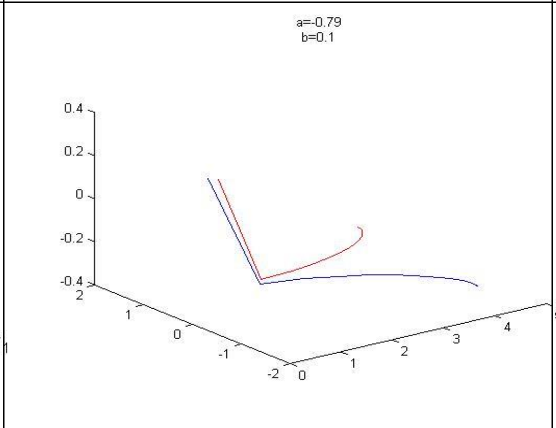
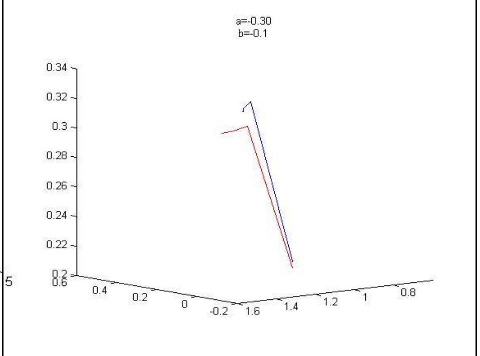
<td>

a=0.99,b=0.01 with initial points (0.7,0.2 ,0.4)and(0.8,0.3,0.2)</td>
<td>

a=0.48,b=0.01 with initial points (0.7,0.2 ,0.4)and(0.8,0.3,0.2)</td>
<td>

a=0.9 ,b=0.01 with initial points (0.7,0.2 ,0.4)and(0.8,0.3,0.2)</td>
</tr>
<tr>
<td>

a=0.98,b=0.01 with initial points (0.7,0.2 ,0.4)and(0.8,0.3,0.2)</td>
<td>

a=0.79,b=0.1 with initial points (0.7,0.2 ,0.4)and(0.8,0.3,0.2)</td>
<td>

a=-0.30,b=-0.1 with initial points (0.7,0.2 ,0.4)and(0.8,0.3,0.2)</td>
</tr>
</table>

Lyapunov exponent was defined of a map f at x by $L^{\pm}(x,v)= \lim_{n\to\infty}\frac{1}{n}\ln\|Df_x^n v\|$ whenever the limit exists. In higher dimensions, for example in R^n the map f will have n Lyapunov exponents,say $L_1^{\pm}(X,V_1)$,$L_2^{\pm}(X,V_2)$, $L_3^{\pm}(X,V_3)$,……,$L_n^{\pm}(X,V_n)$, for a minimum n Lyapunov exponent that is $L^{\pm}(X,V)=\max\{L_1^{\pm}(X,V_1),L_2^{\pm}(X,V_2), L_3^{\pm}(X,V_3),\ldots,L_n^{\pm}(X,V_n)\}$ where $v=(v_1,v_2,\ldots,v_n)$.The usual test for chaos is calculation of the largest Lyapunov exponent [4]. A positive largest Lyapunov exponent indicates chaos. When one has access to the equations generating the chaos, and which measure the rates of separation from the current orbit point along m orthogonal directions .The Lyapunov exponent λ is greater then zero. a quantitative measure of the sensitive dependence on the initial conditions is the Lyapunov exponent λ it's the a averaged rate of divergence(or convergence) of two neighboring trajectories in the phase space .we recall and discuss the Lyapunov exponent first since the Lyapunov exponent are orthogonal quartitive. Representing the average divergence of nearby trajectories in phase space. then is a Lyapunov exponent in the direction of each of the axis (since we can choose our second trajectory to start from a point which denotes some perturbation a long each of the different dimensions of the phase space).so the Lyapunov exponents are relatied to the expanding or contracting of the flow of the system in different direction since the positive (orientation) of the ellipsoid changes as it develops. The directions associated with each exponent vary thronghout the attractor a system is refered to as chaotic.

Proposition (3.15):- Let $I_{a,b}:R^3\rightarrow R^3$ be the Ikeda map either |a|>1 or |b|>1 then the map has positive Lyapunov exponents.

proof:- *if* $|a|>1$ *and by proposition* $|\lambda_{1,2}|=|a|$, $\therefore$ $L_{1,2}\left(\begin{pmatrix}x\\y\\t\end{pmatrix},v1\right)=\lim_{n\to\infty}\frac{1}{n}\left|\left|DI_{a,b}\begin{pmatrix}x\\y\\t\end{pmatrix},V1\right|\right|>\ln|a|>0$ and since $|\lambda_3|=|b|$ then either $L_3\left(\begin{pmatrix}x\\y\\t\end{pmatrix},v3\right)=\lim_{n\to\infty}\frac{1}{n}\left|\left|DI_{a,b}\begin{pmatrix}x\\y\\t\end{pmatrix},V3\right|\right|>\ln|b|<0$ or $L_3\left(\begin{pmatrix}x\\y\\t\end{pmatrix},v3\right)>0$ by definition $L\left(\begin{pmatrix}x\\y\\t\end{pmatrix},v\right)=\max\{L_1^{\pm}(x_1,v_1),L_2^{\pm}(x_2,v_2),L_3^{\pm}(x_3,v_3)\}$ then $L\left(\begin{pmatrix}x\\y\\t\end{pmatrix},v1\right)>0$. In the way, we can prove if |b|<1 then Lyapunov exponent of Ikeda map is positive☼

References

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