Semi Compatibility and Fixed Points For Expansion Mappings In 2-Metric Space

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Abstract
This paper introduces the notion of semi-compatible self- maps in 2- metric spaces and establishes a fixed point theorem for six self maps, satisfying an implicit relation through semi-compatibility of a pair of self-maps.
Key words: semi-compatibility, metric space, 2- metric space,

1. Introduction

The concept of 2- metric space was initially given by Gahler [3] whose abstract properties were suggested by the area of functions in Euclidean space. Iseki [4] set out the tradition of proving fixed point theorem in 2- metric space employing various contractive conditions. Later on, Naidu and Prasad [7] introduced the concept of compatible maps in 2- metric spaces. In [2] Cho, Sharma and Sahu introduced the concept of semi-compatibility maps in d- topological spaces.

They defined a pair of self-maps (S, T) to be semi-compatible if the condition (i) $Sy = Ty \Rightarrow STy = TSy$ (ii) $\{Sx_n\} \to x$, $\{Tx_n\} \to x \Rightarrow STx_n \to Tx$, as $n \to \infty$, holds. However, (ii) implies (i), taking $x_n = y$ and $x = Ty = Sy$. So in a 2- metric space, we define semi-compatibility by the condition (ii) only.

2. Preliminaries

Definition 2.1. Let X be a non-empty set with real-valued function d on X x X x X satisfying the following :

(1) $d(x,y,z) = 0$ if at least two of x, y, z are equal,

(2) $d(x, y, z) = d(p(x, y, z))$ for all $x, y, z \in X$ and each permutation $p(x, y, z)$ of x, y, z.

(3) $d(x, y, z) \leq d(x, y, w) + d(x, w, z) + d(w, y, z)$ for all $x, y, z, w \in X$.

The function 'd' is called a 2- metric on X and the pair (X, d) is called a 2-metric space.

Definition 2.2. A sequence $\{x_n\}$ is said to be 2-convergent to a point $x \in X$ if $\lim_{n\to\infty} d(x_n, x, a) = 0$, and is said to be 2- Cauchy sequence if $\lim_{n,m\to\infty} d(x_n, x_m, a) = 0$ for all 'a' $\in X$. The 2- metric space (X, d) is called complete if every Cauchy sequence in X converges in X converges to a point of X.

Definition 2.3. For a pair of self-maps (S, T) on a 2-metric space (X, d):

(1) (S, T) is said to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n, a) = 0$ for all 'a' $\in X$, whenever the sequence $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x$.

(2) (S, T) is said to be semi-compatible if $\lim_{n\to\infty} d(STx_n, Tx, a) = 0$ for all 'a' $\in X$, whenever the sequence $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = x$.

(3) (S, T) is said to be weak-compatible or coincidently commuting if $Sy = Ty$ for some $y \in X$ then $TSy = STy$.

Proposition 2.1. If S and T are semi-compatible self-maps on a 2-metric space (X, d) then the pair (S, T) is weak- compatible.

Proposition 2.2. If S and T are compatible self-maps on a 2-metric space (X, d) and T is continuous then the pair (S, T) is semi-compatible.

However, weak-compatibility does not imply semi-compatibility. It is clear from following Example 2.1 that the pair of self-maps (A, B) is weak compatible but is not semi-compatible.

Here we give an example of pair of self-maps (I, A) on a 2-metric space, which is compatible but not semi-compatible. Further we see that the semi- compatibility of a pair (A, I) need not imply the semi-compatibility of (I, A).

ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.4, No.4, 2014

Example 2.1. Let $X = \{0, 1/2, 1/4, 1/8, 1/16, \ldots, 1/2^n, \ldots\}$. Define $d : X \times X \times X \to (0, \infty)$ by

$$d(x, y, z) = \begin{cases} 1, & \text{if } x, y, z \text{ are distinct and} \\ 0, & \text{otherwise} \end{cases}$$

Then (X, d) is a 2-metric space. Let 'I' be the identity on X and define a self map 'A' as follows:
$A(1/2^n) = 1/2^{n+3}$, $A(0) = 1/2$ and $x_n = 1/2^n$. Then

$$\lim_{n\to\infty} d(Ix_n, 0, a) = \lim_{n\to\infty} d(x_n, 0, a) = 0,$$

$$\lim_{n\to\infty} d(Ax_n, 0, a) = \lim_{n\to\infty} d(1/2^{n+3}, 0, a) = 0$$

for all 'a' $\in$ X. Thus $\{x_n\}$ and $\{Ax_n\}$ converges to $x = 0$. Now, the pair (I, A) is commuting. Hence it is compatible. But $\{IAx_n\} = \{Ax_n\} \to 0 \neq A(0)$ as $\{Ax_n\} \to 0$, and we get that (I, A) is not semi-compatible.
Also, for any sequence $\{x_n\} \to x$

$$\lim_{n\to\infty} d(AIx_n, Ix, a) = \lim_{n\to\infty} d(Ax_n, x, a) = 0.$$

Thus (A, I) is semi-compatible.

The above example gives an important aspect of semi-compatibility since the pair (I, A) is commuting, hence it is weakly commuting, compatible and weak compatible but it is not semi-compatible.

Note : A pair of self maps (S, T) which is semi-compatible need not to be compatible. Also, semi-compatibility of the pair (S, T) need not imply the semi-compatibility of (T, S).

Definition 2.4.([8]) Let F_4 be the class of upper semi – continuous functions on the right from $(\Re^+)^4 \to \Re$ such that for some $h \in (0,1)$

(1) $F(u, v, u, v) \geq 0$ implies $v \leq hu$.

(2) $F(u, v, v, u) \geq 0$ implies $v \leq hu$.

(3) $F(u, u, 0, 0,) \geq 0$ implies $u = 0$.

REMARK 2.1. It follows from the first two conditions of F_4 that for $F \in F_4$,

(1) $F(0, L, 0, L) \geq 0$ implies $L = 0$.

(2) $F(0, L, L, 0) \geq 0$ implies $L = 0$.

S.L. Singh [9] proved the following.

Lemma2.3. ([9]) Let $\{x_n\}$ be a sequence in a complete 2-metric space X. If there exists a $h \in (0, 1)$ such that

$$d(x_n, x_{n+1}, a) \leq h\, d(x_{n-1}, x_n, a)$$

for all 'a' $\in$ X and all 'n', then $\{x_n\}$ converges to a point in X.

Lemma 2.4. Let A, B, C, D, S and T are six self maps on a complete metric space (X, d) such that

(1) $S(X) \subseteq CD(X)$ & $T(X) \subseteq AB(X)$

(2) For some $F \in F_4$

$$F[d(Sx, Ty, z), d(ABx, CDy, z), d(ABx, Sx, z), d(CDy, Ty, z)] \geq 0,$$

for all $x, y, z \in X$.

Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.4, No.4, 2014

Then the sequence $\{y_n\}$ converges to a point in X , where the sequences $\{x_n\}$ and $\{y_n\}$ are defined by $Sx_{2n} = CDx_{2n+1} = y_{2n+1}$ and $Tx_{2n+1} = AB\, x_{2n+2} = y_{2n+2}$ for n = 0, 1, 2,…

Proof. Let $x_0 \in X$, from condition (1), there exist $x_1, x_2 \in X$ such that

$Sx_0 = CDx_1 = y_0$ and $Tx_1 = ABx_2 = y_1$.

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$Sx_{2n} = CDx_{2n+1} = y_{2n+1}$ and $Tx_{2n+1} = ABx_{2n+2} = y_{2n+2}$ for n = 0, 1, 2…

Put $x = x_{2n}$ and $y = x_{2n+1}$ in (3)

$$F[d(Sx_{2n}, Tx_{2n+1}, z), d(ABx_{2n}, CDx_{2n+1}, z), d(ABx_{2n}, Sx_{2n}, z), d(CDTx_{2n+1}, Tx_{2n+1}, z)] \geq 0,$$

for all x, y, z $\in$ X

$$F[d(y_{2n+1}, y_{2n+2}, z), d(y_{2n+1}, y_{2n}, z), d(y_{2n+1}, y_{2n}, z), d(y_{2n+2}, y_{2n+1}, z)] \geq 0,$$

F[U, V, U, V] $\geq$ 0 implies that V $\leq$ h U, where $U = d(y_{2n}, y_{2n+1}, z)$, and

$V = d(y_{2n+1}, y_{2n+2}, z)$ and $h \in (0,1)$. So

$d(y_{2n+1}, y_{2n+2}, z) \leq h\, d(y_{2n}, y_{2n+1}, z)$, for $h \in (0,1)$.

Similarly, if we take $x = x_{2n}$ and $y = x_{2n-1}$ in (3), then we get

$$F[d(Sx_{2n},Tx_{2n-1}, z), d(ABx_{2n}, CDx_{2n-1}, z), d(ABx_{2n}, Sx_{2n}, z), d(CDx_{2n-1}, Tx_{2n-1}, z)] \geq 0,$$

implies that

$$F[d(y_{2n+1}, y_{2n}, z), d(y_{2n-1}, y_{2n}, z), d(y_{2n+1}, y_{2n}, z), d(y_{2n}, y_{2n-1}, z)] \geq 0,$$

for all z $\in$ X. That is, F[U, V, V, U] $\geq$ 0 implies that V $\leq$ h U, where $U = d(y_{2n}, y_{2n-1}, z)$, and $V = d(y_{2n+1}, y_{2n}, z)$ and $h \in (0,1)$. So

$d(y_{2n},y_{2n-1},z) \leq h\, d(y_{2n-1},y_{2n-2},z)$, for $h \in (0,1)$.

Therefore, for all 'n' even or odd $d(y_n, y_{n+1}, z) \leq h\, d(y_{n-1}, y_n, z)$, for $h \in (0,1)$. By Lemma 2.3, $\{y_n\}$ converges to some $u \in X$..

It has been shown in [3] that, although 'd' is a continuous function of any of its three augments, it need not to be continuous in two augments. If it is continuous in two augments then it is continuous in all three augments. For brevity,'d' which is continuous in all of its augments, will be called continuous.

From now on, the 2-metric'd' is assumed to be continuous.

3. Main results

Theorem 3.1. Let A, B, C, D, S and T are six self maps on a complete 2-metric space (X, d) satisfying

(i) $S(X) \subseteq CD(X)$ and $T(X) \subseteq AB(X)$

(ii) For some $F \in F_4$

$$F[d(Sx,Ty, z), d(ABx, CDy, z), d(ABx, Sx, z), d(CDy, Ty, z)] \geq 0,$$

for all x, y, z $\in$ X.

(iii) AB = BA , CD = DC, SB = BS, TD = DT

(iv) either AB or S is continuous.

(v) (S, AB) is semi-compatible and (T, CR) is weak-compatible.

Then the six self maps A, B, C, D, S and T have a unique common fixed point in X .

Proof. Let $x_0 \in X$, from condition (i) there exist $x_1, x_2 \in X$ such that $Sx_0 = CDx_1 = y_0$, $Tx_1 = ABx_2 = y_1$.

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$Sx_{2n} = CDx_{2n+1} = y_{2n} \text{ and } Tx_{2n+1} = ABx_{2n+2} = y_{2n+1} \text{ for } n = 0, 1, 2, \ldots$$

By Lemma 2.4, $\{y_n\} \to u \in X$. Also its subsequences converges as follows,

i.e. $\{Tx_{2n+1}\} \to u$ and $\{CDx_{2n+1}\} \to u$,

$\{Sx_{2n}\} \to u$ and $\{ABx_{2n}\} \to u$.

Case I:AB is continuous.

As AB is continuous, $(AB)^2 x_{2n} \to ABu$ and $(AB)Sx_{2n} \to ABu$

As (S, AB) is compatible, we have $S(AB)x_{2n} \to ABu$.

Step1: Putting $x = ABx_{2n}$, $y = x_{2n+1}$ in (ii), we get

$$F[d(SABx_{2n}, Tx_{2n+1}, z), d(CDx_{2n+1}, ABABx_{2n}, z), d(SABx_{2n}, ABABx_{2n}, z), d(Tx_{2n+1}, CDx_{2n+1}, z)] \geq 0$$

letting $n \to \infty$, we get

$$F[d(ABu, u, z), d(u, ABu, z), d(ABu, ABu, z), d(u, u, z)] \geq 0$$

implies that

$$F[d(ABu, u, z), d(ABu, u, z), 0,0] \geq 0$$

which gives, $d(ABu, u, z) = 0$. Hence $ABu = u$.

Step2: Putting $x = u$ and $y = x_{2n+1}$ in (ii), we get

$$F[d(Su, Tx_{2n+1}, z), d(CRx_{2n+1}, ABu, z), d(Su, ABu, z), d(Tx_{2n+1}, CRx_{2n+1}, z)] \geq 0$$

letting $n \to \infty$,

$$F[d(Su, u, z), d(u, u, z), d(Su, u, z), d(u, u, z)] \geq 0$$

implies that

$$F[d(Su, u, z), d(Su, u, z), 0, 0] \geq 0$$

which gives , $d(Su, u, z) = 0$. Hence $Su = u$.

Step3: Putting $x = Bu$, $y = x_{2n+1}$ in condition (ii), we get

$$F[d(SBu, Tx_{2n+1}, z), d(CDx_{2n+1}, ABBu, z), d(SBu, ABBu, z),$$
$$d(Tx_{2n+1}, CDx_{2n+1}, z)] \geq 0,$$

as AB = BA, SB = BS, so we have

$S(Bu) = B(Su) = Bu$ and $(AB)(Bu) = B(ABu) = Bu$

letting $n \to \infty$, we get

$$F[d(Bu, u, z), d(u, Bu, z), d(Bu, Bu, z), d(u, u, z)] \geq 0,$$

implies that

$$F[d(Bu, u, z), d(Bu, u, z), 0, 0] \geq 0,$$

that is, $d(Bu, u, z) = 0$ implies that $Bu = u$. Therefore $Au = Bu = Su = u$

As $S(X) \subseteq CD(X)$, therefore there exist $v \in X$ such that $u = Su = CD = v$.

Step 5. Putting $x = x_{2n}$, $y = v$ in condition (ii) , we get

$F[d(Sx_{2n},Tv, z), d(CDv, ABx_{2n}, z), d(Sx_{2n},ABx_{2n}, z),$
$d(Tv,CRv ,z)] \geq 0$

letting $n \to \infty$, we get

$F[d(u,Tv, z), d(u, u, z), d(u, u, z), d(Tv, u, z)] \geq 0$

implies that

$F[d(u, Tv, z), d(u, Tv, z), 0, 0] \geq 0$

that is d(u, Tv, z) = 0 implies that u = Tv. Hence CDv = u = Tv

as (T, CD) is weak- compatible , we have

CDTv = TCDv, thus CDu = Tu .

Step 5. Putting $x = x_{2n,}$ $y = u$ in condition (ii) we get

$F[d(Sx_{2n}, Tu, z), d(CDu, ABx_{2n}, z), d(Sx_{2n}, ABx_{2n}, z),$
$d(Tu, CDu, z)] \geq 0$

letting $n \to \infty$, we get

$F[d(u, Tu, z), d(Tu, u, z), d(u, u, z), d(Tu, Tu, z)] \geq 0$

implies that

$F[d(u, Tu, z), d(Tu, u, z), 0, 0] \geq 0$

which gives, d(u, Tu, z) = 0, implies that u = Tu

Step 6. Putting $x = x_{2n,}$ $y = Du$

$F[d(Sx_{2n},TDu, z), d(CDDu, ABx_{2n}, z), d(Sx_{2n}, ABx_{2n}, z),$
$d(TDu, CDDu, z)] \geq 0$

as TD = DT and CD = DC , therefore TDu = DTu = Du

and CD(Du) = D(CDu) = Du. Letting $n \to \infty$, we get

$F[d(u, Du, z), d(Du, u, z), d(u, u, z), d(Du, Du, z)] \geq 0$

implies that

$F[d(u, Du, z), d(Du, u, z), 0, 0] \geq 0$, that is

d(u, Du, z) = 0 , implies that u = Du, and CDu = Du = u implies that Cu = u .

Hence Cu = Du = Tu = u.

Combining (3) and (4), we get

Au = Bu = Su = Tu = Du = Cu = u.

Hence the six self maps have a common fixed point in this case.

Case II: S is continuous.

As S is continuous, $S^2x_{2n} \to Su$ and $S(ABx^{2n}) \to Su$, and

as (S, AB) is semi compatible, $(AB)Sx_{2n} \to Su$.

Step 7. Putting $x = Sx_{2n}$, $y = x_{2n+1}$ in condition (ii), we have

$F[d(SSx_{2n},Tx_{2n+1},z), d(CD_{2n+1},ABSx_{2n},z), d(SSx_{2n},ABSx_{2n}, z),$
$d(Tx_{2n+1},CDx_{2n+1} ,z)] \geq 0$

letting $n \to \infty$, we get

$F[d(Su, u, z), d(u, Su, z), d(Su, Su, z), d(u, u, z)] \geq 0$

implies that

$F[d(Su, u, z), d(Su, u, z), 0, 0] \geq 0$

which gives, $d(u, Su, z) = 0$ implies that $u = Su$

using steps 4-6 gives us $Tu = CDu = Cu = Du = u$.

Step 8. As $T(X) \subseteq AB(X)$, there exist $w \in X$, such that $u = Tu = ABw$

putting $x = w$, $y = x_{2n+1}$ in condition (ii), we get

$$F[d(Sw, Tx_{2n+1}, z), d(CDx_{2n+1}, ABw, z), d(CDx_{2n+1}, Sw, z), d(ABw, Tx_{2n+1}, z)] \geq 0$$

letting $n \to \infty$, we get

$F[d(Sw, u, z), d(u, u, z), d(u, Sw, z), d(u, u, z)] \geq 0$

implies that

$F[d(Sw, u, z), d(Sw, u, z), 0, 0] \geq 0$

which gives, $Sw = u = ABw$

as (L, AB) is weakly compatible, therefore $Su = ABu$. Also $Bu = u$, follows from step 3.

Thus $Au = Bu = Su = u$ and we obtain u is the common fixed point of the six maps in this case also.

Step 9 (Uniqueness): Let u_1 be another common fixed point of A, B, C, D, S and T. Then, $Au_1 = Bu_1 = Cu_1 = Du_1= Su_1 = Tu_1= u_1$. Put $x = u$ and $y = u_1$ in (ii), we get

$F[d(u, u_1, z), d(u, u_1, z), d(u, u, z), d(u_1,u_1, z)] \geq 0$,

implies that

$F[d(u, u_1, z), d(u, u_1, z), 0, 0] \geq 0$, for all $z \in X$.

Which gives, $d(u, u_1, z) = 0$ implies that $u = u_1$. Hence u is a unique common fixed point of A, B, C, D, S and T.

Proposition 3.2. Let F be a function from $(\Re^+)^4 \to \Re$ such that $F(t_1, t_2, t_3, t_4) = t_1 - a^2t_2 - b^2t_3 - c^4t_4$, where a, b, $c \in \Re^+$ with $b < 1$, $c < 1$ and $a > 1$. Then $F \in F_4$.

Proof. For $u, v \in \Re^+$, $F(u, v, u, v) \geq 0$ implies that $v \leq h_1u$, where $h_1 = \dfrac{1-b^2}{a^2+c^2} < 1$, as $b^2 < 1$, $1 - b^2 > 0$ and as $a^2 + c^2 > 1$ so $1/ a^2 + c^2 < 1$ .Again, $F(u, v, v, u) \geq 0$ implies that $v \leq h_2u$, where $h_2 = \dfrac{1-c^2}{a^2+b^2} < 1$, as $c^2 < 1$, $1 - c^2 > 0$ and as $a^2 + b^2 > 1$ so , $\dfrac{1}{a^2+b^2} < 1$. $F(u, u, 0, 0) \geq 0$ implies that $(1 - a^2) u \geq 0$, which implies that $(a^2 - 1) u \leq 0$, $(a^2 > 1)$. So $u \leq 0$, which gives $u = 0$. Now, take $h = \max\{h_1, h_2\}$. Hence $F \in F_4$.

Corollary 3.3. Let A, B, C, D, S and T be six self-maps of a complete 2-metric space (X, d) satisfying (i), (iii), (iv) and (v) for some a, b, c, $\in \Re^+$, $a > 1$, $b < 1$, $c < 1$,

$d(ABx, CDy, z) \geq a\, d(Sx, Ty, z) + b\, d(Sx, ABx, z) + c\, d(Ty, CDy, z)$

for all $x, y, z \in X$,

or

(vi) $d(ABx, CDy, Z) \geq (Sx, Ty, z)$ for all $x, y, z \in X$ and some $a > 1$.

Then the six self-maps A, B, C, D, S and T have a unique common fixed point in X.

Theorem 3.4. Let S, T, C and A be four self-maps of a complete 2-metric space (X, d) satisfying following conditions

(i) S is continuous

(ii) $S(X) \subset C(X)$ and $T(X) \subset A(X)$

(iii) for some $F \in F_4$,

$$F[d(Sx, Ty, z), d(Ax, Cy, z), d(Ax, Sx, z), d(Cy, Ty, z)] \geq 0 \text{ for all } x, y, z \in X.$$

(iv) (S, A) is semi-compatible and the pair (C, T) is weak-compatible.

Then S, T, C and A have a unique common fixed point.

Proof. The result follows from Theorem 3.1, by taking

$$CD = C \text{ and } AB = A.$$

Corollary 3.5. Let A, S and T be three self-maps of a complete 2-metric space (X, d) satisfying (i), (ii), (iii) and (iv) for some a, b, c $\in R^+$, a >1, b < 1, c < 1,

$$d(Sx, Ty, z) \geq a\, d(Ax, Ay, z) + b\, d(Ax, Sx, z) + c\, d(Ay, Ty, z)$$

or

(v) $d(Sx, Ty, z) \geq a\, d(Ax, Ay, z)$ for all x, y, z $\in$ X and some a >1

Then A, S and T have a unique common fixed point.

Corollary 3.6. Let A, S and C be three self-maps of a complete 2-metric space (X, d). If S is continuous and

(i) $S(X) \subset A(X) \cap C(X)$

(ii) for some $F \in F_4$,

$$F[d(Sx, Sy, z),d(Ax, Cy, z),d(Ax, Sx, z),d(Cy, Sy, z)] \geq 0$$

for all x, y, z $\in$ X.

(iii) (S, C) is semi-compatible and (A, S) is weak- compatible.

Then C, A and S have a unique common fixed point.

Proof. The result follows from Theorem 3.4, by taking T = S.

Corollary 3.7. Let A and T be two surjective self-maps of a complete 2-metric space (X, d) such that

(i) for some a, b, c $\in R^+$, a >1 and b, c $\in$ [0,1],

$$d(Sx, Ty, z) \geq a\, d(x, y, z) + b\, d(x, Sx, z) + c\, d(y, Ty, z)$$

(ii) $d(Sx, Ty, z) \geq a\, d(x, y, z)$ for all x, y, z $\in$ X and some a > 1.

Then A and T have a unique common fixed point.

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