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On Solution of Fractional Differential Equations Using Osler Definition in Hilbert Space

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Abstract: In this paper we consider the fractional differential equations with constant coefficients, using Osler definition

$$x(t)^{\beta} + x(t)^{\alpha} + x(t) = 0 \qquad \ldots(1)$$

Where $, \beta > \alpha > 0, 0 < t < T$

We study the necessary and sufficient conditions for the existence of solutions of these kinds of differential equations by using the functional analysis tools.

Keywords: Some operators, Osler definition fractional, differential equations.

1-Introduction

The field of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) is almost as old as calculus itself.(The idea where known-Leibniz (1859) mentions in a letter to L'Hospital in (1695)) [5].

The earliest more or less systematic studies seem to have been made in the beginning and middle of the 19th century by Liouville (1832a),Riemann (1953) ,and Holmgren (1864),although Euler(1730),Lagrange(1772),and others made some contributions even earlier[5] .

Through the last decades the usefulness of this mathematical theory in applications as well as its merits in pure mathematics has become more and more evident .Possibly the easiest access to the idea of the non-integer differential and integral operators studied in the field of fractional calculus is given by Cauchy's well known representation of an n-fold integral as a convolution integral [7]

$$J^{n}y(x) = \int_0^x \int_0^{x_{n-1}} \ldots \int_0^{x_1} y(x) \, dx_o \ldots dx_{n-2} dx_{n-1}$$

$$= \frac{1}{(n-1)!} \int_0^x \frac{1}{(x-t)^{1-n}} y(t) dt, \qquad n \in \mathbb{N}, x \in \mathbb{R}_{+,}$$

Where $J^n$is the n-fold integral operator with $J^{\,0}y(x) = y(x)$**.**Replacing the discreat factorial (n-1)! with Euler's continuous gamma function $\Gamma(n)$, which satisfies (n-1)! = $\Gamma(n)$ for $n \in \mathbb{N}$, one obtains a definition of a non-integer order integral, i.e.

$$J^{\alpha}y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{1}{(x-t)^{1-\alpha}} y(t) dt, \qquad \alpha, x \in \mathbb{R}_{+,}$$

The fact that there is obviously more than one way to define non-integer order derivatives is one of the challenging and rewarding aspects of this mathematical field.

The object of chief interest in the study of a Hilbert space is not the vectors in the space but the operators on it.

Most people who say they study the theory of Hilbert spaces in fact study operator theory, the reason is that the algebra and geometry of vectors, linear functional, quadratic forms, subspaces and the like are easier than operator theory and are pretty well worked out.

In mathematics, the L^p spaces are function spaces defined using a natural generalization of the P-norm for finite dimentional vector spaces, they are sometimes called Lebesgue spaces named after Henri Lebesgue (Dunford & Schwartz 1958,III.3)although according to the Bourbaki group (Bourbaki 1987)they were first introduced by Frigyes Riesz(Riesz 1910).

L^p spaces form an important class in functional analysis and have applications in physics,statistics,engineering and other disciplines .

2- Preliminaries:

Many definitions of fractional derivatives and fractional integrals (fractional differintegration for short) were introduced. These definitions agree when the order is integer and some of them are different when the order is not integer. Some are equivalent such that Riemann- Liouville and Grűnwald(1867) [3]. Several authors have considered different methods for calculating fractional derivatives of a given function [3 p.89- 90]. Now we review some definitions and theorems.

Definition, [6]: Euler Gamma function F (z) is defined by the so called Euler integral of the second kind

$$F(z) = \int_0^\infty t^{z-1} e^{-t} \, dt \qquad R(z) > 0, \text{Where } t^{z-1} = e^{(z-1)\log t}$$

This integral is convergent for all complex $z \in \mathbb{C}(\; R(z) > 0)$.

Definition, [8]: Osler's derivative of order v of a function f(z) is

$$D_{z-a}^{v} f(z) = \frac{\Gamma(v+1)}{2\pi i} \int_a^{z^+} (t-z)^{-v-1} f(t)\, dt \qquad , \quad v \notin z^- \ldots\ldots (1)$$

Where he made a branch cut from z to a and the integral curve is an open contour which starts from a and encloses z in positive sense and return to a.

Definition, [7]: let $\Omega = [a,b]$ where $(-\infty \le a \le b \le \infty)$be a finite or infinite interval of the real axis R= $(-\infty,\infty)$; $L_p(a,b)(1 \le p \le \infty)$ is the set of lebesgue complex valued functions f on Ω for which

$$\|f\|_p = \left(\int_a^b |f(t)|^p \right)^{1/p} \quad (1 \le p \le \infty),$$

Remark (2.1): When p=2 which is very important case we have L_2 space which is going to be Hilbert space.

Theorem (2.2), [1]: Let H be a Hilbert space and T∈ L(H)with $\lim \left(\|T^n\|^{1/n}\right) < 1$ then $\sum T^n$ converges in L (H) and $(I - T))$ is regular.

Theorem (2.3), [1]: Let H be a Hilbert space and $T \epsilon L(H)$. Let $\lambda \epsilon \mathbb{C}$ and $|\lambda| > \lim \|T^n\|^{1/n}$ then $(\lambda I - T))$ is regular and $\|\lambda I - T\|^{-1} = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$

3- Main result:

In this section we give two theorems concerning the existence of the inverse operator and the solution of the problem.

Consider the following fractional differential equation With Osler definition

$$x(t)^\beta + x(t)^\alpha + x(t) = 0 \quad , \text{Where } , \beta > \alpha > 0. \qquad \ldots. (2)$$

Rewrite it in the integral form to get,

$$\frac{\Gamma(\beta+1)}{2\pi i}\int_0^t (\tau-t)^{-\beta-1}\ x(\tau)d\tau+\frac{\Gamma(\alpha+1)}{2\pi i}\int_0^t (\tau-t)^{-\alpha-1}+x(t)=0$$

i.e. ,

$$\int_0^t [\frac{\Gamma(\beta+1)}{2\pi i}(\tau-t)^{-\beta-1}+\frac{\Gamma(\alpha+1)}{2\pi i}(\tau-t)^{-\alpha-1}]x(\tau)d\tau+x(t)=0\ \dots\ (2)$$

Rewrite (2) in the form

$$\int_0^t k(t,\tau)x(\tau)\,d\tau+x(t)=0 \qquad \dots (3)$$

Where,

$$k(t,\tau)=\frac{\Gamma(\beta+1)}{2\pi i}(\tau-t)^{-\beta-1}+\frac{\Gamma(\alpha+1)}{2\pi i}(\tau-t)^{-\alpha-1}$$

, is the kernal on the square G=J × J, where J = [0, T] is given interval and x(t) is the unknown function on [0, T].

Let:

$$Sx(t)=\int_0^t k(t,\tau)x(\tau)d\tau, \qquad \dots (4),$$

Then

$$Tx(t)=Sx(t)+x(t) \qquad \dots\dots(5),$$

Which is equivalent to our equation (1).

Note: We denote T is a constant while T is an operator.

We study the kernal $k(t,\tau)$ to give some existence results of Sx(t) .Many researchers studied operators having singularties such as

T.S.Aleroev [9] take $A^{[\alpha,\beta]}u(x)=c_{\alpha,\beta}x^{1/\alpha}\,(1-t)^{1/\beta-1}u(t)dt+\ \ c_\gamma\int_0^x(x-t)^{1/\gamma-1}u(t)dt$,where α,β,γ are positive reals while $c_{\alpha,\beta}$and c_γ are real and consider$A^{[\alpha,\beta]}\ \in L^2(0,1)$ for $\alpha=\beta=\gamma=p,\ 0<p<2$.

Mare Weibeer [7]prove $\int_a^x(x-t)^{\alpha-1}f(t)dt$ in $L_1[a,b]$, where $\alpha\in(0,1)$ by theorem(4.1.1).

Hilm.E [2] used $\{0=t_\circ<t_1<\cdots<t_p<t_{p+1}=1\}$. J.Cabollero [4] have $\lim_{t\to 0^+}f(t,.)=\infty$ (i.e f is singular at t=0) using fixed point theorem in apartialy order sets and green function.
Now we give our main results.

Lemma (3.1): $k\,(t,\tau)$ is bounded.

Proof (1):

We prove for all $t,\tau\in k(t,\tau)$, $\quad 0<t<\tau<T$, there exist $L>0$

Such that $\quad |k(t,\tau)|\le L,\quad L=\frac{\Gamma(\beta+1)}{\pi}$

$$|k(t,\tau)|=\left|\frac{\Gamma(\beta+1)}{2\pi i}\ (\tau-t)^{-\beta-1}+\frac{\Gamma(\alpha+1)}{2\pi i}(\tau-t)^{-\alpha-1}\right|$$

$$\le\frac{\Gamma(\beta+1)}{2\pi}\left|\ (\tau-t)^{-\beta-1}\ \right|+\frac{\Gamma(\alpha+1)}{2\pi}\ |(\tau-t)^{-\alpha-1}|$$

$$\le\frac{\Gamma(\beta+1)}{2\pi}\ e^{(-\beta-1)\ln(\tau-t)}\quad+\frac{\Gamma(\alpha+1)}{2\pi}e^{\,(-\alpha-1)\ln(\tau-t)}$$

by a property of, $\qquad [\frac{x-1}{x}\le\ln x\le x-1]$,

$$(\beta+1)\ln(\tau-t) \geq (\beta+1)\left(\frac{\tau-t-1}{\tau-t}\right) \qquad \geq (\beta+1)\left(1-\frac{1}{\tau-t}\right)$$

$$(\tau-t)\,\epsilon[0,T], \quad \ln(\tau-t) > 0$$

$$-(\beta+1)\ln(\tau-t) < -(\beta+1)\left(1-\frac{1}{\tau-t}\right)$$

$$e^{-(\beta+1)\ln(\tau-t)} < e^{-(\beta+1)\left(1-\frac{1}{\tau-t}\right)} < e^{-m\left(1-\frac{1}{\tau-t}\right)}$$

, m >0, m=$(\beta+1)$, similary

$$(\alpha+1)\ln(\tau-t) \geq (\alpha+1)\left(\frac{\tau-t-1}{\tau-t}\right) \rightarrow$$

$$-\alpha-1\ln(\tau-t) < -(\alpha+1)\left(1-\frac{1}{\tau-t}\right) \ ,$$

$$(\tau-t)\epsilon[0,T], \quad \ln(\tau-t) > 0.$$

$$e^{-(\alpha+1)\ln(\tau-t)} < e^{-(\alpha+1)\left(1-\frac{1}{\tau-t}\right)} < e^{-s\left(1-\frac{1}{\tau-t}\right)}$$

$$s > 0, s = (\ \alpha+1)$$

$$|k(\tau,t)| \quad < \frac{\Gamma(\beta+1)}{2\pi}e^{-m\left(1-\frac{1}{\tau-t}\right)} + \frac{\Gamma(\alpha+1)}{2\pi}\ e^{-s\left(1-\frac{1}{\tau-t}\right)}$$

$$< 2\frac{\Gamma(\beta+1)}{2\pi}e^{-m\left(1-\frac{1}{\tau-t}\right)}$$

$$< \ \frac{\Gamma(\beta+1)}{\pi}e^{-m\left(1-\frac{1}{\tau-t}\right)} \ ,$$

$$< \quad \frac{\Gamma(\beta+1)}{\pi}e^{0} < \frac{\Gamma(\beta+1)}{\pi} \leq \quad L \,,$$

Where,L $= \frac{\Gamma(\beta+1)}{\pi}$, L >0.

Proof(2)
We also prove it by using another property of

$$x > y \rightarrow \ln x > \ln y, \ \ln x = \int_1^x \frac{1}{s}\,ds \rightarrow \ln(\tau-t) = \int_1^{\tau-t}\frac{1}{s}\,ds$$

For

$$(\tau-t) > 1 \qquad \rightarrow \quad \ln(\tau-t) > \ln 1 \rightarrow$$

$$(\beta+1)\ln(\tau-t) > (\beta+1)\ln 1$$

$$-(\beta+1)\ln(\tau-t) < -(\beta+1)\ln 1$$

$$e^{-(\beta+1)\ln(\tau-t)} < e^{-(\beta+1)\ln 1} < e^{0} < 1$$

$$e^{-(\alpha+1)\ln(\tau-t)} < e^{-(\alpha+1)\ln 1} < e^{0} < 1$$

$$|k(\tau,t)| \quad < \frac{\Gamma(\beta+1)}{2\pi}e^{0} + \frac{\Gamma(\alpha+1)}{2\pi}\ e^{0}$$

$$< \ 2\frac{\Gamma(\beta+1)}{2\pi} < \frac{\Gamma(\beta+1)}{\pi} < L$$

Where, L>0 , $L = \frac{\Gamma(\beta+1)}{\pi}$

Lemma (3.2): Consider the Hilbert space (H,‖ ‖) , where H=$L_2[0,T]$, $\|u\|_2 = (\int_a^t |u(t)|^2 dt)^{\frac{1}{2}}$,

Let X= $\{f(t,\tau): J \times H \longrightarrow H;\ f(t,\tau) = (\tau - t)^{-v-1}$, v is an order}. Consider the partition on, $J: 0 = \tau_\circ < \tau_1 < \cdots < \tau_{p=}T, p \in N$, T is a constant, then the kernal belonge to H.

It's mean that for K: $[0,T] \times [0,T] \rightarrow \mathbb{C}$, $\int_0^T \int_0^t |k(t,\tau)|^2\, dtdt \le \infty$

Proof:

From lemma [3.1], it is proved that $|k(\tau,t)| < L$, L >0

$$\int_0^T \int_0^t |k(t,\tau)|^2\, d\tau dt \le \int_0^T \int_0^t L^2\, d\tau dt = \frac{L^2T^2}{2} = \frac{(LT)^2}{2} \le \infty.$$

Remark(3.3): from lemma (2.1)and lemma (2.2)we get S_x in (4) is an operator and Therefore $Tx(t) = Sx(t) + Ix(t)$ is an operator too.

Now we study the properties of the operator Tx(t) to ensure the existence of it's inverse

Lemma (3.4): Let Ω be a closed convex and nonempty subset of a Hilbert space H. Then the operator $Sx(t) = \int_0^t k(t,\tau)x(\tau)d\tau$, for all t $\epsilon[0,T]$ is bounded.

Proof :

$$\|Sx\|_2^2 = \int_0^T |\ Sx(t)|^2\, dt$$

$$|Sx(t)| = \left|\int_0^t k(t,\tau)x(\tau)d\tau\right| \le \int_0^t |k(t,\tau)|\, |x(\tau)| d\tau$$

$$\le \left(\int_0^t |k(t,\tau|^2\, d\tau\right)^{1/2} \left(\int_0^t |x(\tau)|^2\, d\tau\right)^{1/2}$$

$$= \|x\|_2 \left(\int_0^t |k(t,\tau|^2\, d\tau\right)^{1/2}$$

$$|Sx(t)|^2 \le \|x\|_2^2 \int_0^t |k(t,\tau|^2\, d\tau$$

$$\int_0^T |Sx(t)|^2 dt \le \int_0^T \|x\|_2^2 \int_0^t |k(t,\tau|^2\, d\tau\, dt$$

$$\le \int_0^T \int_0^t |k(t,\tau|^2 d\tau dt\, \|x\|_2^2$$

$$\le \frac{L^2T^2}{2}\ \|x\|_2^2$$

Hence, $\|Sx\|_2 \le \frac{LT}{2^{1/2}}\ \|x\|_2$

Which implies that

$$\|S\|_2 = \mathrm{SUP}\{\ \|Sx\|: x \in D(s) \text{and } \|x\| = 1\}$$

$$= \frac{LT}{2^{1/2}}$$

i.e., S is bounded linear operator.

$$\|S\|_2 = K, \quad K = \frac{LT}{2^{1/2}}$$

$$\Rightarrow \text{if } (LT) < 2^{1/2} \text{ then } K < 1 \quad \Rightarrow \|S\|_2 < 1$$

$$\Rightarrow (I - S)^{-1} \text{exist},$$

$$\Rightarrow (I - (I - T)^{-1} \text{ exist},$$

$$\Rightarrow (T)^{-1} \text{ exist}.$$

Therefore the inverse operator exist when we use Osler definition and when the norm of the operator S less than one, we have the norm of the operator T less than delta.

$$\|T\|_2 < \delta \quad \Rightarrow \quad \|(\delta I - T)\|^{-1} = \sum_{n=0}^{\infty} \frac{T^n}{\delta^{n+1}}$$

Which represent the solution $x(t)$ in the form

$$x(t) = \sum_{n=0}^{\infty} \frac{T^n}{\delta^{n+1}}.$$

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