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Solving Continuous and Weakly Singular Linear Volterra Integral Equations of the second kind by Laplace Transform Method

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Abstract

This work provides solutions to some continuous and weakly singular linear Volterra integral equations of the second kind by Laplace transform method. With the basic definition of convolution integral of two functions and Volterra fundamental theorems, the Laplace transform method gives an efficient and remarkable performance. Test problems are presented to show the efficiency and reliability of the method.

Keywords: Volterra Integral equations, continuous and weakly singular Kernels , Laplace method

1. Introduction

Vito Volterra,an Italian Mathematician(1860-1940) was the first Mathematician to work on Integral equations which are now called Volterra Integral equations. In his work, he came out with useful fundamental theorems that are still useful till today. Mathematical formulations of physical phenomena from fluid dynamics, biological models and chemical kinetics also result in integral and integro-differential equations.

Cooke[1],in his work on epidemic equation with immigration, formulated a model of single-specie population growth in which there is immigration into the population at any prescribed rate and age distribution. His model resulted in non linear,,non homogeneous integral equation with delay which is also shown to be model for growth of capital and certain epidemics. Orsi[2],also worked on product integration of Volterra integral equations of the second kind with weakly singular kernels. He introduced a new numerical approach for solving Volterra equation of the second kind when the kernel contains a mild singularity. He gave a convergence result and presented numerical examples which show the performance and efficiency of the method. Baratella and Orsi [3] worked on a new approach to the numerical solution of weakly singular Volterra integral equations of the second kind. By transformation of the unknown function, they obtain another weakly singular equation whose solution is smooth when solved by standard product integration method. Odibat[4] solved Volterra equations with separable kernels by using differential transform method . Approximate solutions of the equations are calculated in form of series with easily computable terms. The exact solutions of the linear and non linear integral equations are investigated and the results illustrate the reliability and performance of the method.

Geng and Shen[5] also solved a Volterra integral equation with weakly singular kernel in the producing kernel space. Exact solutions are presented in the producing kernel space in the form of series. They obtained the nth term approximation $u_n(t)$ of the exact solution u(t) and also gave some numerical examples to demonstrate the accuracy of their method . They compared the result from their method with the exact solution and found it to be in good agreement.

In this work, we provide solutions to some continuous and weakly singular linear Volterra integral equations(V.I.E) of the second kind by by using Laplace transform method.

2. Mathematical Formulation

Definition 2.1

A linear Volterra integral equation(VIE) of the second kind is a functional equation of the form

$u(t) = f(t) + \int_0^\infty K(t,s)u(s)ds, t \in I := [0,T]$.......(2.1.1)

where $f(t), K(t,s)$ are given functions and $u(t)$ is an unknown function. The function $K(t,s)$ is called the kernel of the Volterra integral equation.

Definition 2.2

The Laplace transform of a function f(t), is defined by

$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)dt$..............(2.2.1)

Whenever integral in (2.2.1) exists (converges) or does not exist(diverges) where, t ≥0, s is real and $\mathcal{L}$ is the Laplace transform operator.

Definition 2.3
Let $f(t)$ and $g(t)$ be piecewise continuous functions on $[0,\infty]$,then the convolution integral of $f(t)$ and $g(t)$ is

$f(t)*g(t) == \int_0^\infty f(t-s)g(s)ds,$……….(2.3.1)

Note that $f(t)*g(t) = \int_0^\infty f(t-s)g(s)ds = \int_0^t g(t-s)f(s)ds = g(t)*f(t)$….(2.3.2)

Now if $\mathcal{L}[f(t)] = F(s)$ and $\mathcal{L}[G(t)] = G(s)$ then
$\mathcal{L}[f(t)*g(t)] = F(s)*G(s) = \mathcal{L}[f(t)]*\mathcal{L}[g(t)]$
and $\mathcal{L}^{-1}[F(s)*G(s))] = f(t)*g(t) = \int_0^t f(t-s)g(s)ds$ ……….(2.3.3)

Volterra Theorem 2.4
Assume that the kernel $K(t,s)$ of the linear Volterra integral equation (2.1.1) is continuous on $D := \{(t,s) : 0 \le s \le t \le T\}$. Then, for any function $g(t)$ that is continuous on I ,then the Volterra integral equation possesses a unique solution $u(t) \in C(I)$ which can be written in the form

$$u(t) = f(t) + \int_0^\infty R(t,s)f(s)ds, t \in I$$

for some $R \in C(D)$ .The function $R = R(t,s)$ is called the **resolvent kernel** of the given kernel $K(t,s)$

Definition 2.5
A Volterra integral equation with weakly singular kernels is defined as

$u(t) = f(t) + \int_0^\infty K_\alpha(t,s)u(s)ds, 0 < \alpha < 1$ ……….(2.5.1)

The kernel $K_\alpha = (t-s)^{-\alpha}K(t,s)$ is a weakly singular kernel called the Integrable kernel. It is bounded when $t = s$ but its integral over any bounded interval $[0,T]$ is finite. Also, $K(t,s)$ is continuous on D and satisfies $K(t,t) \ne 0, t \in I$

Remark 2.6
Volterra integral equations with weakly singular kernels $K_\alpha(t,s) = (t-s)^{-\alpha}K(t,s), 0 < \alpha < 1$ are called Abel Integral equations. It is named after a Norwegian Mathematician, Niels Henrik Abel(1802-1829) who was the first Mathematician to study such integrals.

Volterra Theorem 2.7
Let $0 < \alpha < 1$ and assume that $g \in C^d(I), K \in C^d(D)$ for some $d \ge 0$

(a) If $d = 0$, then the Volterra integral equation (2.5.1), $t \in I$, possesses a unique solution $u(t) \in C(I)$ which can be written in the form

$u(t) = f(t) + \int_0^\infty R_\alpha(t,s)f(s)ds, t \in I := [0,T]$ …….(2.7.1)

where the resolvent kernel $R_\alpha(t,s)$ of the kernel $K_\alpha(t,s)$ has the form

$R_\alpha(t,s) = (t-s)^{-\alpha}Q_\alpha(t,s)$ and $Q_\alpha(t,s)$ is continuous on D.

(b) If $d \ge 1$, then every non trivial solution has the property that $u(t) \notin C^1(I)$ and as $t \to 0^+$,the solution behaves like $u'(t) \sim Ct^{-\alpha}$

Corollary 2.8
Assume that $K \in C(I)$,then for any given $g \in C(I)$, the Volterra integral equation

$u(t) = f(t) + \int_0^\infty K(t-s)f(s)ds, t \in I$ ………(2..8.1)

Possesses a unique solution given by

$u(t) = f(t) + \int_0^\infty r(t-s)f(s)ds, t \in I$ ……….(2.8.2)

Where the resolvent kernel $R(t,s)$ of $K(t,s) = K(t-s)$ has the convolution form $R(t,s) = r(t-s)$

3. Method of Solution

Consider a linear Volterra Integral equation is of the second kind

$$a_i u + \sum_{i=0}^{k} b_i \int_a^t K_i(t,s)u(s)ds + f(t) = 0 \tag{3.1}$$

where f and the kernel, $K(t,s)$, of the integral is defined on the triangle $a \leq u \leq t \leq b$) are given, and u is sought with prescribed conditions and $a_i, b_i\, are\ constant.$ When $K(t,s)$ and $f(t)$ are continuous, then (3.1) has a unique continuous solution as in Theorem (2.4).

Then, using definition (2.3), we obtain

$$a_i \mathcal{L}[u] + \sum_{i=0}^{k} b_i \mathcal{L}[K_i(t,s) * \frac{d^i u}{dt^i}] + \mathcal{L}[f(t)] = 0 \tag{3.1}$$

$$a_i \bar{u}[s] + \sum_{i=0}^{k} b_i \bar{K}_i(s) * \frac{d^i \overline{u_i}(t,s)}{dt^i}] + f(t)] = 0 \tag{3.2}$$

Replacing the function $\bar{u}[s]\ by\ F(s)$ in (3.2) and solving the algebraic equations for the Laplace transforms to obtain u(t) which is obtained by taking the inverse Laplace transform of $F(s)$ to obtain a solution of (3.1)

Test Problem1: Solve the linear Volterra Integral Equation with continuous kernel

$$f(t) = e^t + 2\int_0^t cos(x-t)f(t)dt$$

Taking the Laplace transform of both sides

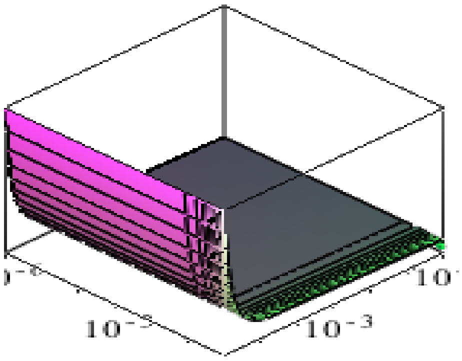
$$\mathcal{L}[f(t)] = \mathcal{L}[e^t] + 2 * F(s) * \mathcal{L}[\cos\,(t)]$$

$$F[s] = \frac{1}{s-1} + \frac{2F(s)*s}{s^2+1} \tag{3.3}$$

Solving (3.3) to obtain $F(s) = \frac{s^2+1}{(s-1)(s^2-2s+1)}$ (3.4)

Using definition (2.4) to obtain the inverse $\mathcal{L}^{-1}$ of (3.4) as

$$u(t) = e^t(t+1)^2 \tag{3.5}$$



Test Problem2: Solve the linear Volterra Integral Equation with continuous kernel

$$f(t) = \cos\,(t) - \int_0^t (x-t)cos(x-t)f(t)dt$$

Taking the Laplace transform of both sides,

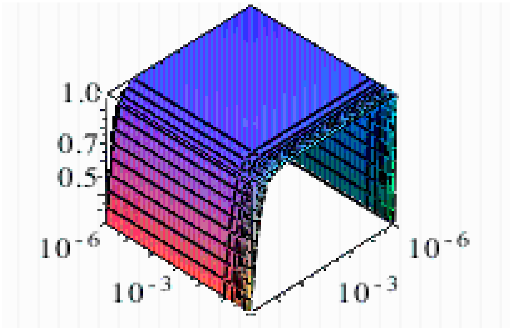
$$\mathcal{L}[f(t)] = \mathcal{L}[\cos(t)] - \mathcal{L}[t * \cos(t)] * F(s)]$$

$$F(s) = \frac{s}{s^2+1} - \frac{F(s)\,(s^2-1)}{(s^2+1)^2} \tag{3.6}$$

Solving (3.6) to obtain $F(s) = \frac{s^2+1}{s^2(s+3)}$ (3.7)

Using definition (2.4) to get the inverse of $\mathcal{L}^{-1}$ of (3.7) as

$$u(t) = \frac{2}{3}\cos\,(\sqrt{3t}) + \frac{1}{3} \tag{3.8}$$



Test Problem 3: Solve the weakly singular Kernel(Abel's Equation)

$$\int_0^t \frac{f(t)}{(x-t)^{1/3}}\,dt = \frac{\pi}{2}t + t^2$$

Taking the Laplace transform of both sides

$$\mathcal{L}\left[t^{-\frac{1}{3}}\right] * F(s)] = \mathcal{L}[\frac{\pi}{2}t + t^2\Big] \qquad (3.7)$$

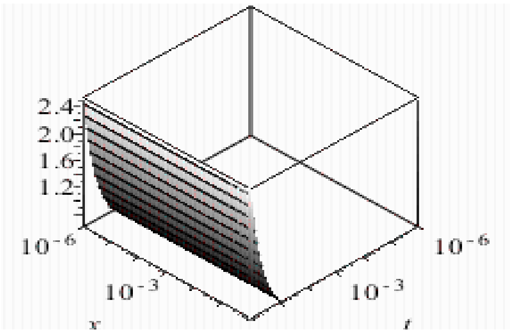
$$\frac{F(s)\Gamma(\frac{2}{3})}{s^{\frac{2}{3}}} = \frac{\pi}{2s^2} + \frac{2}{s^3} \qquad (3.8)$$

(3.9)

Solving (3.8) to obtain $F(s) = \frac{1}{2}\frac{\pi s+4}{s^{\frac{7}{3}}\Gamma(\frac{2}{3})}$

Using definition (2.4) to get the inverse of $\mathcal{L}^{-1}$ of (3.9) as

$$u(t) = \frac{3}{4}\frac{t^{1/3}}{\pi}\sqrt{3}(3t+\pi)$$



4. Conclusions and Discussion

In this work, we have successfully applied Laplace transform method to find the exact solutions of linear Volterra equations with continuous and weakly singular kernels. Some test problems are presented and our results show a remarkable performance and this shows the reliability and efficiency of the method. Hence, this method is applicable to many linear Volterra integral equations of the second kind with continuous and weakly singular kernels

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