

Numerical Solution of Linear Volterra-Fredholm Integral Equations Using Lagrange Polynomials

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Abstract:
In this paper, new algorithms for finding numerical solution of Linear Volterra-Fredholm integral equations (LVFIE's) of the second kind are introduced. The methods based upon Lagrange polynomial approximation, Barycentric Lagrange polynomial approximation, and Modified Lagrange polynomial approximation. Also, some examples are included to improve the validity and applicability of the techniques. Finally a comparison between the proposed methods and other methods were used to solve this kind of equations.

Keywords: Volterra-Fredholm integral equation, Lagrange polynomial, Barycentric Lagrange Polynomial, Modified Lagrange polynomial.

1. Introduction:

Integral equations are encountered in various fields of science and numerous applications in elasticity, plasticity, heat and mass transfer, approximation theory, fluid dynamics, filtration theory, electrostatics, electrodynamics, biomechanics, game theory, control, queuing theory, electrical engineering, economics, medicine, etc. [3].

Many researchers studied and discuss the linear Volterra-Fredholm integral equations, Majeed, S.J. and Omran, H.H. [7] used some numerical methods to solve LVFIE's of the first and second kinds namely the repeated Trapezoidal method and the repeated Simpson's 1/3 method, Al-Jarrah, Y. and Lin E.-B. [2] used Scaling function interpolation method to solve Linear Volterra-Fredholm integral equations (VFIE's), were scaling functions and Wavelet functions are the key element of Wavelet methods which are a very useful tool in solving integral equations .

Lagrange interpolation polynomial used to solve integral equations: Adibi, H. and Rismani, A.M. [1] applied Legendre-spectral method to solve functional integral equations where the Legendre Gauss points are used as collocation nodes and Lagrange scheme is employed to interpolate the quantities needed, Shahsavaran A. [8] presented a numerical method for solving nonlinear VFIE's based upon Lagrange functions approximations together with the Gaussian Quadrature rule and then utilized to reduce the VFIE to the solution of algebraic equations.

In this work, Lagrange polynomial, and Barycentric Lagrange Polynomial are used to solve LVFIE's numerically. The remainder of the paper is organized as follows: the methods of the solution (Lagrange polynomial, and Barycentric Lagrange Polynomial), test examples are investigated and the corresponding tables are presented. Finally, the report ends with a brief conclusion.

2. Methods of solution:

The Linear Volterra-Fredholm integral equation (LVFIE) of the second kind is:

$$u(x) = f(x) + \int_a^x k_1(x,t)u(t)dt + \int_a^b k_2(x,t)u(t)dt \qquad (1)$$

Where $a \le x \le b$; $f(x), k_1(x,t)$, and $k_2(x,t)$ are continuous functions and $u(x)$ is the unknown function to be determined.

Now, to solve Eq. (1) using Lagrange polynomial method, Barycentric Lagrange Polynomial method, and Modified Lagrange polynomial method; the derivation of the methods are showing as follows:

2.1 Lagrange Polynomial Method:

First, define Lagrange formula for a set of n+1 data points $\{(x_0,t_0),(x_1,t_1),\ldots,(x_n,t_n)\}$as [5]:

$$L_n(x) = \sum_{m=0}^{n} u_m\, L_{n,m}(x) \tag{2}$$

Where

$$L_{n,m}(x) = \prod_{\substack{k=0\\ m\neq k}}^{n} \frac{(x - x_k)}{(x_m - x_k)} \tag{3}$$

In order to solve LVFIE using Lagrange polynomial, we substitute Eq. (2) in Eq. (1), to get:

$$u_0\frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}+u_1\frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}+\cdots+u_n\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}=f(x)$$

$$+\int_a^x k_1(x,t)\,[u_0\frac{(t-x_1)(t-x_2)\ldots(t-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}+u_1\frac{(t-x_0)(t-x_2)\ldots(t-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}+\cdots+u_n\frac{(t-x_0)(t-x_1)\ldots(t-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}]dt$$

$$+\int_a^b k_2(x,t)\,[u_0\frac{(t-x_1)(t-x_2)\ldots(t-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}+u_1\frac{(t-x_0)(t-x_2)\ldots(t-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}+\cdots+u_n\frac{(t-x_0)(t-x_1)\ldots(t-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}]dt$$

Now, simplify the last equation, to get:

$$\frac{u_0}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}\{\,(x-x_1)(x-x_2)\ldots(x-x_n)-[\int_a^x k_1(x,t)(t-x_1)(t-x_2)\ldots(t-x_n)dt$$

$$+\int_a^b k_2(x,t)(t-x_1)(t-x_2)\ldots(t-x_n)dt]\,\}+\frac{u_1}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}\{\,(x-x_0)(x-x_2)\ldots(x-x_n)$$

$$-[\int_a^x k_1(x,t)(t-x_0)(t-x_2)\ldots(t-x_n)dt+\int_a^b k_2(x,t)(t-x_0)(t-x_2)\ldots(t-x_n)dt]\,\}+\cdots$$

$$+\frac{u_n}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}\{\,(x-x_0)(x-x_1)\ldots(x-x_{n-1})-[\int_a^x k_1(x,t)(t-x_0)(t-x_1)\ldots(t-x_{n-1})dt$$

$$+\int_a^b k_2(x,t)(t-x_0)(t-x_1)\ldots(t-x_{n-1})dt]\,\}=f(x)$$

Putting $x=x_i$, for $i=0,1,\ldots,n$, to get a system of n+1 equations, which is:

$$A\vec{u} = \vec{B} \tag{4}$$

When $A = a_{ij}$, $\vec{B} = b_i$.

$$b_i = f_i\,, i = 0,1,\ldots,n \tag{5}$$

And

$$a_{ij} = \begin{cases} 1-\dfrac{1}{p}*\left[\displaystyle\int_a^{x_i} k_1(x_i,t)*p_1dt+\int_a^b k_2(x_i,t)*p_1dt\right] & ,if\ i=j \\ -\dfrac{1}{p}*\left[\displaystyle\int_a^{x_i} k_1(x_i,t)*p_1dt+\int_a^b k_2(x_i,t)*p_1dt\right] & ,if\ i\neq j \end{cases} \tag{6}$$

Where

$$p = \prod_{\substack{k=0\\ j\neq k}}^{n} (x_j - x_k) \tag{7}$$

And

$$p_1 = \prod_{k=0}^{n}(t - x_k) \tag{8}$$

For all $i, j = 0,1,\ldots,n$.

The Algorithm:

The numerical solution of LVFIE's, by using Lagrange polynomial, is obtained as follows:

Step1:

Put $h = \frac{b-a}{n}$, $n\epsilon\mathbb{N}$.

Step2:

Set $x_i = a + ih$, with $x_0 = a$ and $x_n = b$, i=0,1,…,n.

Step3:

Use step1, step2 and Eq. (6) to find a_{ij} (note that for integral in Eq. (6), we use the exact value).

Step4:

Compute b_i using Eq. (5).

Step5:

Solve the system Eq. (4) using step3 and step4 and Gaussian Elimination Method.

2.2 Barycentric Lagrange Polynomial Method:

First, define Barycentric Lagrange formula for a set of n+1 data points $\{(x_0,t_0),(x_1,t_1),\ldots,(x_n,t_n)\}$as [4]:

$$p(x) = \frac{\sum_{j=0}^{n} \frac{w_j}{x - x_j}\, u_j}{\sum_{j=0}^{n} \frac{w_j}{x - x_j}} \tag{9}$$

Where

$$w_j = \frac{1}{\prod_{j \neq k}(x_j - x_k)} \qquad , k, j = 0,1,\ldots,n \tag{10}$$

In order to solve LVFIE using Barycentric Lagrange polynomial, we substitute Eq. (9) in Eq. (1), to get:

$$\frac{(x-x_1)(x-x_2)\ldots(x-x_n)w_0\, u_0}{(x-x_1)(x-x_2)\ldots(x-x_n)w_0 + (x-x_0)(x-x_2)\ldots(x-x_n)w_1 + \cdots + (x-x_0)(x-x_1)\ldots(x-x_{n-1})w_n}$$

$$\frac{(x-x_0)(x-x_2)\ldots(x-x_n)w_1\, u_1}{(x-x_1)(x-x_2)\ldots(x-x_n)w_0 + (x-x_0)(x-x_2)\ldots(x-x_n)w_1 + \cdots + (x-x_0)(x-x_1)\ldots(x-x_{n-1})w_n} + \cdots$$

$$+\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})w_n\, u_n}{(x-x_1)(x-x_2)\ldots(x-x_n)w_0 + (x-x_0)(x-x_2)\ldots(x-x_n)w_1 + \cdots + (x-x_0)(x-x_1)\ldots(x-x_{n-1})w_n} = f(x)$$

$$+\int_a^x k_1(x,t)[\frac{(t-x_1)(t-x_2)\ldots(x-x_n)w_0\, u_0}{(t-x_1)(t-x_2)\ldots(t-x_n)w_0 + (t-x_0)(t-x_2)\ldots(t-x_n)w_1 + \cdots + (t-x_0)(t-x_1)\ldots(t-x_{n-1})w_n}$$

$$+\frac{(t-x_0)(t-x_2)\ldots(t-x_n)w_1\, u_1}{(t-x_1)(t-x_2)\ldots(t-x_n)w_0 + (t-x_0)(t-x_2)\ldots(t-x_n)w_1 + \cdots + (t-x_0)(t-x_1)\ldots(t-x_{n-1})w_n} + \cdots$$

$$+\frac{(t-x_0)(t-x_1)\ldots(t-x_{n-1})w_n\, u_n}{(t-x_1)(t-x_2)\ldots(t-x_n)w_0 + (t-x_0)(t-x_2)\ldots(t-x_n)w_1 + \cdots + (t-x_0)(t-x_1)\ldots(t-x_{n-1})w_n}]dt$$

$$+\int_a^b k_2(x,t)[\frac{(t-x_1)(t-x_2)\ldots(t-x_n)w_0\, u_0}{(t-x_1)(t-x_2)\ldots(t-x_n)w_0 + (t-x_0)(t-x_2)\ldots(t-x_n)w_1 + \cdots + (t-x_0)(t-x_1)\ldots(t-x_{n-1})w_n}$$

$$+\frac{(t-x_0)(t-x_2)\ldots(t-x_n)w_1\, u_1}{(t-x_1)(t-x_2)\ldots(t-x_n)w_0 + (t-x_0)(t-x_2)\ldots(t-x_n)w_1 + \cdots + (t-x_0)(t-x_1)\ldots(t-x_{n-1})w_n} + \cdots$$

$$+\frac{(t-x_0)(t-x_1)\ldots(t-x_{n-1})w_n\,u_n}{(t-x_1)(t-x_2)\ldots(t-x_n)w_0+(t-x_0)(t-x_2)\ldots(t-x_n)w_1+\cdots+(t-x_0)(t-x_1)\ldots(t-x_{n-1})w_n}]dt$$

Now, putting $x=x_i$, for $i=0,1,\ldots,n$, yields a system of n+1 equations, which is:

$$A\vec{u}=\vec{B} \qquad (11)$$

Where $A=\ a_{ij},\ \vec{B}=\ b_i$.

$$b_i\ =\ f_i\ , i=0,1,\ldots,n \qquad (12)$$

And

$$a_{ij}=\begin{cases}1-\left[\int_a^{x_i}k_1(x_i,t)\left(\frac{p}{p_1}\right)dt+\int_a^{b}k_2(x_i,t)\left(\frac{p}{p_1}\right)dt\right] & ,if\ i=j\\ -1*\left[\int_a^{x_i}k_1(x_i,t)\left(\frac{p}{p_1}\right)dt+\int_a^{b}k_2(x_i,t)\left(\frac{p}{p_1}\right)dt\right] & ,if\ i\neq j\end{cases} \qquad (13)$$

Where

$$p=\prod_{\substack{k=0\\ j\neq k}}^{n}\frac{t-x_k}{x_j-x_k} \qquad (14)$$

And

$$p_1=\sum_{j\neq k}\left(\prod_{k=0}^{n}(t-x_k)\right)*w_j \qquad (15)$$

For all $i, j=0,1,\ldots,n$.

The Algorithm:

The numerical solution of LVFIE's, by using Barycentric Lagrange polynomial, is obtained as follows:

Step1:

Put $h=\frac{b-a}{n},\quad n\epsilon\mathbb{N}$.

Step2:

Set $x_i=a+ih$, with $x_0=a$ and $x_n=b$, $i=0,1,\ldots,n$.

Step3:

Use step1, step2 and Eq. (13) to find a_{ij} (note that for integral in Eq. (13), we use the exact value).

Step4:

Compute b_i using Eq. (12).

Step5:

Solve the system Eq. (11) using step3 and step4 and Gaussian Elimination Method.

2.3 Modified Lagrange Polynomial Method:

First, define Modified Lagrange formula for a set of n+1 data points $\{(x_0,t_0),(x_1,t_1),\ldots,(x_n,t_n)\}$as [6]:

$$P_n(x)=l(x)\sum_{j=0}^{n}\frac{w_j}{x-x_j}u_j \qquad (16)$$

Where

$$l(x)=\prod_{j=0}^{n}(x-x_j) \qquad (17)$$

And

$$w_j=\frac{1}{\prod_{k\neq j}(x_j-x_k)} \qquad ,k,j=0,1,\ldots,n \qquad (18)$$

In order to solve LVFIE using Modified Lagrange polynomial, we substitute Eq. (16) in

Eq. (1), to get:

$$(x-x_1)(x-x_2)\dots(x-x_n)w_0u_0+(x-x_0)(x-x_2)\dots(x-x_n)w_1u_1+\cdots+(x-x_0)(x-x_1)\dots(x-x_{n-1})w_nu_n=f(x)$$
$$+\int_a^x k_1(x,t)\,[(t-x_1)(t-x_2)\dots(t-x_n)w_0u_0+(t-x_0)(t-x_2)\dots(t-x_n)w_1u_1+\cdots+(t-x_0)(t-x_1)\dots(t-x_{n-1})w_nu_n]dt$$
$$+\int_a^b k_2(x,t)\,[(t-x_1)(t-x_2)\dots(t-x_n)w_0u_0+(t-x_0)(t-x_2)\dots(t-x_n)w_1u_1+\cdots+(t-x_0)(t-x_1)\dots(t-x_{n-1})w_nu_n]dt$$

Now, putting $x=x_i$, for $i=0,1,\dots,n$, yields a system of n+1 equations, which is:

$$A\vec{u}=\vec{B} \tag{19}$$

Where $A=a_{ij}$, $\vec{B}=b_i$.

$$b_i=f_i\,,i=0,1,\dots,n \tag{20}$$

And

$$a_{ij}=\begin{cases}1-\left[\int_a^{x_i}k_1(x_i,t)*p\,dt+\int_a^b k_2(x_i,t)*p\,dt\right] & ,if\ i=j\\ -1*\left[\int_a^{x_i}k_1(x_i,t)*p\,dt+\int_a^b k_2(x_i,t)*p\,dt\right] & ,if\ i\neq j\end{cases} \tag{21}$$

Where

$$p=\prod_{\substack{k=0\\ j\neq k}}^{n}\frac{t-x_k}{x_j-x_k} \tag{22}$$

For all $i, j=0,1,\dots,n$.

The Algorithm:

The numerical solution of LVFIE's, by using Modified Lagrange polynomial, is obtained as follows:

Step1:

Put $h=\frac{b-a}{n}$, $n\epsilon\mathbb{N}$.

Step2:

Set $x_i=a+ih$, with $x_0=a$ and $x_n=b$, $i=0,1,\dots,n$.

Step3:

Use step1, step2 and Eq. (21) to find a_{ij} (note that for integral in Eq. (21), we use the exact value).

Step4:

Compute b_i using Eq. (20).

Step5:

Solve the system Eq. (19) using step3 and step4 and Gaussian Elimination Method.

3. Test Examples:

In this section, we give some of the numerical examples to illustrate the above methods for solving the linear Volterra-Fredholm integral equations of the second kind.

The exact solution is known and used to show that the numerical solution obtained with our methods is correct. We used MATLAB v 7.6 to solve the examples.

Example 1: Consider the LVFIE [7]:

$$u(x)=f(x)+\int_0^x (x-t)u(t)dt+\int_0^2 (xt)u(t)dt \tag{23}$$

Where $f(x) = 2\cos(x) - x\cos(2) - 2x\sin(2) + x - 1$

With the exact solution $u(x) = \cos(x)$.

Tables 1 and 2 represent the absolute error by using Lagrange polynomial, Barycentric Lagrange polynomial, and Modified Lagrange polynomial with n=5 and n=9 respectively, Table 3 represent the absolute error by using Lagrange polynomial, and Modified Lagrange polynomial with n=10 and the methods of [7], the repeated Trapezoidal method and the repeated Simpson's 1/3 method, where $\|err\|_\infty$ is the maximum absolute error, and R.T. represent the running time.

Table (1)
The Absolute Error of Example 1 by using Lagrange polynomial, Barycentric Lagrange polynomial, and Modified Lagrange polynomial with n=5

x	*The Exact Solution*	*Lagrange approximation Error*	*Barycentric Lagrange approximation Error*	*Modified Lagrange approximation Error*
0.0	**1.00000000000e+00**	**0**	**0**	**0**
0.4	**9.210609940028e-01**	**4.592865900376e-06**	**4.592865900376e-06**	**4.592865900376e-06**
0.8	**6.96706709347e-01**	**8.388707957385e-06**	**8.388707957385e-06**	**8.388707957385e-06**
1.2	**3.623577544766e-01**	**1.377961597220e-05**	**1.377961597220e-05**	**1.377961597220e-05**
1.6	**-2.91995223012e-02**	**2.152697485147e-05**	**2.152697485147e-05**	**2.152697485147e-05**
2.0	**-4.16146836547e-01**	**3.179163671773e-05**	**3.179163671773e-05**	**3.179163671773e-05**
$\|err\|_\infty$	**-**	**3.179163671773e-05**	**3.179163671773e-05**	**3.179163671773e-05**
R.T.	**-**	**3.394608272777e+00**	**7.204834975904e-01**	**7.883790785975e-01**

Table (2)
The Absolute Error of Example 1 by using Lagrange polynomial, Barycentric Lagrange polynomial, and Modified Lagrange polynomial with n=9

x	*The Exact Solution*	*Lagrange approximation Error*	*Barycentric Lagrange approximation Error*	*Modified Lagrange approximation Error*
0	**1.000000000000e+00**	**0**	**0**	**0**
0.2222	**9.754100854384e-01**	**5.694289484381e-11**	**5.694289484381e-11**	**5.694289484381e-11**
0.4444	**9.028496695499e-01**	**9.868994510497e-11**	**9.868983408267e-11**	**9.869005612728e-11**
0.6667	**7.858872611891e-01**	**1.477969968632e-10**	**1.477966637963e-10**	**1.477973299301e-10**
0.8889	**6.302750516130e-01**	**2.036726343135e-10**	**2.036724122689e-10**	**2.036727453358e-10**
1.1111	**4.436660226979e-01**	**2.697739254209e-10**	**2.697739254209e-10**	**2.697743139989e-10**
1.3333	**2.352375745989e-01**	**3.493415279276e-10**	**3.493410838384e-10**	**3.493417777278e-10**
1.5556	**1.524018277769e-02**	**4.457734754242e-10**	**4.457730642948e-10**	**4.457736315494e-10**
1.7778	**-2.055067186283e-01**	**5.662834645647e-10**	**5.662833535424e-10**	**5.662837976316e-10**
2.0000	**-4.161468347285e-01**	**7.011122238331e-10**	**7.011116687216e-10**	**7.011126679223e-10**
$\|err\|_\infty$	**-**	**7.011122238331e-10**	**7.011116687216e-10**	**7.011126679223e-10**
R.T.	**-**	**1.677031976048e+01**	**2.1318e+03**	**2.693039623271e+00**

Table (3)
***The Absolute Error of Example1 by using Lagrange polynomial, Modified Lagrange polynomial with n=10, and the methods of* [7]**

x	***Lagrange approximation Error***	***Modified Lagrange approximation Error***	**[7]**	
			Repeated Trapezoidal Error	***Repeated Simpson's 1/3 Error***
0.0	**0**	**0**	**0**	**0**
0.2	**5.0703885534e-13**	**5.0714987764e-13**	**3.36697e-4**	**8.35541e-5**
0.4	**2.2122303988e-12**	**2.2125634657e-12**	**8.17363e-4**	**3.06842e-4**
0.6	**3.8533620738e-12**	**3.8535841184e-12**	**1.45337e-3**	**1.08547e-4**
0.8	**5.6827875738e-12**	**5.6832316630e-12**	**2.25741e-3**	**6.32625e-4**
1.0	**7.7304829204e-12**	**7.7313710988e-12**	**3.24451e-3**	**1.91537e-4**
1.2	**1.0088707647e-11**	**1.0089984403e-11**	**4.43334e-3**	**9.79100e-4**
1.4	**1.2861656184e-11**	**1.2862655385e-11**	**5.84776e-3**	**9.06532e-5**
1.6	**1.6114169026e-11**	**1.6116111917e-11**	**7.51871e-3**	**1.36428e-3**
1.8	**2.0181245563e-11**	**2.0182494564e-11**	**9.48653e-3**	**2.64724e-4**
2.0	**2.3753443656e-11**	**2.3755164502e-11**	**1.18036e-2**	**1.83138e-3**
$\|\|err\|\|_\infty$	**2.3753443656e-11**	**2.3755164502e-11**	**1.18036e-2**	**1.83138e-3**
R.T.	**2.2065155750e+01**	**3.3047332893e+00**	**-**	**-**

Example 2: Consider the LVFIE:

$$u(x) = f(x) + \int_0^x (x^2 - t)u(t)dt + \int_0^1 (xt + x)u(t)dt \qquad (24)$$

Where $f(x) = e^x + e^x(x-1) - xe - x^2(e^x - 1) + 1$

With the exact solution $u(x) = e^x$.

Tables 4 and 5 represent the absolute error by using Lagrange polynomial, Barycentric Lagrange polynomial, and Modified Lagrange polynomial with n=5 and n=9 respectively, where $||err||_\infty$ is the maximum absolute error, and R.T. represent the running time.

Table (4)
The Absolute Error of Example 2 by using Lagrange polynomial, Barycentric Lagrange polynomial, and Modified Lagrange polynomial with n=5

x	***The Exact Solution***	***Lagrange approximation Error***	***Barycentric Lagrange approximation Error***	***Modified Lagrange approximation Error***
0.0	**1.00000000000e+00**	**0**	**0**	**0**
0.2	**1.221402758160e+00**	**1.263201389489e-06**	**1.263201389489e-06**	**1.263201389489e-06**
0.4	**1.491824697641e+00**	**2.554577710256e-06**	**2.554577710256e-06**	**2.554577710256e-06**
0.6	**1.822118800390e+00**	**3.878669883494e-06**	**3.878669883494e-06**	**3.878669883494e-06**
0.8	**2.225540928492e+00**	**5.505856810472e-06**	**5.505856810472e-06**	**5.505856810472e-06**
1.0	**2.718281828459e+00**	**7.751094385444e-06**	**7.751094385444e-06**	**7.751094385444e-06**
$\|\|err\|\|_\infty$	**-**	**7.751094385444e-06**	**7.751094385444e-06**	**7.751094385444e-06**
R.T.	**-**	**5.560824136215e+00**	**1.903141251640e+00**	**9.581812516600e-01**

Table (5)
The Absolute Error of Example 2 by using Lagrange polynomial, Barycentric Lagrange polynomial, and Modified Lagrange polynomial with n=9

x	The Exact Solution	Lagrange approximation Error	Barycentric Lagrange approximation Error	Modified Lagrange approximation Error
0.0	1.00000000000e+00	0	0	0
0.1111	1.11751906861e+00	9.13269460056e-013	9.12825370846e-013	9.132694600566e-13
0.2222	1.24884886872e+00	1.84230408706e-012	1.84163795324e-012	1.842304087062e-12
0.3333	1.39561242462e+00	2.75268696725e-012	2.75179878883e-012	2.752686967255e-12
0.4444	1.55962349691e+00	3.67816888058e-012	3.67728070216e-012	3.678390925188e-12
0.5556	1.74290899766e+00	4.63762361846e-012	4.63562521701e-012	4.637623618464e-12
0.6667	1.94773403975e+00	5.68478597529e-012	5.68278757384e-012	5.684785975290e-12
0.7778	2.17662993002e+00	6.86961598717e-012	6.86783963033e-012	6.870504165590e-12
0.8889	2.43242545212e+00	8.29158963711e-012	8.28936919106e-012	8.292033726320e-12
1.0	2.71828182574e+00	1.00524033541e-011	1.00488506404e-011	1.005240335416e-11
$\|err\|_\infty$	-	1.00524033541e-011	1.00488506404e-011	1.005240335416e-11
R.T.	-	1.676966823049e+01	1.489754612529e+03	2.693308515848e+00

Example 3: Consider the LVFIE [2]:

$$u(x) = f(x) + \int_0^x (xt)u(t)dt + \int_0^1 (xt)u(t)dt \qquad (25)$$

Where $f(x) = \frac{2}{3}x - \frac{1}{3}x^4$

With the exact solution $u(x) = x$.

Tables 6 and 7 represent the absolute error by using Lagrange polynomial, Barycentric Lagrange polynomial, and Modified Lagrange polynomial with n=5 and n=9 respectively, Table 8 represent the absolute error by using Lagrange polynomial, and Modified Lagrange polynomial with n=10 and the methods of [2], scaling function interpolation method, where $\|err\|_\infty$ is the maximum absolute error, and R.T. represent the running time.

Table (6)
The Absolute Error of Example 3 by using Lagrange polynomial, Barycentric Lagrange polynomial, and Modified Lagrange polynomial with n=5

x	The Exact Solution	Lagrange approximation Error	Barycentric Lagrange approximation Error	Modified Lagrange approximation Error
0.0	0	0	0	0
0.2	2.000000000000e-01	2.775557561562e-17	2.775557561562e-17	2.775557561562e-17
0.4	4.000000000000e-01	0	0	0
0.6	6.000000000000e-01	0	0	0
0.8	8.000000000000e-01	2.220446049250e-16	2.220446049250e-16	2.220446049250e-16
1.0	1.000000000000e+00	0	0	0
$\|err\|_\infty$	-	2.220446049250e-16	2.220446049250e-16	2.220446049250e-16
R.T.	-	3.371779640281e+00	8.931815158864e-01	8.261619751411e-01

Table (7)
The Absolute Error of Example 3 by using Lagrange polynomial, Barycentric Lagrange polynomial, and Modified Lagrange polynomial with n=9

x	*The Exact Solution*	*Lagrange approximation Error*	*Barycentric Lagrange approximation Error*	*Modified Lagrange approximation Error*
0.0	**1.000000000000e+00**	**0**	**0**	**0**
0.1111	**1.117519068617e+00**	**2.77555756156e-017**	**1.38777878078e-017**	**2.775557561562e-17**
0.2222	**1.248848868724e+00**	**5.55111512312e-017**	**2.77555756156e-017**	**2.775557561562e-17**
0.3333	**1.395612424620e+00**	**5.55111512312e-017**	**5.55111512312e-017**	**5.551115123125e-17**
0.4444	**1.559623496913e+00**	**1.11022302462e-016**	**1.11022302462e-016**	**1.110223024625e-16**
0.5556	**1.742908997665e+00**	**1.11022302462e-016**	**1.11022302462e-016**	**1.110223024625e-16**
0.6667	**1.947734039756e+00**	**1.11022302462e-016**	**1.11022302462e-016**	**1.110223024625e-16**
0.7778	**2.176629930023e+00**	**0**	**1.11022302462e-016**	**0**
0.8889	**2.432425452125e+00**	**3.33066907387e-016**	**0**	**3.330669073875e-16**
1.0	**2.718281825740e+00**	**6.66133814775e-016**	**1.11022302462e-016**	**6.661338147750e-16**
$\|\|err\|\|_\infty$	**-**	**6.66133814775e-016**	**1.11022302462e-016**	**6.661338147750e-16**
R.T.	**-**	**1.602750678597e+01**	**2.016615692573e+03**	**2.663894952085e+00**

Table (8)
***The Absolute Error of Example 3 by using Lagrange polynomial, Modified Lagrange polynomial with n=10, and the methods of* [2]**

x	*Lagrange approximation Error*	*Modified Lagrange approximation Error*	[2]		
			j = -2	*j = -1*	*j = 0*
0.1	**0**	**0**	**3.348e-7**	**1.032e-7**	**2.817e-7**
0.2	**2.7755575615628e-17**	**1.387778780781e-17**	**1.263e-7**	**5.75e-8**	**2.971e-7**
0.3	**0**	**2.775557561562e-17**	**1.905e-7**	**3.789e-8**	**4.913e-8**
0.4	**5.5511151231257e-17**	**5.551115123125e-17**	**2.564e-8**	**1.758e-7**	**4.506e-7**
0.5	**0**	**5.551115123125e-17**	**1.316e-8**	**8.553e-8**	**1.323e-7**
0.6	**0**	**1.110223024625e-16**	**1.876e-7**	**5.004e-7**	**1.243e-7**
0.7	**1.1102230246251e-16**	**0**	**6.735e-7**	**3.977e-7**	**5.035e-8**
0.8	**3.3306690738754e-16**	**2.220446049250e-16**	**2.064e-7**	**4.912e-7**	**4.879e-8**
0.9	**2.2204460492503e-16**	**3.330669073875e-16**	**5.589e-7**	**4.063e-7**	**2.472e-7**
1.0	**2.2204460492503e-16**	**2.220446049250e-16**	**5.887e-7**	**2.745e-7**	**7.36e-8**
$\|\|err\|\|_\infty$	**3.3306690738754e-16**	**3.330669073875e-16**	**6.735e-7**	**5.004e-7**	**4.506e-7**
R.T.	**2.099612464838e+01**	**3.326378526033e+00**	**-**	**-**	**-**

4. Conclusion:

In this work, we applied Lagrange polynomial, Barycentric Lagrange polynomial, and Modified Lagrange polynomial for solving the LVFIE of the second kind. According to the numerical results which obtain from the illustrative examples, we conclude that:

- The approximate solutions obtained by MATLAB software show the validity and efficiency of the proposed methods.
- The Barycentric Lagrange polynomial gives better accuracy than other polynomials.
- The faster method is Modified Lagrange polynomial.
- As n (the number of knots) increase, the error term is decreased in all the used polynomials.
- The methods can be extended and applied to nonlinear VFIE.

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