Some Chaotic Properties of Ikeda Map

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Abstract

We study the dynamical system of Ikeda map on three dimension, we find some of the general properties, and we show some chaotic properties of it. We prove the Lyapunov exponent of Ikeda map is positive and Ikeda map has sensitivity dependence to initial condition. Finally we use the Matlab program to draw the sensitivity of Ikeda map.

1. Introduction

The Ikeda map occurs in the modeling of optical recording media (crystals) the numerical results obtained to date show that under certain parameter values the Ikeda map exhibits highly complicated dynamical behavior [2] for chaotic systems, which exhibit sensitive dependence on initial conditions this problem is especially important [3]. When we iterate the map using a computer the inevitable rounding errors cause that the trajectory generated by a computer and the true one diverge exponentially and after a certain number of iterations become uncorrelated. In this work we study the Ikeda map which has form I_{a,b}. In the literature there are some authors study Ikeda map, they study the bifurcation of it.

2. Preliminaries

Let F: X→X be a map and let x_0 be a point in the domain of F, then F(x_0)= the first iterate of x_0 for F and F(F(x_0))= the second iterate of x_0 for F. More generally, if n any positive integer, and x_n is the nth iterate of x_0 for F, then F^{n+1}(x_0) is the (n+1)th iterate of x_0 for F. Let V be a subset of R^3. Let GL (3,R) be the set of all 3×3 matrices, M = [a b c]
[d e f]
[r s t]
where a, b, c, d, e, f, r, s, t ∈ R such that det(M)=±1 The map F can always be represented in the form F(v) = [f_1(v) f_2(v) f_3(v)]
, for all v in V when f_1 , f_2 , f_3 are real valued coordinate map of F. In our work, Any P = [P_1 P_2 P_3]

for which , f_1 (P) = P_1 , f_2 (P) = P_2 , f_3 (P) = P_3 is called a fixed point of the three dynamical system. Let P = [P_1 P_2 P_3] be a fixed point of F, then is attracting fixed point. if and only if there is a disk centered of P = [P_1 P_2 P_3] such that F^n = [x y t] → [p_1 p_2 p_3] as n→∞ , for every [x y t] in the disk. x_0 is an eventually fixed point of F if there is a positive integer m such that F^m (x_0) is a fixed point of F. Assume that the first partials of the coordinate maps f_1 , f_2 and f_3 of F exit at v_0, the differential of F at v_0 is the nonlinear map DF(v_0)
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\[
\frac{\partial f_1(v_0)}{\partial x} \quad \frac{\partial f_1(v_0)}{\partial y} \quad \frac{\partial f_1(v_0)}{\partial t}
\]
\[
\frac{\partial f_2(v_0)}{\partial x} \quad \frac{\partial f_2(v_0)}{\partial y} \quad \frac{\partial f_2(v_0)}{\partial t}
\]
\[
\frac{\partial f_3(v_0)}{\partial x} \quad \frac{\partial f_3(v_0)}{\partial y} \quad \frac{\partial f_3(v_0)}{\partial t}
\]

defined on \( \mathbb{R}^3 \) by: \( DF(v_0) = \begin{bmatrix}
\frac{\partial f_1}{\partial x}(v_0) & \frac{\partial f_1}{\partial y}(v_0) & \frac{\partial f_1}{\partial t}(v_0) \\
\frac{\partial f_2}{\partial x}(v_0) & \frac{\partial f_2}{\partial y}(v_0) & \frac{\partial f_2}{\partial t}(v_0) \\
\frac{\partial f_3}{\partial x}(v_0) & \frac{\partial f_3}{\partial y}(v_0) & \frac{\partial f_3}{\partial t}(v_0)
\end{bmatrix} \), for all \( v_0 \) in \( \mathbb{R}^3 \). The determinant of \( DF(v_0) \) is called the Jacobian of \( F \) at \( 0 \) and is denoted by \( J = \det DF(v_0) \).

If \( 0 \leq \det DF(v_0) < 1 \), then \( F \) is said to be area-contracting at \( v_0 \), and if \( \det DF(v_0) > 1 \), then \( F \) is said to be area-expanding at \( v_0 \).

3. The General Properties of Ikeda Map

In this section, we introduce some general properties of Ikeda map \( I_{a,b} \). If \( a \neq 0, 1 \) and \( b \neq 0, 1 \) then \( I_{a,b} \) has unique fixed point, we show that by proposition below:-

Proposition (3.1): If \( a \neq 1 \) and \( b \neq 1 \) then Ikeda map \( I_{a,b} \) has unique fixed point

Proof:- by the definition of fixed point, we get:-

\[
\begin{bmatrix}
1 + ax \cos t - ay \sin t \\
ax \sin t + ay \cos t \\
bt
\end{bmatrix} =
\begin{bmatrix}
x \\
y \\
t
\end{bmatrix}, \quad bt = t \text{ by hypothesis } b \neq 1 \text{ so } t = 0 \text{ then } t = 0 \text{ so } ax \sin(t) + ay \cos(t) = y \text{ then } ay = t.
\]

Since \( a \neq 1 \) we get \( y = 0 \)

\[
1 + ax \cos t - ay \sin t = x. \quad \text{Hence } x = \frac{1}{1 - a} \text{ then }
\begin{bmatrix}
1 \\
1 - a \\
0
\end{bmatrix}
\]

is the fixed point. \( \Box \)

Proposition (3.2):-

If \( a = 1, b = 1 \) and \( t \neq n\pi \) then Ikeda map has infinite fixed points.

If \( a \neq 0,1, b = 1 \) and \( t \neq n\pi \) and \( t \neq \cos^{-1}(\frac{b^2 + 1}{2a}) \) then Ikeda map has infinite fixed points.

Proof:-

1. By the definition of fixed point, we get:-

\[
\begin{bmatrix}
1 + x \cos t - y \sin t \\
x \sin t + y \cos t \\
bt
\end{bmatrix} =
\begin{bmatrix}
x \\
y \\
t
\end{bmatrix}, \quad \text{then } t = t \text{ so } x \sin t + y \cos t = y
\]

then \( x \sin t = y(1 - \cos t) \) since \( t \neq n\pi \) therefore \( x = \frac{y(1 - \cos t)}{2t} \) so \( 1 + \left(\frac{y(1 - \cos t)}{2t}\right) \cos t - y \sin t = \frac{y(1 - \cos t)}{\sin t} \), then

\[
\frac{\sin t + y \cos t(1 - \cos t) - y \sin t}{\sin t} = 0, \text{ then}
\]
\[\sin t + 2y\cos t - 2y = 0\] therefore \[y = \frac{\sin t}{2(1-\cos t)}\] then \[x = \frac{\sin (t+\cos t)}{2(1-\cos t)}\], \[\frac{1}{\sin t} = \frac{1}{2}\], then the fixed point of Ikeda map is \(\left(\frac{1}{2\sin t}, \frac{1}{t}\right)\), \(\forall t \in \mathbb{R}\).

2. By the definition of fixed point, we get:
\[\begin{bmatrix} 1 + ax \cos t - ay \sin t \\ ax \sin t + ay \cos t \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}\] then \(t = t\), so \(a \times \sin t + a \times \cos t = y\) then \(a \times \sin t = y - a \times \cos t\) since \(a \neq 0\), and \(t \neq n\pi\) therefore \(y = \frac{y(1-\cos t)}{a\sin t}\), so
\[1 + \frac{y(1-\cos t)}{a\sin t} - a \times \sin t = \frac{y(1-\cos t)}{a\sin t}\] then \(a \times \sin t + a \times \cos t = a^2 \times \cos^2 t - a^2 \times \sin^2 t - y + a \times \cos t = 0\) since \(t \neq \cos^{-1}\left(\frac{a^2+1}{2a}\right)\) therefore \(y = (a \times \sin t) / (1+a^2-2a \times \cos t)\) and \(x=(1-a \cos t) / (1+a^2-2a \times \cos t)\) then Ikeda map has infinite fixed points.

**Proposition (3.3):** The Jacobian of Ikeda map \(I_{a,b}\) is \(a^2 b\).

**Proof:** The differential matrix of Ikeda is

\[
\begin{bmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial t} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial t} \\
\frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial t}
\end{bmatrix}
\]

\[
DI_{a,b}(v_0) = \begin{bmatrix} a \cos t & -a \sin t & -ax \sin t - ay \cos t \\ a \sin t & a \cos t & ax \cos t - ay \sin t \\
0 & 0 & b
\end{bmatrix}
\]

Then
\[
J = \det DI_{a,b}(v_0) = b \det \begin{bmatrix} a \cos t & -a \sin t \\ a \sin t & a \cos t \\ 0 & 1
\end{bmatrix} = ba^2
\]

In some conditions the Ikeda map is area contracting at each point in \(\mathbb{R}^3\) but in other conditions it is expanding area.

**Proposition (3.4):**

Let \(I_{a,b}\) be the Ikeda map :-

1. If \(|a|<1\) and \(|b|<1\) then \(I_{a,b}\) is area contraction at \(v:\forall v \in \mathbb{R}^3\).

2. If \(|b|>1, b \neq 0\) and \(|a|^2 > |b|\) then \(I_{a,b}\) is area expanding at \(v:\forall v \in \mathbb{R}^3\).

3. If \(|a|>1, a \neq 0\) and \(|b|^2 > |a|^2\) then \(I_{a,b}\) is area expanding at \(v:\forall v \in \mathbb{R}^3\).

**Proof:**

1. If \(|a|<1\) and \(|b|<1\) then \(|ba^2|<1\) so the absolute Jacobian of Ikeda map is least than 1 so from definition of area contracting.
2. If $b \neq 0$ since $|J| = |\text{det}(D I_{a,b})| = |a^2b| = |a|^2|b|$ by hypothesis $|b| > 1$ and so $J \frac{1}{|b|}, |b| > 1$. Then $I_{a,b}$ is area expending at $v \in \mathbb{R}^3$.

3. Similarity proof (2).

Now, we study the conditions which Ikeda map is onto.

**Proposition (3.5)**

If $a \neq 0$ and $b \neq 0$ then $I_{a,b}$ is onto.

**Proof**:

**Case (1)**: If $t \neq (\frac{1}{2} + n)\pi$, Let

\[
\begin{pmatrix}
v \\
w \\
s
\end{pmatrix}
\]

any element in $\mathbb{R}^3$ such that

\[
\begin{pmatrix}
v \\
w \\
s
\end{pmatrix} = \begin{pmatrix}
1 + ax \cos t - ay \sin t \\
ax \sin t + ay \cos t \\
b t
\end{pmatrix}.
\]

Then $v = 1 + ax \cos t - ay \sin t$, since $a \neq 0$ and $t \neq (\frac{1}{2} + n)\pi$ then $x = \frac{v - 1 + ax \sin t}{acost}$ so

$w = ax \sin t + ay \cos t$ then $w = a \left(\frac{v-1+ax \sin t}{acost}\right) \sin(t) + a \cos(t)$ therefore $w = v \tan(t) - \tan(t) + (a \sin(t) \tan(t) + a \cos(t) \tan(t) + (a \cos(t) \tan(t))$ then $w - v \tan(t) + \tan(t) = (a \tan(t) \sin(t) + a \cos(t) \tan(t))$ therefore $Y = \frac{w - v \tan(t) + \tan(t)}{(a \tan(t) \sin(t) + a \cos(t))}$ then $b t = s$ since $b \neq 0$ therefore $t = \frac{s}{b}$ then there exist

\[
\begin{pmatrix}
x \\
y \\
s
\end{pmatrix}
\]

such that $I_{a,b}$ is onto.

**Case (2)**: If $t = (n+\frac{1}{2})\pi$ Let

\[
\begin{pmatrix}
v \\
w \\
s
\end{pmatrix}
\]

any element in $\mathbb{R}^3$ such that

\[
\begin{pmatrix}
v \\
w \\
s
\end{pmatrix} = \begin{pmatrix}
1 - ay \\
ax \\
b t
\end{pmatrix},
\]

$s = bt$ then $t = \frac{s}{b}$ so

$w = x \text{a since } a \neq 0 \text{ then } x = \frac{w}{a}$ so $v = 1 + a \sin t$ then $y = \frac{1-a \sin t}{a}$ such that there exist

\[
\begin{pmatrix}
w \\
1-a \sin t \\
1-a \sin t
\end{pmatrix}
\]

$\in \mathbb{R}^3$ then $I_{a,b}$ is onto.
Remark:-
1. If \( a=0, b\neq 0 \) and \( t\neq (\frac{1}{2} + n)\pi \) then \( I_{a,b} \) is not onto.
2. If \( a\neq 0, b=0 \) and \( t\neq (\frac{1}{2} + n)\pi \) then \( I_{a,b} \) is not onto.
3. If \( a=0, b=0 \) and \( t\neq (\frac{1}{2} + n)\pi \) then \( I_{a,b} \) is not onto.

Remark:- If \( a=0 \) then \( I_{a,b} \) is not one to one, so it is not diffeomorphism.

Now, we study the conditions which Ikeda map is one to one.

**Proposition (3.6):**- If \( a\neq 0 \) and \( b\neq 0 \) then Ikeda map is one to one.

**Proof:**- If \( a\neq 0 \) and \( b\neq 0 \) then

\[
0 = at - ay \sin t, \quad 0 = at + ay \cos t, \quad b = 0, \quad t = 0 \quad \text{so} \quad y = 0 \quad \text{since} \quad a \neq 0 \quad \text{therefore} \quad x = \frac{-1}{a}
\]

Then the kernel of \( I_{a,b} \) is unique set.

**Proposition (3.7):**- If \( a\neq 0 \) and \( b\neq 0 \) Ikeda map is diffeomorphism

**Proof:**- By proposition (3.6) and (3.7) Ikeda map is onto and one to one.

Let \( I_{a,b} \) be Ikeda map and \( a\neq 0, b\neq 0 \) then

**Proposition (3.8):**- If \( a\neq 1 \) and \( b\neq 1 \) then The eigenvalues of Ikeda map are \( \lambda_1, \lambda_2 = a, \lambda_3 = b \) at the fixed point.

**Proof:**- By proposition (3.1) Ikeda map has unique fixed point \( v_0 = \begin{bmatrix} 1 \\ 1-a \\ 0 \\ 0 \end{bmatrix} \)

\[
\det (a - \lambda I)v_0 = \det \begin{bmatrix} a - \lambda & 0 & 0 \\ 0 & a - \lambda & 0 \\ 0 & 0 & b - \lambda \end{bmatrix} = (a - \lambda)(a - \lambda)(b - \lambda) = 0
\]

Then \( \lambda_1, \lambda_2, \lambda_3 = a, b. \)

**Proposition (3.9):**- Let \( I_{a,b} \) be Ikeda map and \( a\neq 0, b\neq 0 \) then

1. If \( |a|<1 \) and \( |b|<1 \) then the fixed point of Ikeda map is attracting fixed point.
2. If \( |a|>1 \) and \( |b|>1 \) then the fixed point of Ikeda map is repelling fixed point.
3. If \( |a|>1 \) and \( |b|<1 \) then the fixed point of Ikeda map is saddle fixed point.
4. If \( |a|<1 \) and \( |b|>1 \) then the fixed point of Ikeda map is saddle fixed point.

**Proof:**- By proposition (3.4) and definition it's satisfying (1, 2, 3, 4)
Proposition (3.10): 1. \( \forall \begin{bmatrix} x \\ y \\ t \end{bmatrix} \in \mathbb{R}^3, \mathbf{D}_{a,b} \begin{bmatrix} x \\ y \\ t \end{bmatrix} \) is a hyperbolic matrix if and only if \( |ba^2| = 1 \).

Proof: \( \implies \) Let \( \mathbf{D}_{a,b} \begin{bmatrix} x \\ y \\ t \end{bmatrix} \) be a hyperbolic matrix then in view of definition \( \mathbf{D}_{a,b} \begin{bmatrix} x \\ y \\ t \end{bmatrix} \in \text{GL}(3, \mathbb{R}) \) then

\[
\det \left( \mathbf{D}_{a,b} \begin{bmatrix} x \\ y \\ t \end{bmatrix} \right) = ba^2 = \mp 1 \text{, hence } |ba^2| = 1.
\]

\( \Leftarrow \) Let \( |ba^2| = 1 \) then \( \det \left( \mathbf{D}_{a,b} \begin{bmatrix} x \\ y \\ t \end{bmatrix} \right) = ba^2 = \mp 1 \), \( \mathbf{D}_{a,b} \begin{bmatrix} x \\ y \\ t \end{bmatrix} \in \text{GL}(3, \mathbb{R}) \) and by the relation between roots and coefficients, \( |ba^2| = |\pm 1| = 1 \) so if \( |b| \neq 1 \) and \( |a| \neq 1 \) then \( |a| = \frac{1}{|b|} \) such that \( |b| 
eq 0 \) and by proposition(3.4) either \( \lambda_1, \lambda_2 \) are real numbers and since: \( |\lambda_1, \lambda_2| = \begin{bmatrix} a \end{bmatrix} \neq 1 \) and \( |\lambda| = |b| \neq 1 \) and since \( R \) is totally ordered set either \( |a| > 1 \) or \( |b| < 1 \). If \( |a| > 1 \) then \( |b| = \frac{1}{|a|} < 1 \) and if \( |b| < 1 \) then

\[
|b|^2 = \frac{1}{|b|} > 1. \text{ } \Diamond
\]

Proposition (3.11): If \( a \neq 1 \) and \( b \neq 1 \) the eventually fixed points set of Ikeda map is the set of fixed points.

Proof: By proposition (3.1), we have only one fixed points and clearly they is eventually fixed point \( \begin{bmatrix} 1 \\ 1-a \\ 0 \end{bmatrix} \).

Suppose that there exists an eventually fixed point \( \begin{bmatrix} x \\ y \\ t \end{bmatrix} \) for \( \mathbf{I}_{a,b} \) and \( y \neq 0 \) then by definition of eventually fixed point. Then exists a positive integer number \( n \) such that \( \mathbf{I}_{a,b}^n \begin{bmatrix} x \\ y \\ t \end{bmatrix} \) is a fixed point of \( \mathbf{I}_{a,b} \) so then
is \( n \in \mathbb{Z}^+ \) such that \( I^n_{a,b} \begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \). We get if \( n = 1 \), then 

\[
I_{a,b} \begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} 1 + ax \cos t - ay \sin t \\ ax \sin t + ay \cos t \\ bt \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

So \( bt = 0 \) then \( t = 0 \), we get \( ax \sin(0) + ay \cos(0) = 0 \) this imply \( y = 0 \), hence \( 1 + ax \cos(0) = \frac{1}{1-a} \) then \( x = \frac{1}{1-a} \) we generalize this relation by induction 

\[
I^n_{a,b} \begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} \frac{1}{1-a} \\ 0 \\ 0 \end{bmatrix}
\]

Proposition (3.12): If \( a \neq 1 \) and \( b \neq 1 \) then The Ikeda map has periodic points of period two

Proof: Suppose that there exists a periodic point of period 2 for \( I_{a,b} \) then for different we have

\[
I^2_{a,b} \begin{bmatrix} x \\ y \\ t \end{bmatrix} = \begin{bmatrix} x \\ y \\ t \end{bmatrix} \]

so

\[
\begin{bmatrix} 1 + a(1 + ax \cos(0) - ay \sin(0)) \cos(0) - a(ax \sin(0) + ay \cos(0)) \sin(0) + a(1 + ax \cos(0) - ay \sin(0)) \sin(0) + a(ax \sin(0) + ay \cos(0)) \cos(0) \\ a(1 + ax \cos(0) - ay \sin(0)) \sin(0) + a(ax \sin(0) + ay \cos(0)) \cos(0) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}
\]

since \( b \neq 1 \), \( bt = t \) then \( t = 0 \) so

\[
a(1 + ax \cos(0) - ay \sin(0)) + a(ax \sin(0) + ay \cos(0)) = y \]

therefore \( y = 0 \) so

\[
1 + a + ax = x \]

\[
1 + a = x - ax \]

since \( a \neq 1 \)

therefore \( x = \frac{1 + a}{1 - a} \) then \( I^2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1 + a}{1 - a} \\ \frac{1 - a}{1 - a} \end{bmatrix} \)

Hence \( \begin{bmatrix} \frac{1 + a}{1 - a} \\ \frac{1 - a}{1 - a} \end{bmatrix} \) is the periodic points of periodic 2

Proposition (3.13): The Ikeda map has periodic points of period 3

Proof: Suppose that there exists a periodic point of three for \( I_{a,b} \) then we have

\[
\begin{bmatrix} 1 + a + a(1 + ax \cos(0) - ay \sin(0)) \cos(0) - a(ax \sin(0) + ay \cos(0)) \sin(0) + a(1 + ax \cos(0) - ay \sin(0)) \sin(0) + a(ax \sin(0) + ay \cos(0)) \cos(0) \\ a(1 + ax \cos(0) - ay \sin(0)) \sin(0) + a(ax \sin(0) + ay \cos(0)) \cos(0) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}
\]

since \( b \neq 1 \), \( bt = t \) then \( t = 0 \) so

\[
a(1 + ax \cos(0) - ay \sin(0)) + a(ax \sin(0) + ay \cos(0)) = y \]

therefore \( y = 0 \) so

\[
1 + a + ax = x \]

\[
1 + a = x - ax \]

since \( a \neq 1 \)

therefore \( x = \frac{1 + a + ax}{1 - a^2} \) then
4. Sensitivity Dependence on Initial Condition of Ikeda Map:

The function $f : X → X$ is said to be sensitive dependence on initial conditions if there exists $ε > 0$ such that for any $x_0 ∈ X$ and any open set $U ⊆ X$ containing $x_0$, there exists $y_0 ∈ U$ and $n ∈ \mathbb{Z}^+$ such that $d(f^n(x_0), f^n(y_0)) > ε$. That is, $\exists ε > 0, \forall x, \exists δ > 0, \exists y \in B(δ(x)) \ni d(f^n(x_0), f^n(y_0)) ≥ ε$. A dynamical system has sensitive dependence on initial conditions on subset $x ⊆ X$ if there is $ε > 0$ such that for every $x ∈ x$ and $δ > 0$ there exists $y ∈ Y$ and $n ∈ \mathbb{N}$ for which $d(x, y) < δ$ and $d(f^n(x), f^n(y)) > ε$. Although there is no universal agreement on definition of chaos, it is generally agreed that a chaotic dynamical system should exhibit sensitive dependence on initial conditions chaotic.

Al-Shara’a and Al-Yaseen [1] defined an order on $\mathbb{R}^n$ as: let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) ∈ \mathbb{R}^n$ we write $x < y$ if and only if $x_i < y_i$, $∀ i = 1, \ldots, n$.

Proposition (3.14): If $|a| > 1$ or $|b| > 1$ then $I_{a,b}$ is sensitive dependence on initial conditions

**Proof:** Let $x = \begin{bmatrix} x \\ y \\ t \end{bmatrix}$ be a point in $\mathbb{R}^3$, since the sine and cose maps are bounded then $I_{a,b}(X) = \begin{bmatrix} 1 + ax - ay \\ ax + ay \\ bt \end{bmatrix}$.

Case (1): If $\left| x \right| ≤ 1$ then by hypothesis and by definition of the order $a \ b$.

$I_{a,b}(X) < \begin{bmatrix} 1 - ay \\ ay \\ bt \end{bmatrix}$ and $I_{a,b}^2(X) < \begin{bmatrix} 1 - a^2y \\ a^2y \\ bt \end{bmatrix}$, that is, $I_{a,b}^n(X) < \begin{bmatrix} 1 - a^ny \\ a^ny \\ bt \end{bmatrix}$.

Thus if $|a| > 1$ or $|b| > 1$ then $n → ∞$ then $I_{a,b}(x) → ∞$, let $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} ∈ \mathbb{R}^3$ such that $d(x, y, t) < δ$.

$$d(I_{a,b}(x), I_{a,b}(y), I_{a,b}(t)) = \sqrt{(1-ay)^2 + (ay)^2 + (bt)^2},$$

$$d(I_{a,b}^2(x), I_{a,b}^2(y), I_{a,b}^2(t)) = \sqrt{(1-a^2y)^2 + (a^2y)^2 + (bt)^2} = \sqrt{b^2t^2 + 1 - 2a^2y + a^4y + a^4y},$$

$$\sqrt{b^2t^2 + 1 - 2a^2y - 2a^4y}.$$

And by induction $d(I_{a,b}^n(x), I_{a,b}^n(y), I_{a,b}^n(t)) = \sqrt{(1-ay)^{2n} + (ay)^{2n} + (bt)^n}$.

If $|a^2b| > 1$ and $n → ∞$, $d(I_{a,b}^n(x), I_{a,b}^n(y), I_{a,b}^n(t))$ hence $I_{a,b}$ has sensitive dependence an initial condition.
Case (2):- If $|x|>1$ from of the Ikeda map, then the iterates of the Ikeda map are divergent. Thus it has sensitive dependence on initial condition. 

Then we study the sensitivity to initial condition of map by varying the point $(x_i, y_i, z_i)$ as follows (i.e. 1, 2) control parameters $(a, b)$ by using (Matlab) to analysis of view for sensitivity dependent on initial condition. This work shows as in figure (1) Now consider the map we get sensitivity to initial condition on the initial

5. The Lyapunov Exponents of Ikeda Map

Let $f:X \to X$ be continuous differential map where $X$ be any space then all $x$ in $X$ in direction $V$, the
Lyapunov exponent was defined of a map \( f \) at \( x \) by 
\[
L^f(x,v) = \lim_{n \to \infty} \frac{1}{n} \ln ||Df^v||
\]
whenever the limit exists. In higher dimensions, for example in \( \mathbb{R}^n \) the map \( f \) will have \( n \) Lyapunov exponents, say \( L^f_1(X,V_1), L^f_2(X,V_2), \ldots, L^f_n(X,V_n) \), for a minimum \( n \) Lyapunov exponent that is 
\[
L^f(X,V) = \max\{L^f_1(X,V_1), L^f_2(X,V_2), \ldots, L^f_n(X,V_n)\}
\]
where \( v = (v_1, v_2, \ldots, v_n) \). The usual test for chaos is calculation of the largest Lyapunov exponent \([4]\). A positive largest Lyapunov exponent indicates chaos. When one has access to the equations generating the chaos, and which measure the rates of separation from the current orbit point along \( m \) orthogonal directions. The Lyapunov exponent \( \lambda \) is greater then zero. A quantitative measure of the sensitive dependence on the initial conditions is the Lyapunov exponent \( \lambda \), it’s the average rate of divergence (or convergence) of two neighboring trajectories in the phase space. We recall and discuss the Lyapunov exponent first since the Lyapunov exponent are orthogonal quadratrive. Representing the average divergence of nearby trajectories in phase space, then is a Lyapunov exponent in the direction of each of the axis (since we can choose our second trajectory to start from a point which denotes some perturbation a long each of the different dimensions of the phase space). So the Lyapunov exponents are related to the expanding or contracting of the flow of the system in different direction since the positive (orientation) of the ellipsoid changes as it develops. The directions associated with each exponent vary throughout the attractor a system is referred to as chaotic.

**Proposition (3.15):** Let \( I_{a,b} \colon \mathbb{R}^3 \to \mathbb{R}^3 \) be the Ikeda map either \( |a|>1 \) or \( |b|>1 \) then the map has positive Lyapunov exponents.

**Proof:** if \( |a|>1 \) and by proposition \( |\lambda_{1,2}|=|a| \). \( L_{1,2} \left( \begin{pmatrix} x \\ y \\ t \end{pmatrix} , v_1 \right) = \lim_{n \to \infty} \frac{1}{n} \ln ||D_{a,b} \left( \begin{pmatrix} x \\ y \\ t \end{pmatrix} , V_1 \right)|| > \ln |a| > 0 \) and since \( |\lambda_3|=|b| \) then either \( L_3 \left( \begin{pmatrix} x \\ y \\ t \end{pmatrix} , v_3 \right) = \lim_{n \to \infty} \frac{1}{n} \ln ||D_{a,b} \left( \begin{pmatrix} x \\ y \\ t \end{pmatrix} , V_3 \right)|| > |\ln |b| | < 0 \) or \( L_3 \left( \begin{pmatrix} x \\ y \\ t \end{pmatrix} , v_3 \right) > 0 \) by definition 
\[
L \left( \begin{pmatrix} x \\ y \\ t \end{pmatrix} , v \right) = \max\{L_1^+ (x_1,v_1),L_2^+ (x_2,v_2),L_3^+ (x_3,v_3)\}
\]
then \( L \left( \begin{pmatrix} x \\ y \\ t \end{pmatrix} , v_1 \right) > 0 \). In the way, we can prove if \( |b|<1 \) then Lyapunov exponent of Ikeda map is positive. 

**References**


