A Novel Numerical Approach for Odd Higher Order Boundary Value Problems

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Abstract

In this paper, we investigate numerical solutions of odd higher order differential equations, particularly the fifth, seventh and ninth order linear and nonlinear boundary value problems (BVPs) with two point boundary conditions. We exploit Galerkin weighted residual method with Legendre polynomials as basis functions. Special care has been taken to satisfy the corresponding homogeneous form of boundary conditions where the essential types of boundary conditions are given. The method is formulated as a rigorous matrix form. Several numerical examples, of both linear and nonlinear BVPs available in the literature, are presented to illustrate the reliability and efficiency of the proposed method. The present method is quite efficient and yields better results when compared with the existing methods.

Keywords: Galerkin method, fifth, seventh and ninth order linear and nonlinear BVPs, Legendre Polynomials.

1. Introduction

From the literature on the numerical solutions of BVPs, it is observed that the higher order differential equations arise in some branches of applied mathematics, engineering and many other fields of advanced physical sciences. Particularly, the solutions of fifth order BVPs arise in the mathematical modeling of viscoelastic flows in (Davis, Karageorghis & Philips, 1988), the seventh order BVPs arise in modeling induction motors with two rotor circuits in (Richards & Sarma, 1994), and the ninth order boundary value problems are known to arise in hydrodynamic, hydro magnetic stability and applied sciences. The performance of the induction motor behavior is modeled by a fifth order differential equation (Siddiqi, Akram & Iftikhar, 2012). This model is constructed with two stator state variables, two rotor state variables and one shaft speed. Generally, two more variables must be added to account for the effects of a second rotor circuit representing deep bars, a starting cage or rotor distributed parameters (Siddiqi & Iftikhar, 2013) . For neglecting the computational burden of additional state variables when additional rotor circuits are needed, the model is often bounded to the fifth order. The rotor impedance is done under the assumption that the frequency of rotor currents depends on rotor speed. This process is appropriate for the steady state response with sinusoidal voltage, but it does not hold up during the transient conditions, when the rotor frequency is not a single value (Siddiqi, Akram & Iftikhar, 2012). So this behavior is modeled in the seventh order differential equation . Recently, the BVPs of ninth order have been developed due to their mathematical importance and the potential for applications in hydrodynamic, hydro magnetic stability. Agarwal (1986) discussed extensively the existence and uniqueness theorem of solutions of such BVPs in his book without any numerical examples. Caglar & Twizell (1999) used sixth degree B-spline functions for the numerical solution of fifth order BVPs where their approach is divergent and unexpected situation is found near the boundaries of the interval. The spline methods have been discussed for the solution of other higher order BVPs. Siddiqi & Twizell (1997, 1998, 1996) applied twelfth, tenth, eighth and sixth degree splines for solving linear BVPs of orders 12, 10, 8 and 6 successively. Kasi Viswanadham, Murali Krishna & Prabhakara (2010) presented the numerical solution of fifth order BVPs by collocation method with sixth order B-splines. Lamnii, Mraoui, Sbibih & Tijini (2008) derived sextic spline solution of fifth order BVPs. The numerical solution of fifth order BVPs by the decomposition method was developed in (Wazwaz, 2001). Recently, Erturk (2007) has investigated differential transformation method for solving nonlinear fifth order BVPs. Siddiqi, Akram & Iftikhar (2012) presented the solutions of seventh order BVPs by the differential transformation method and variational iteration technique respectively. Very recently the solution of seventh order BVPs was developed in (Siddiqi & Iftikhar, 2013) using variation of parameters method. The modified variational iteration method has been applied for solving tenth and ninth order BVPs in (Mohyud-Din &

Yildirim, 2010). Nadjafi & Zahmatkesh (2010) also investigated the homotopy perturbation method for solving higher order BVPs.

In the present paper, first we shall employ the Galerkin weighted residual method (Reddy, 1991) with Legendre polynomials (Davis &Robinowitz, 2007) as basis functions for the numerical solution of fifth, seventh and ninth order linear BVPs of the following form:

$$c_5\frac{d^5u}{dx^5}+c_4\frac{d^4u}{dx^4}+c_3\frac{d^3u}{dx^3}+c_2\frac{d^2u}{dx^2}+c_1\frac{du}{dx}+c_0u=r,\ a<x<b \tag{1a}$$

$$u(a)=A_0, u(b)=B_0, u'(a)=A_1, u'(b)=B_1, u''(a)=A_2 \tag{1b}$$

$$c_7\frac{d^7u}{dx^7}+c_6\frac{d^6u}{dx^6}+c_5\frac{d^5u}{dx^5}+c_4\frac{d^4u}{dx^4}+c_3\frac{d^3u}{dx^3}+c_2\frac{d^2u}{dx^2}+c_1\frac{du}{dx}+c_0u=r,\ a<x<b \tag{2a}$$

$$u(a)=A_0, u(b)=B_0, u'(a)=A_1, u'(b)=B_1, u''(a)=A_2, u''(b)=B_2, u'''(a)=A_3 \tag{2b}$$

and

$$c_9\frac{d^9u}{dx^9}+c_8\frac{d^8u}{dx^8}+c_7\frac{d^7u}{dx^7}+c_6\frac{d^6u}{dx^6}+c_5\frac{d^5u}{dx^5}+c_4\frac{d^4u}{dx^4}+c_3\frac{d^3u}{dx^3}+c_2\frac{d^2u}{dx^2}+c_1\frac{du}{dx}+c_0u=r,\quad a<x<b \tag{3a}$$

$$u(a)=A_0, u(b)=B_0, u'(a)=A_1, u'(b)=B_1, u''(a)=A_2, u''(b)=B_2,\ u'''(a)=A_3, u'''(b)=B_3,\quad u^{(iv)}(a)=A_4 \tag{3b}$$

where $A_i, i=0,1,2,3,4$ and $B_j, j=0,1,2,3$ are finite real constants and $c_i, i=0,1,\cdots 9$ and r are all continuous functions defined on the interval $[a, b]$.

However, in section 2 of this paper, we give a short description on Legendre polynomials. In section 3, the formulations are presented for solving linear fifth, seventh and ninth order BVPs by the Galerkin weighted residual method. Numerical examples and results for both linear and nonlinear BVPs are considered to verify the proposed formulation and the solutions are compared with the existing methods in the literature in section 4.

2. Legendre Polynomials

The general form of the Legendre polynomials of degree n is defined by

$$P_n(x)=\frac{(-1)^n}{2^n(n!)}\frac{d^n}{dx^n}[(1-x^2)^n],\ n\ge 1$$

Now we modify the above Legendre polynomials as

$$L_n(x)=\left[\frac{1}{n!}\frac{d^n}{dx^n}\left(x^2-x\right)^n-(-1)^n\right](x-1),\quad n\ge 1$$

We write first few modified Legendre polynomials over the interval [0, 1]:

$L_1(x)=2x(x-1)$, $L_2(x)=6x(x-1)^2$, $L_3(x)=2x(x-1)(10x^2-15x+6)$,

$L_4(x)=20x-110x^2+230x^3-210x^4+70x^5$

$L_5(x)=-30x+240x^2-770x^3+1190x^4-882x^5+252x^6$,

$L_6(x)=42x-462x^2+2100x^3-4830x^4+5922x^5-3696x^6+924x^7$

$L_7(x)=-56x+812x^2-4956x^3+15750x^4-28182x^5+28644x^6-15444x^7+3432x^8$

$L_8(x)=72x-1332x^2+10500x^3-43890x^4+106722x^5-156156x^6+135564x^7-64350x^8+12870x^9$

$$L_9(x) = -90x + 2070x^2 - 20460x^3 + 108570x^4 - 342342x^5 + 672672x^6 - 832260x^7 + 630630x^8 - 267410x^9 + 48620x^{10}$$

$$L_{10}(x) = 110x - 3080x^2 + 37290x^3 - 244530x^4 + 966966x^5 - 2438436x^6 + 4015440x^7 - 4302870x^8 + 2892890x^9 - 1108536x^{10} + 184756x^{11}$$

Since the modified Legendre polynomials have special properties at $x = 0$ and $x = 1: L_n(0) = 0$ and $L_n(1) = 0$, $n \geq 1$ respectively, so that they can be used as set of basis function to satisfy the corresponding homogeneous form of the *Dirichlet* boundary conditions to derive the matrix formulation in the Galerkin method to solve a BVP over the interval [0, 1].

3. Matrix Formulation

In this section, we first derive the matrix formulation rigorously for fifth-order linear BVP and then we extend our idea for solving seventh and ninth order linear BVPs. To solve the BVPs (1) by the Galerkin weighted residual method we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} a_i L_{i,n}(x), \; n \geq 1 \tag{4}$$

Here $\theta_0(x)$ is specified by the essential boundary conditions, $L_{i,n}(x)$ are the Legendre polynomials which must satisfy the corresponding homogeneous boundary conditions such that $L_{i,n}(a) = L_{i,n}(b) = 0$ for each $i = 1,2,3,\ldots,n$.

Using eqn. (4) into eqn. (1a), the Galerkin weighted residual equations are

$$\int_a^b \left[c_5 \frac{d^5\tilde{u}}{dx^5} + c_4 \frac{d^4\tilde{u}}{dx^4} + c_3 \frac{d^3\tilde{u}}{dx^3} + c_2 \frac{d^2\tilde{u}}{dx^2} + c_1 \frac{d\tilde{u}}{dx} + c_0\tilde{u} - r \right] L_{j,n}(x)dx = 0 \tag{5}$$

Integrating by parts the terms up to second derivative on the left hand side of (5), we get

$$\int_a^b c_5 \frac{d^5\tilde{u}}{dx^5} L_{j,n}(x)dx = \left[c_5 L_{j,n}(x) \frac{d^4\tilde{u}}{dx^4} \right]_a^b - \int_a^b \frac{d}{dx}\left[c_5 L_{j,n}(x)\right] \frac{d^4\tilde{u}}{dx^4} dx$$

$$= -\left[\frac{d}{dx}\left[c_5 L_{j,n}(x)\right] \frac{d^3\tilde{u}}{dx^3} \right]_a^b + \int_a^b \frac{d^2}{dx^2}\left[c_5 L_{j,n}(x)\right] \frac{d^3\tilde{u}}{dx^3} dx \quad \text{[Since } L_{j,n}(a) = L_{j,n}(b) = 0\text{]}$$

$$= -\left[\frac{d}{dx}\left[c_5 L_{j,n}(x)\right] \frac{d^3\tilde{u}}{dx^3} \right]_a^b + \left[\frac{d^2}{dx^2}\left[c_5 L_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} \right]_a^b - \int_a^b \frac{d^3}{dx^3}\left[c_5 L_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} dx$$

$$= -\left[\frac{d}{dx}\left[c_5 L_{j,n}(x)\right] \frac{d^3\tilde{u}}{dx^3} \right]_a^b + \left[\frac{d^2}{dx^2}\left[c_5 L_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} \right]_a^b - \left[\frac{d^3}{dx^3}\left[c_5 L_{j,n}(x)\right] \frac{d\tilde{u}}{dx} \right] + \int_a^b \frac{d^4}{dx^4}\left[c_5 L_{j,n}(x)\right] \frac{d\tilde{u}}{dx} dx \tag{6}$$

In the same way of equation (6), we have

$$\int_a^b c_4 \frac{d^4\tilde{u}}{dx^4} L_{j,n}(x)dx = -\left[\frac{d}{dx}\left[c_4 L_{j,n}(x)\right] \frac{d^2\tilde{u}}{dx^2} \right]_a^b + \left[\frac{d^2}{dx^2}\left[c_4 L_{j,n}(x)\right] \frac{d\tilde{u}}{dx} \right]_a^b - \int_a^b \frac{d^3}{dx^3}\left[c_4 L_{j,n}(x)\right] \frac{d\tilde{u}}{dx} dx \tag{7}$$

$$\int_a^b c_3 \frac{d^3\tilde{u}}{dx^3} L_{j,n}(x)dx = -\left[\frac{d}{dx}\left[c_3 L_{j,n}(x)\right] \frac{d\tilde{u}}{dx} \right]_a^b + \int_a^b \frac{d^2}{dx^2}\left[c_3 L_{j,n}(x)\right] \frac{d\tilde{u}}{dx} dx \tag{8}$$

$$\int_a^b c_2 \frac{d^2\tilde{u}}{dx^2} L_{j,n}(x)dx = -\int_a^b \frac{d}{dx}\left[c_2 L_{j,n}(x)\right] \frac{d\tilde{u}}{dx} dx \tag{9}$$

Substituting eqns. (6) – (9) into eqn. (5) and using approximation for $\tilde{u}(x)$ given in eqn. (4) and after imposing the boundary conditions given in eqn. (1b) and rearranging the terms for the resulting equations we get a system of equations in matrix form as

$$\sum_{i=1}^{n} D_{i,j}\, a_i = F_j, \; j = 1,2,\ldots,n \tag{10a}$$

where

$$D_{i,j}=\int_a^b\left\{\left[\frac{d^4}{dx^4}\left[c_5L_{j,n}(x)\right]-\frac{d^3}{dx^3}\left[c_4L_{j,n}(x)\right]+\frac{d^2}{dx^2}\left[c_3L_{j,n}(x)\right]-\frac{d}{dx}\left[c_2L_{j,n}(x)\right]+c_1L_{j,n}(x)\right]\frac{d}{dx}[L_{i,n}(x)]\right.$$
$$\left.+c_0L_{i,n}(x)L_{j,n}(x)\right\}dx-\left[\frac{d}{dx}\left[c_5L_{j,n}(x)\right]\frac{d^3}{dx^3}[L_{i,n}(x)]\right]_{x=b}+\left[\frac{d}{dx}\left[c_5L_{j,n}(x)\right]\frac{d^3}{dx^3}[L_{i,n}(x)]\right]_{x=a}$$
$$+\left[\frac{d^2}{dx^2}\left[c_5L_{j,n}(x)\right]\frac{d^2}{dx^2}[L_{i,n}(x)]\right]_{x=b}-\left[\frac{d}{dx}\left[c_4L_{j,n}(x)\right]\frac{d^2}{dx^2}[L_{i,n}(x)]\right]_{x=b} \quad (10b)$$

$$F_j=\int_a^b\left\{rL_{j,n}(x)+\left[-\frac{d^4}{dx^4}\left[c_5L_{j,n}(x)\right]+\frac{d^3}{dx^3}\left[c_4L_{j,n}(x)\right]-\frac{d^2}{dx^2}\left[c_3L_{j,n}(x)\right]+\frac{d}{dx}\left[c_2L_{j,n}(x)\right]\right.\right.$$
$$\left.\left.-c_1L_{j,n}(x)\right]\frac{d\theta_0}{dx}-c_0\theta_0L_{j,n}(x)\right\}dx+\left[\frac{d}{dx}\left[c_5L_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=b}-\left[\frac{d}{dx}\left[c_5L_{j,n}(x)\right]\frac{d^3\theta_0}{dx^3}\right]_{x=a}$$
$$-\left[\frac{d^2}{dx^2}\left[c_5L_{j,n}(x)\right]\frac{d^2\theta_0}{dx^2}\right]_{x=b}+\left[\frac{d}{dx}\left[c_4L_{j,n}(x)\right]\frac{d^2\theta_0}{dx^2}\right]_{x=b}+\left[\frac{d^2}{dx^2}\left[c_5L_{j,n}(x)\right]\right]_{x=a}\times A_2$$
$$+\left[\frac{d^3}{dx^3}\left[c_5L_{j,n}(x)\right]\right]_{x=b}\times B_1-\left[\frac{d^3}{dx^3}\left[c_5L_{j,n}(x)\right]\right]_{x=a}\times A_1+\left[\frac{d}{dx}\left[c_4L_{j,n}(x)\right]\right]_{x=a}\times A_2$$
$$-\left[\frac{d^2}{dx^2}\left[c_4L_{j,n}(x)\right]\right]_{x=b}\times B_1+\left[\frac{d^2}{dx^2}\left[c_4L_{j,n}(x)\right]\right]_{x=a}\times A_1+\left[\frac{d}{dx}\left[c_3L_{j,n}(x)\right]\right]_{x=b}\times B_1$$
$$-\left[\frac{d}{dx}\left[c_3L_{j,n}(x)\right]\right]_{x=a}\times A_1 \quad (10c)$$

Solving the system (10a), we obtain the values of the parameters a_i and then substituting these parameters into eqn. (4), we get the approximate solution of the desired BVP (1).

Similarly, for seventh order linear BVP given in eqn. (2), and for ninth order linear BVP given in eqn. (3), we can obtain equivalent system of equations in matrix form. For nonlinear BVP, we first compute the initial values on neglecting the nonlinear terms and using the system (10). Then using the Newton's iterative method we find the numerical approximations for desired nonlinear BVP. This formulation is described through the numerical examples in the next section.

4. Numerical Examples and Results

To test the applicability of the proposed method, we consider both linear and nonlinear problems which are available in the literature. For all the examples, the solutions obtained by the proposed method are compared with the exact solutions. All the calculations are performed by ***MATLAB* 10**. The convergence of linear BVP is calculated by

$$E=\left|\tilde{u}_{n+1}(x)-\tilde{u}_n(x)\right|<\delta$$

where $\tilde{u}_n(x)$ denotes the approximate solution using *n*-th polynomials and δ (depends on the problem) which is less than 10^{-12}. In addition, the convergence of nonlinear BVP is calculated by the absolute error of two consecutive iterations such that

$\left|\tilde{u}_n^{N+1}-\tilde{u}_n^{N}\right|<\delta$, where δ is less than 10^{-10} and N is the Newton's iteration number.

Example 1: Consider the linear BVP of fifth order (Viswanadham, Krishna & Prabhakara, 2010):

$$\frac{d^5u}{dx^5}+xu=19x\cos x+2x^3\cos x+41\sin x-2x^2\sin x,\ -1\le x\le 1 \quad (11a)$$

$$u(-1)=u(1)=\cos 1,\ u'(-1)=-u'(1)=-4\cos 1+\sin 1, u''(-1)=3\cos 1-8\sin 1 \quad (11b)$$

The analytic solution of the above system is $u(x)=(2x^2-1)\cos x.$

In **Table 1**, we list the maximum absolute errors for this problem to compare with the existing methods.

Table 1: Maximum absolute errors for the example 1

x	Exact Results	12 Legendre Polynomials	
		Approximate	Abs. Error
-1.0	0.5403023059	0.5403023059	0.0000000E+000
-0.8	0.1950778786	0.1950778786	4.7337134E-013
-0.6	-0.2310939722	-0.2310939722	3.3925640E-013
-0.4	-0.6263214759	-0.6263214759	6.9122486E-013
-0.2	-0.9016612516	-0.9016612516	1.3414825E-012
0.0	-1.0000000000	-1.0000000000	1.5528689E-012
0.2	-0.9016612516	-0.9016612516	6.3637984E-013
0.4	-0.6263214759	-0.6263214759	6.4226402E-013
0.6	-0.2310939722	-0.2310939722	1.0645096E-012
0.8	0.1950778786	0.1950778786	6.7196249E-013
1.0	0.5403023059	0.5403023059	0.0000000E+000

On the contrary, the maximum absolute error has been obtained by Kasi Viswanadham, Murali Krishna & Prabhakara (2010) is 4.5162×10^{-4}.

Now the exact and approximate solutions are depicted in Fig. 1 of example 1 for $n=12$.

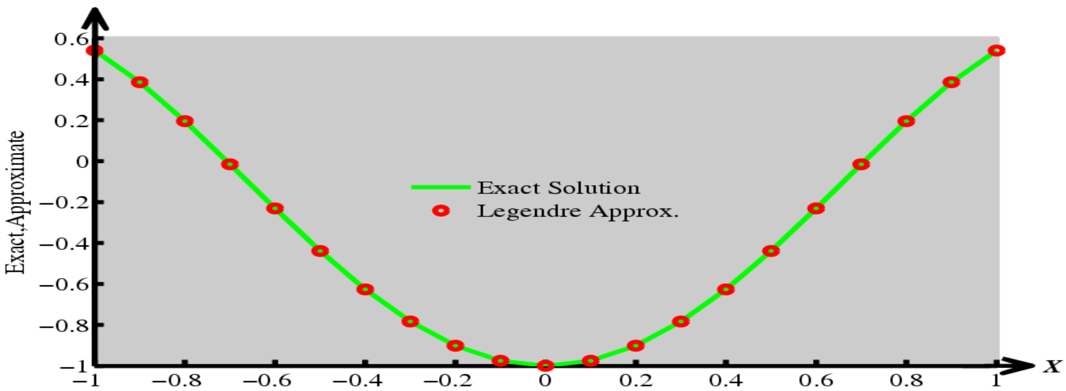

Fig. 1: Graphical representation of exact and approximate solutions of example 1.

Example 2: Consider the linear BVP of seventh order [Siddiqi & Iftikhar, 2013]

$$\frac{d^7u}{dx^7} = -u - e^x(2x^2 + 12x + 35),\ 0 \le x \le 1 \tag{12a}$$

$$u(0) = u(1) = 0,\ u'(0) = 1, u'(1) = -e, u''(0) = 0, u''(1) = -4e, u'''(0) = -3. \tag{12b}$$

The analytic solution of the above system is $u(x) = x(1-x)e^x$.

The maximum absolute errors for this problem are shown in **Table 2** to compare with the existing methods.

On the other hand, the maximum absolute error has been found by Siddiqi & Iftikhar (2013) is 7.482×10^{-10}. In Fig. 2, the exact and approximate solutions are given for example 2.

Table 2: Maximum absolute errors for the example 2

x	Exact Results	12 Legendre Polynomials	
		Approximate	Abs. Error
0.0	0.0000000000	0.0000000000	6.2597190E-026
0.1	0.0994653826	0.0994653826	5.8911209E-014
0.2	0.1954244413	0.1954244413	1.0269563E-014
0.3	0.2834703496	0.2834703496	1.0608181E-013
0.4	0.3580379274	0.3580379274	6.8500761E-014
0.5	0.4121803177	0.4121803177	9.6977981E-014
0.6	0.4373085121	0.4373085121	7.1442852E-014
0.7	0.4228880686	0.4228880686	6.1228800E-014
0.8	0.3560865486	0.3560865486	5.0015547E-014
0.9	0.2213642800	0.2213642800	5.9674488E-015
1.0	0.0000000000	0.0000000000	0.0000000E+000

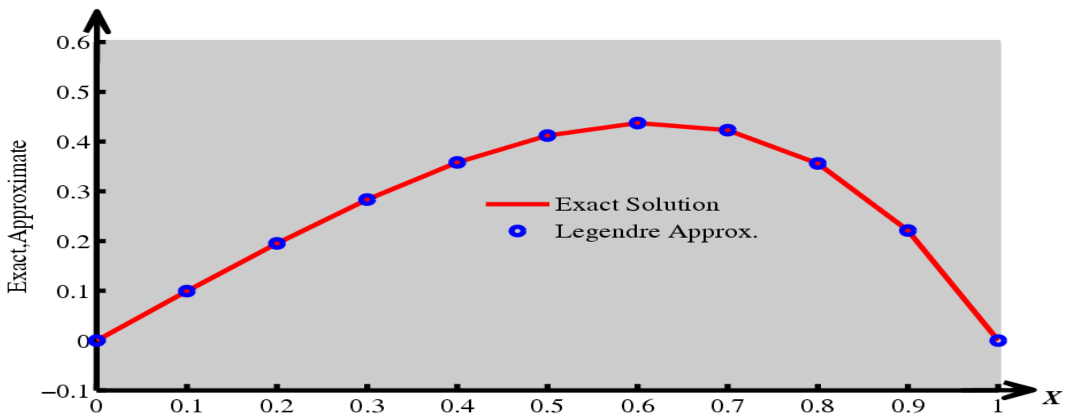

Fig. 2: Graphical representation of exact and approximate solutions of example 2.

Example 3: Consider the following ninth-order linear BVP (Nadjafi and Zahmatkesh (2010), Mohy-ud-Din & Yildirim (2010), Wazwaz, 2000)

$$\frac{d^9u}{dx^9} = u - 9e^x, \quad 0 \le x \le 1 \tag{13a}$$

$$u(0)=1,\ u(1)=0,\ u'(0)=0, u'(1)=-e, u''(0)=-1, u''(1)=-2e, u'''(0)=-2, u'''(1)=-3e, u^{(iv)}(0)=-3 \tag{13b}$$

Table 3: Maximum absolute errors for the example 3

x	Exact Results	12 Legendre Polunomials	
		Approximate	Abs. Error
0.0	1.0000000000	1.0000000000	0.0000000E+000
0.1	0.9946538263	0.9946538263	1.4432899E-015
0.2	0.9771222065	0.9771222065	3.2196468E-015
0.3	0.9449011653	0.9449011653	4.5519144E-015
0.4	0.8950948186	0.8950948186	4.8849813E-015
0.5	0.8243606354	0.8243606354	3.1086245E-015
0.6	0.7288475202	0.7288475202	5.6621374E-015
0.7	0.6041258122	0.6041258122	4.4408921E-016
0.8	0.4451081857	0.4451081857	3.1641356E-015
0.9	0.2459603111	0.2459603111	4.4408921E-016
1.0	0.0000000000	0.0000000000	0.0000000E+000

The analytic solution of the above system is $u(x)=(1-x)e^x$. The numerical results for this problem are summarized in **Table 3**. On the other hand, the accuracy is found nearly the order 10^{-10} by Mohy-ud-Din and Yildirim (2010) and by Wazwaz (2000) and nearly the order 10^{-9} by Nadjafi and Zahmatkesh. In Fig. 3, the exact and approximate solutions are given of example 3 for $n=12$.

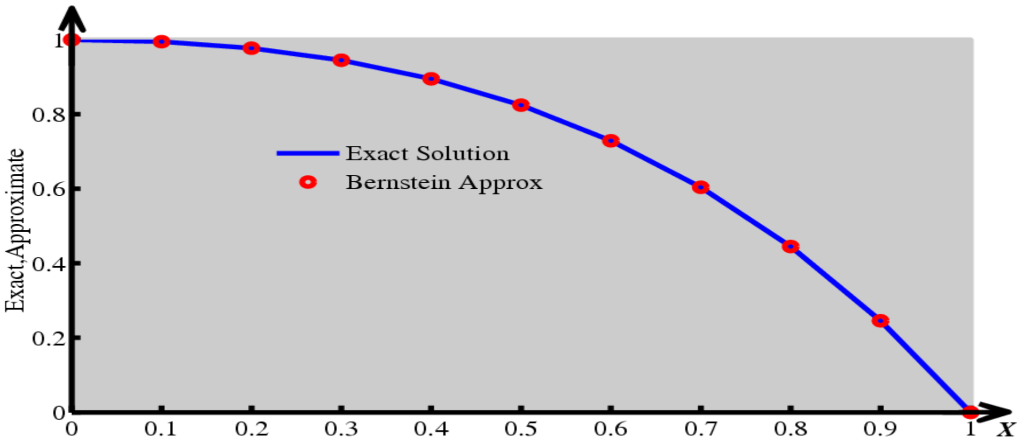

Fig. 3: Graphical representation of exact and approximate solutions of example 3.

Example 4: Consider the **nonlinear** BVP of fifth order (Erturk, 2007 and Wazwaz, 2001)

$$\frac{d^5u}{dx^5} = u^2 e^{-x},\ 0 \le x \le 1 \tag{14a}$$

$$u(0)=1,\ u(1)=e,\ u'(0)=1, u'(1)=e, u''(0)=1 \tag{14b}$$

The exact solution of this BVP is $u(x)=e^x$.

Consider the approximate solution of $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n} a_i\, L_{i,n}(x),\ \ n \ge 1 \tag{15}$$

Here $\theta_0(x) = 1 - x(1-e)$ is specified by the essential boundary conditions in (14b). Also $L_{i,n}(0) = L_{i,n}(1) = 0$ for each $i = 1,2,\ldots,n$. Using eqn. (15) into eqn. (14a), the Galerkin weighted residual equations are

$$\int_0^1 \left[\frac{d^5\tilde{u}}{dx^5} - \tilde{u}^2 e^{-x} \right] L_{k,n}(x)dx = 0, k = 1,2,\cdots,n \tag{16}$$

Integrating first term of (16) by parts, we obtain

$$\int_0^1 \frac{d^5\tilde{u}}{dx^5} L_{k,n}(x)\,dx = -\left[\frac{dL_{k,n}(x)}{dx} \frac{d^3\tilde{u}}{dx^3} \right]_0^1 + \left[\frac{d^2L_{k,n}(x)}{dx^2} \frac{d^2\tilde{u}}{dx^2} \right]_0^1 - \left[\frac{d^3L_{k,n}(x)}{dx^3} \frac{d\tilde{u}}{dx} \right]_0^1 + \int_0^1 \frac{d^4L_{k,n}(x)}{dx^4} \frac{d\tilde{u}}{dx} dx \tag{17}$$

Putting eqn. (17) into eqn. (16) and using approximation for $\tilde{u}(x)$ given in eqn. (15) and after applying the boundary conditions given in eqn. (14b) and rearranging the terms for the resulting equations, we obtain

$$\sum_{i=1}^{n} \left[\int_0^1 \left[\frac{d^4L_{k,n}(x)}{dx^4} \frac{dL_{i,n}(x)}{dx} - 2\theta_0 e^{-x} L_{i,n}(x) L_{k,n}(x) - \sum_{j=1}^{n} a_j \left(L_{i,n}(x) L_{j,n}(x) L_{k,n}(x)\right) e^{-x} \right] dx \right.$$
$$\left. - \left[\frac{dL_{k,n}(x)}{dx} \frac{d^3L_{i,n}(x)}{dx^3} \right]_{x=1} + \left[\frac{dL_{k,n}(x)}{dx} \frac{d^3L_{i,n}(x)}{dx^3} \right]_{x=0} + \left[\frac{d^2L_{k,n}(x)}{dx^2} \frac{d^2L_{i,n}(x)}{dx^2} \right]_{x=1} \right] a_i$$
$$= \int_0^1 \left[-\frac{d^4L_{k,n}(x)}{dx^4} \frac{d\theta_0}{dx} + \theta_0^2 e^{-x} L_{k,n}(x) \right] dx + \left[\frac{dL_{k,n}(x)}{dx} \frac{d^3\theta_0}{dx^3} \right]_{x=1} - \left[\frac{dL_{k,n}(x)}{dx} \frac{d^3\theta_0}{dx^3} \right]_{x=0}$$
$$- \left[\frac{d^2L_{k,n}(x)}{dx^2} \frac{d^2\theta_0}{dx^2} \right]_{x=1} + \left[\frac{d^2L_{k,n}(x)}{dx^2} \right]_{x=0} + \left[\frac{d^3L_{k,n}(x)}{dx^3} \right]_{x=1} \times e - \left[\frac{d^3L_{k,n}(x)}{dx^3} \right]_{x=0} \tag{18}$$

The above equation (18) is equivalent to matrix form as

$$(D+B)A = G \tag{19a}$$

where the elements of *A*, *B*, *D*, *G* are $a_i, b_{i,k}, d_{i,k}$ and g_k respectively, given by

$$d_{i,k}=\int_0^1\left[\frac{d^4L_{k,n}(x)}{dx^4}\frac{dL_{i,n}(x)}{dx}-2\theta_0e^{-x}L_{i,n}(x)L_{k,n}(x)\right]dx-\left[\frac{dL_{k,n}(x)}{dx}\frac{d^3L_{i,n}(x)}{dx^3}\right]_{x=1}$$
$$+\left[\frac{dL_{k,n}(x)}{dx}\frac{d^3L_{i,n}(x)}{dx^3}\right]_{x=0}+\left[\frac{d^2L_{k,n}(x)}{dx^2}\frac{d^2L_{i,n}(x)}{dx^2}\right]_{x=1} \tag{19b}$$

$$b_{i,k}=-\sum_{j=1}^{n}a_j\int_0^1(L_{i,n}(x)L_{j,n}(x)L_{k,n}(x))e^{-x}\,dx \tag{19c}$$

$$g_k=\int_0^1\left[-\frac{d^4L_{k,n}(x)}{dx^4}\frac{d\theta_0}{dx}+\theta_0{}^2e^{-x}L_{k,n}(x)\right]dx+\left[\frac{dL_{k,n}(x)}{dx}\frac{d^3\theta_0}{dx^3}\right]_{x=1}-\left[\frac{dL_{k,n}(x)}{dx}\frac{d^3\theta_0}{dx^3}\right]_{x=0}$$
$$-\left[\frac{d^2L_{k,n}(x)}{dx^2}\frac{d^2\theta_0}{dx^2}\right]_{x=1}+\left[\frac{d^2L_{k,n}(x)}{dx^2}\right]_{x=0}+\left[\frac{d^3L_{k,n}(x)}{dx^3}\right]_{x=1}\times e-\left[\frac{d^3L_{k,n}(x)}{dx^3}\right]_{x=0} \tag{19d}$$

The initial values of these coefficients a_i are obtained by applying Galerkin method to the BVP neglecting the nonlinear term in (14a). That is, to find initial coefficients we solve the system

$$DA=G \tag{20a}$$

whose matrices are constructed from

$$d_{i,k}=\int_0^1\frac{d^4L_{k,n}(x)}{dx^4}\frac{dL_{i,n}(x)}{dx}dx-\left[\frac{dL_{k,n}(x)}{dx}\frac{d^3L_{i,n}(x)}{dx^3}\right]_{x=1}+\left[\frac{dL_{k,n}(x)}{dx}\frac{d^3L_{i,n}(x)}{dx^3}\right]_{x=0}$$
$$+\left[\frac{d^2L_{k,n}(x)}{dx^2}\frac{d^2L_{i,n}(x)}{dx^2}\right]_{x=1} \tag{20b}$$

$$g_k=\int_0^1-\frac{d^4L_{k,n}(x)}{dx^4}\frac{d\theta_0}{dx}dx+\left[\frac{dL_{k,n}(x)}{dx}\frac{d^3\theta_0}{dx^3}\right]_{x=1}-\left[\frac{dL_{k,n}(x)}{dx}\frac{d^3\theta_0}{dx^3}\right]_{x=0}-\left[\frac{d^2L_{k,n}(x)}{dx^2}\frac{d^2\theta_0}{dx^2}\right]_{x=1}$$
$$+\left[\frac{d^2L_{k,n}(x)}{dx^2}\right]_{x=0}+\left[\frac{d^3L_{k,n}(x)}{dx^3}\right]_{x=1}\times e-\left[\frac{d^3L_{k,n}(x)}{dx^3}\right]_{x=0} \tag{20c}$$

Once the initial values of a_i are obtained from eqn. (20a), they are substituted into eqn. (19a) to obtain new estimates for the values of a_i. This iteration process continues until the converged values of the unknown parameters are obtained. Substituting the final values of the parameters into eqn. (15), we obtain an approximate solution of the BVP (14). The maximum absolute errors for this problem are shown in **Table 4** with 6 iterations. On the other hand, the maximum absolute errors have been obtained by Wazwaz (2001) and Erturk (2007) are 4.1×10^{-8} and 1.52×10^{-10}, respectively. We depict the exact and approximate solutions in **Fig. 4** of example 4 for $n=12$.

Example 5: Consider the **nonlinear** BVP of seventh order (Siddiqi, Akram & Iftikhar, 2012)

$$\frac{d^7u}{dx^7}=u^2e^{-x},\ 0\le x\le 1 \tag{21a}$$

$$u(0)=1,\ u(1)=e,\ u'(0)=1, u'(1)=e, u''(0)=1, u''(1)=e, u'''(0)=1. \tag{21b}$$

The exact solution of this BVP is $u(x)=e^x$. Following the proposed method illustrated in section 3 as well as in example 4; the maximum absolute errors for this problem are summarized in **Table 5.**

Table 4: Maximum absolute errors of example 4 using 6 iterations

x	Exact Results	12 Legendre Polynomials	
		Approximate	Abs. Error
0.0	1.0000000000	1.0000000000	0.0000000E+000
0.1	1.1051709181	1.1051709181	2.9918024E-012
0.2	1.2214027582	1.2214027582	3.4379395E-011
0.3	1.3498588076	1.3498588076	1.1878055E-012
0.4	1.4918246976	1.4918246976	2.3980112E-011
0.5	1.6487212707	1.6487212707	3.4213951E-012
0.6	1.8221188004	1.8221188004	3.6519182E-012
0.7	2.0137527075	2.0137527075	2.8570975E-012
0.8	2.2255409285	2.2255409285	1.4395748E-011
0.9	2.4596031112	2.4596031112	2.8618543E-011
1.0	2.7182818285	2.7182818285	1.2759373E-000

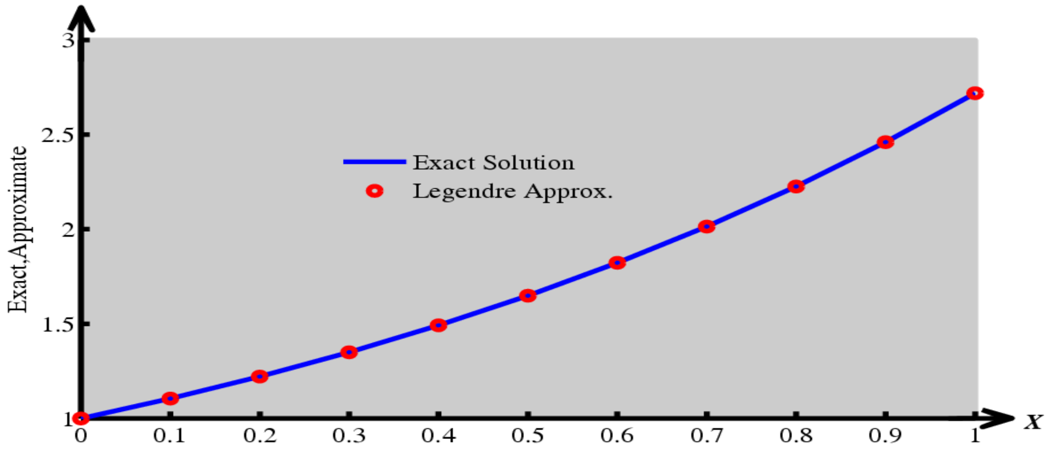

Fig. 4: Graphical representation of exact and approximate solutions of example 4.

Table 5: Maximum absolute errors of example 5 using 6 iterations

x	Exact Results	12 Legendre Polynomials	
		Approximate	Abs. Error
0.0	1.0000000000	1.0000000000	0.0000000E+000
0.1	1.1051709181	1.1051709181	1.6084911E-011
0.2	1.2214027582	1.2214027580	1.3298251E-011
0.3	1.3498588076	1.3498588075	5.0746074E-011
0.4	1.4918246976	1.4918246976	7.1547213E-012
0.5	1.6487212707	1.6487212707	6.2061467E-011
0.6	1.8221188004	1.8221188004	2.9898306E-012
0.7	2.0137527075	2.0137527075	1.8429702E-013
0.8	2.2255409285	2.2255409282	6.5281114E-011
0.9	2.4596031112	2.4596031112	1.6604496E-011
1.0	2.7182818285	2.7182818285	0.0000000E+000

On the contrary, the maximum absolute error has been obtained by Siddiqi , Akram & Iftikhar (2012) is 7.586×10^{-10}. We depict the exact and approximate solutions in **Fig. 5** of example 5 for $n=12$.

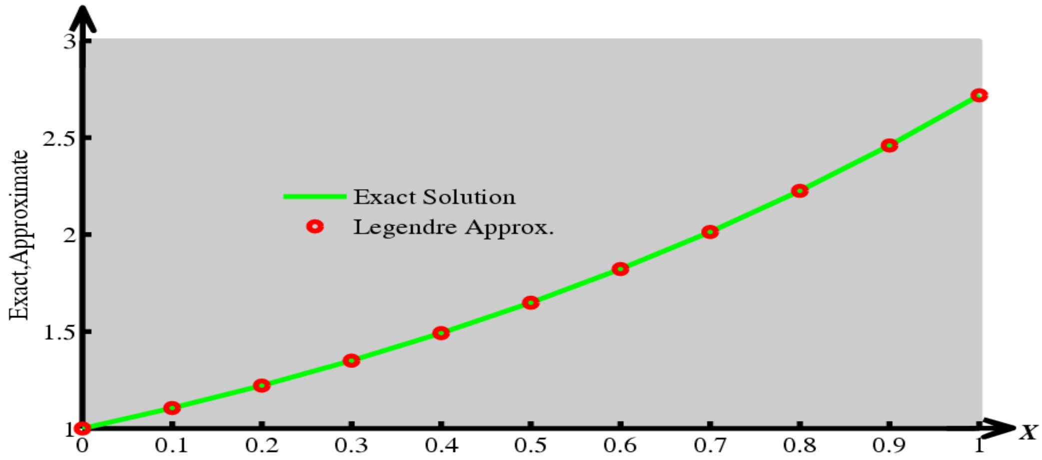

Fig. 5: Graphical representation of exact and approximate solutions of example 5 for.

5. Conclusions

In this paper, Galerkin method has been used for finding the numerical solutions of fifth, seventh and ninth order linear and nonlinear BVPs with Legendre polynomials as basis functions. The numerical examples available in the literature have been considered to verify the proposed method. We see from the tables that the numerical results obtained by our method are better than other existing methods. It may also notice that the numerical solutions coincide with the exact solution even Legendre polynomials are used in the approximation which are shown in Figs. [1-5]. The algorithm can be coded easily and may be used for solving any higher order BVP.

6. References

[1] Agarwal, R.P. (1986). Boundary Value Problems for Higher Order Differential Equations, *World Scientific*, Singapore.

[2] Caglar H.N., Caglar S.H., & Twizell E.H. (1999). The numerical solution of fifth order boundary value problems with sixth degree B-spline functions, *Appl. Math. Lett.*, **12**, 25-30.

[3] Davis P.J., & Robinowitz P. (2007). Methods of Numerical Integration, *Dover Publications*, Inc, 2nd Edition.

[4] Davis A.R., Karageorghis A., & Phillips T.N. (1988). Spectral Galerkin methods for the primary two point boundary value problem in modeling viscoelastic flows, *International J. Numer. Methods Eng*. **26**, 647-662.

[5] Karageorghis A., Phillips T.N. & Davis A.R., (1988). Spectral Collocation methods for the primary two point boundary value problem in modeling viscoelastic flows, *International J. Numer. Methods Eng*. **26**, 805-813.

[6] Erturk V.S. (2007). Solving nonlinear fifth order boundary value problems by differential transformation method, *Selcuk J. Appl. Math*., **8** (1), 45-49.

[7] Kasi Viswanadham K.N.S., Murali Krishna P., & Prabhakara Rao, C. (2010). Numerical solution of fifth order boundary value problems by collocation method with sixth order B-splines, *International Journal of Applied Science and Engneering*, 8 (2), 119-125.

[8] Lamnii A., Mraoui H., Sbibih D., & Tijini A. (2008). Sextic spline solution of fifth order boundary value problems, *Math. Computer Simul*., **77**, 237-246.

[9] Mohyud-Din S.T., & Yildirim A. (2010). Solutions of tenth and ninth order boundary value problems by modified variational iteration method, *Application Applied Math*., **5** (1), 11-25.

[10] Nadjafi, J.S., & Zahmatkesh, S. (2010). Homotopy perturbation method (HPM) for solving higher order boundary value problems, *Applied Math. Comput.* Sci., **1** (2), 199-224.

[11] Reddy, J.N. (1991). Applied Functional Analysis and Variational Methods in Engineering, Krieger Publishing Co. Malabar, Florida.

[12] Richards, G. & Sarma, P.R.R. (1994). Reduced order models for induction motors with two rotor circuits, *IEEE Transactions Energy Conversion*, 9 (4), 673-678.

[13] Siddiqi, S. S., & Twizell E.H. (1997). Spline solutions of linear twelfth order boundary value problems, *J. Comp. Appl. Maths*. **78**, 371-390.

[14] Siddiqi, S. S., Twizell, E.H. (1998). Spline solutions of linear tenth order boundary value problems, *Intern. J. Computer Math*. **68** (3), 345-362.

[15] Siddiqi, S. S., & Twizell, E.H. (1996). Spline solutions of linear eighth order boundary value problems, *Comp. Meth. Appl. Mech. Eng*. **131**, 309-325.

[16] Siddiqi, S. S., & Twizell, E.H. (1996). Spline solutions of linear sixth order boundary value problems, *Intern. J. Computer Math*. **60 (**3), 295-304.

[17] Siddiqi, S. S., Ghazala Akram & Muzammal Iftikhar (2012). Solution of seventh order boundary value problem by differential transformation method, *World Applied Sciences Journal*, 16 (11), 1521-1526.

[18] Siddiqi S. S., Ghazala Akram & Muzammal Iftikhar (2012). Solution of seventh order boundary value problems by variational iteration technique, *Applied Mathematical Sciences*, **6**, (94), 4663-4672.

[19] Siddiqi, S. S. & Muzammal Iftikhar (2013). Solution of seventh order boundary value problem by variation of parameters method, *Research Journal of Applied Sciences, Engineering and Technology*, 5 (1), 176-179.

[20] Wazwaz, A.M. (2001). The numerical solution of fifth order boundary value problems by the decomposition method, *J. Comput. Appl. Math*., **136**, 259-270.

[21] Wazwaz A.M. (2000). Approximate solutions to boundary value problems of higher order by the modified decomposition method, *Comput. Math. Appl*., **40**, 679-691.