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Approximation of Entire Function of Slow Growth in Several Complex Variables

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Abstract
In the present paper, we study the polynomial approximation of entire function in Banach space ($B(p,q,k)$ space, Hardy space and Bergman space). The coefficient characterizations of generalized type of entire function of slow growth in several complex variables have been obtained in terms of the approximation errors.
Keyword: Entire function, generalized order, generalized type, approximation error.

1.Introduction

Let $f(z)=\sum_{n=0}^{\infty} a_n z^n$ be an entire function and $M(r,f)=\max_{|z|=r}|f(z)|$ be its maximum modulus. The growth of $f(z)$ is measured in terms of its order ρ and type τ defined as under

$$\lim_{r\to\infty}\sup\frac{\ln\ln M(r,f)}{r^{\rho}}=\rho \tag{1.1}$$

$$\lim_{r\to\infty}\sup\frac{\ln\ln M(r,f)}{r^{\rho}}=\tau \tag{1.2}$$

for $0<\rho<\infty$. Various workers have given different characterizations for entire function of fast growth ($\rho=\infty$). M. N. Seremeta [6] defined the generalized order and generalized type with the help of general functions as follows.

Let L^0 denoted the class of functions h satisfying the following conditions
(i) $h(x)$ is defined on [a,∞) and is positive, strictly increasing, differentiable and tend to ∞ as $x\to\infty$,
(ii)

$$\lim_{x\to\infty}\frac{h\{(1+1/\psi(x))x\}}{h(x)}=1,$$

for every function $\psi(x)$ such that $\psi(x)\to\infty$ as $x\to\infty$.
Let Λ denoted the class of function h satisfying condition (i) and

$$\lim_{x\to\infty}\frac{h(cx)}{h(x)}=1$$

for every c > 0, that is, $h(x)$ is slowly increasing.

For the entire function $f(z)$ and function $\alpha(x)\in\Lambda$, $\beta(x)\in L^0$, the generalized order of an entire function in the terms of maximum modulus is defined as

$$\rho(\alpha,\beta,f)=\lim_{r\to\infty}\sup\frac{\alpha[\ln M(r,f)]}{\beta(\ln r)}. \tag{1.3}$$

Further, for $\alpha(x)\in L^0$, $\beta^{-1}(x)\in L^0$, $\gamma(x)\in L^0$, generalized type of an entire function f of finite generalized order ρ is defined as

$$\tau(\alpha,\beta,f)=\lim_{r\to\infty}\sup\frac{\alpha[\ln M(r,f)]}{\beta\left[(\gamma(r))^{\rho}\right]}. \tag{1.4}$$

where $0 < \rho < \infty$ is a fixed number.

Above relation were obtained under certain conditions which do not hold if $\alpha = \beta$. To overcome this difficulty, G. P. Kapoor and Nautiyal [4] defined generalized order $\rho(\alpha; f)$ of slow growth with the help of general functions as follows

Let Ω be the class of functions $h(x)$ satisfying (i) and

(iv) there exists a $\delta(x) \in \Omega$ and x_0, K_1 and K_2 such that

$$0 < K_1 \le \frac{d(h(x))}{d(\delta(\log x))} \le K_2 < \infty \text{ for all } x > x_0.$$

Let $\overline{\Omega}$ be the class of functions $h(x)$ satisfying (i) and

(v)

$$\lim_{x\to\infty} \frac{d(h(x))}{d(\log x)} = K, \quad 0 < \mathrm{K} < \infty.$$

Kapoor and Nautiyal [4] showed that class Ω and $\overline{\Omega}$ are contained in Λ. Further, $\Omega \cap \overline{\Omega} = \phi$ and they defined the generalized order $\rho(\alpha; f)$ for entire function $f(z)$ of slow growth as

$$\rho(\alpha; f) = \lim_{r\to\infty} \sup \frac{\alpha(\ln M(r, f))}{\alpha(\ln r)},$$

where $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$.

Let $f(z_1, z_2, \ldots, z_n)$ be an entire function, $z = (z_1, z_2, \ldots, z_n) \in C^n$. Let G be a full region in R_+^n (positive hyper octant). Let $G_R \subset C^n$ denoted the region obtained from G by a similarity transformation about the origin, with ratio of similitude R. let $d_t(G) = \sup_{z\in G} |z|^t$, where $|z|^t = |z_1|^{t_1} |z_2|^{t_2} \ldots |z_n|^{t_n}$, and let ∂G denoted the boundary of the region G. Let

$$f(z) = f(z_1, z_2, \ldots, z_n) = \sum_{t_1, t_2, \ldots, t_n = 0}^{\infty} a_{t_1 \ldots t_n} z_1^{t_1} \ldots z_n^{t_n} = \sum_{\|t\|=0}^{\infty} a_t z^t,$$

$\|t\| = t_1 + t_2 + \cdots + t_n$, be the power series expansion of the function $f(z)$. Let $M_G(R, f) = \max_{z \in G_R} |f(z)|$. To characterize the growth of f, order (ρ_G) and type (σ_G) of f are defined as [2]

$$\rho_G = \lim_{R\to\infty} \sup \frac{\ln \ln M_G(R, f)}{\ln R},$$

$$\sigma_G = \lim_{R\to\infty} \sup \frac{\ln M_G(R, f)}{R^{\rho_G}}.$$

For the entire function, $f(z) = \sum_{\|t\|=0}^{\infty} a_t z^t$, A. A. Gol'dberg [3, Th .1] obtained the order and type in terms of the coefficients of its Taylor expansion as

$$\rho_G = \lim_{\|t\|\to\infty} \frac{\|t\| \ln \|t\|}{-\ln |a_t|}. \tag{1.5}$$

$$(e \rho_G \sigma_G)^{1/\rho_G} = \lim_{\|t\|\to\infty} \sup \left\{ \|t\|^{1/\rho_G} [|a_t| d_t(G)]^{1/\|t\|} \right\}, (0 < \rho_G < \infty) \tag{1.6}$$

where $d_t(G) = \max_{r \in G} r^t; \quad r^t = r_1^{t_1} r_2^{t_2} \ldots r_n^{t_n}$.

for an entire function of several complex variables $f(z)=\sum_{\|t\|=0}^{\infty} a_t z^t$, and functions $\alpha(x)\in\Lambda$, $\beta(x)\in L^0$, Seremeta [6, Th .1] proved that

$$\rho=\limsup_{R\to\infty}\frac{\alpha[\ln M_G(R,f)]}{\beta(\ln R)}=\limsup_{\|k\|\to\infty}\frac{\alpha(\|t\|)}{\beta[-\frac{1}{\|t\|}\ln(|a_t|d_t(G))]}. \tag{1.7}$$

Further, for $\alpha(x)\in L^0$, $\beta^{-1}(x)\in L^0$, $\gamma(x)\in L^0$, Seremeta [6, Th .1] proved that

$$\sigma=\limsup_{R\to\infty}\frac{\alpha[\ln M_G(R,f)]}{\beta[(\gamma(R))^{\rho}]}=\limsup_{\|t\|\to\infty}\frac{\alpha(\frac{\|t\|}{\rho})}{\beta[(\gamma\{e^{1/\rho}[|a_t|d_t(G)]^{-1/\|t\|}\})^{\rho}]}. \tag{1.8}$$

where $0<\rho<\infty$ is a fixed number.

And let H_q' denote the Bergman space of functions $f(z)$ satisfying the condition

$$\|f\|_{H_q'}=\left\{\frac{1}{A}\iint_{z\in G}|f(z)|^q d\sigma_1 d\sigma_2\ldots d\sigma_n\right\}^{1/\rho}<\infty,$$

where $z=(z_1,z_2,\ldots,z_n)$, $d\sigma_j=dx_j dy_j$, $z_j=x_j+iy_j$, $j=1,2,\ldots,n.$ and A is the area of G. For q = ∞, let $\|f\|_{H_\infty'}=\|f\|_{H_\infty}=\sup\{|f(z)|,z\in U\}$. Then H_q and H_q' are Banach space for $q\geq 1$. In analogy with spaces of functions of one variable, we call H_q and H_q' the Hardy and Bergman spaces respectively.

The function $f(z)$ analytic in U belong to the space B(*p*, *q*, *k*), where $0<p<q\leq\infty$, and $0<k\leq\infty$, if

$$\|f\|_{p,q,k}=\left\{\int_0^1(1-R)^{k(1/p-1/q)-1}M_{q,G}^k(R,f)\,dR\right\}^{1/k}<\infty.$$

And

$$\|f\|_{p,q,\infty}=\sup\{(1-R)^{1/p-1/q}M_{q,G}(R,f);0<R<1\}<\infty.$$

It is known [1] that B(*p*, *q*, *k*) is a Banach space for *p* > 0 and $q,k\geq 1$, otherwise it is a Freachet space. Further , we have

$$H_q\subseteq H_q'=B(\frac{q}{2},q,q),\ 1\leq q<\infty.$$

Let $P_m=\{p:p=\sum_{\|t\|\leq m}a_t z^t\}$ be the class polynomials of degree at most m and let X denote one of the Banach spaces defined. Then we defined error of an entire function f on the region G as

$$E_{\|t\|}(f)=E_{\|t\|}(f,G,X)=\inf\{\|f-p\|_X:p\in P_m\}.$$

Vakarchuk and Zhir [5] obtained the characterizations of generalized order and generalized type of $f(z)$ in terms of the errors $E_{\|t\|}(f)$ defined above. These characterizations do not hold good when α = β = γ. i.e. for entire functions of slow growth. In this paper we have tried to fill this gap. We define the generalized type $\tau(\alpha;f)$ of an entire function $f(z)$ having finite generalized order as

$$\tau(\alpha;f)=\limsup_{r\to\infty}\frac{\alpha(\ln M_G(R,f))}{[\alpha(\ln R)]^{\rho}}.$$

Where $\alpha(x)$ either belongs to Ω or to $\overline{\Omega}$.

2. Main Results

Theorem 2.1: Let $\alpha(x)\in\overline{\Omega}$, then the entire function $f(z)$ of generalized order ρ, $1<\rho<\infty$, is of generalized type τ if and only if

$$\tau=\limsup_{R\to\infty}\frac{\alpha(\ln M_G(R,f))}{[\alpha(\ln R)]^{\rho}}=\limsup_{\|t\|\to\infty}\frac{\alpha\left(\frac{\|t\|}{\rho}\right)}{\{\alpha[\frac{\rho}{\rho-1}\ln\left(|a_t|d_t(G)\right)^{-1/\|t\|}]\}^{\rho-1}}, \tag{2.1}$$

provided $dF(x;\tau,\rho)/d\ln x=O(1)$ as $x\to\infty$ for all τ, $0<\tau<\infty$.

Proof. Let

$$\limsup_{R\to\infty}\frac{\alpha(\ln M_G(R,f))}{[\alpha(\ln R)]^{\rho}}=\tau .$$

We suppose $\tau<\infty$. Then for every $\varepsilon>0$, $\exists R(\varepsilon)\ni$

$$\frac{\alpha(\ln M_G(R,f))}{[\alpha(\ln R)]^{\rho}}\le\tau+\varepsilon=\bar{\tau},\quad \forall R\ge R(\varepsilon) .$$

$$\text{(or)}\qquad \ln M_G(R,f)\le(\alpha^{-1}\{\bar{\tau}[\alpha(\ln R)]^{\rho}\}) .$$

Choose $R=R(t)$ to be the unique root of the equation

$$t=\frac{\rho}{\ln R}F[\ln R;\ \bar{\tau},\frac{1}{\rho}]. \tag{2.2}$$

Then

$$\ln R=\alpha^{-1}[(\frac{1}{\bar{\tau}}\alpha(\frac{\|t\|}{\rho}))^{1/(\rho-1)}]=F[\frac{\|t\|}{\rho};\frac{1}{\bar{\tau}},\rho-1]. \tag{2.3}$$

By Cauchy's inequality,

$$|a_t|d_t(G)\le R^{-\|t\|}M_G(R,f)$$
$$\le\exp\{-\|t\|\ln R+(\alpha^{-1}\{\bar{\tau}[\alpha(\ln R)]^{\rho}\})\}$$

By using (2.5) and (2.6), we get

$$|a_t|d_t(G)\le\exp\{-\|t\|F+\frac{\|t\|}{\rho}F\}$$

or

$$\frac{\rho}{\rho-1}\ln(|a_t|d_t(G))^{-1/\|t\|}\ge\alpha^{-1}\{[(\frac{1}{\bar{\tau}}\alpha(\frac{\|t\|}{\rho}))^{1/(\rho-1)}]\}$$

or

$$\bar{\tau}=\tau+\varepsilon\ge\frac{\alpha(\frac{\|t\|}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\ln(|a_t|d_t(G))^{-1/\|t\|}]\}^{\rho-1}}.$$

Now proceeding to limits, we obtain

$$\tau\ge\limsup_{\|t\|\to\infty}\frac{\alpha(\frac{\|t\|}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\ln(|a_t|d_t(G))^{-1/\|t\|}]\}^{\rho-1}}. \tag{2.4}$$

Inequality (2.4) obviously holds when $\tau=\infty$.

Conversely, let

$$\limsup_{\|t\|\to\infty}\frac{\alpha(\frac{\|t\|}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\ln\left(|a_t|d_t(G)\right)^{-1/\|t\|}]\}^{\rho-1}}=\sigma .$$

Suppose $\sigma<\infty$. Then for every $\varepsilon>0$ and for all $\|t\|\ge N(\varepsilon)$, we have

$$\frac{\alpha(\frac{\|t\|}{\rho})}{\{\alpha[\frac{\rho}{\rho-1}\ln\left(|a_t|d_t(G)\right)^{-1/\|t\|}]\}^{\rho-1}} \le \sigma+\varepsilon=\overline{\sigma}$$

$$|a_t|\,d_t(G) \le \frac{1}{\exp\{(\rho-1)\frac{\|t\|}{\rho}F[\frac{\|t\|}{\rho};\frac{1}{\overline{\sigma}},\rho-1]}. \tag{2.5}$$

The inequality

$$\sqrt[\|t\|]{|a_t|\,d_t(G)\,R^{\|t\|}} \le \mathrm{Re}^{-\left(\frac{\rho-1}{\rho}\right)F[\frac{\|t\|}{\rho};\frac{1}{\overline{\sigma}},\rho-1]} \le \frac{1}{2} \tag{2.6}$$

is fulfilled beginning with some $\|t\|=m(R)$. Then

$$\sum_{\|t\|=m(R)+1}^{\infty}|a_t|\,d_t(G)\,R^{\|t\|} \le \sum_{\|t\|=m(R)+1}^{\infty}\frac{1}{2^{\|t\|}} \le 1. \tag{2.7}$$

We now express m® in terms of R. From inequality (2.6),

$$2R \le \exp\{(\frac{\rho-1}{\rho})F[\frac{\|t\|}{\rho};\frac{1}{\overline{\sigma}},\rho-1]\},$$

we can take $m(R)=E[\rho\alpha^{-1}\{\overline{\tau}(\alpha(\ln R+\ln 2))^{\rho-1}\}]$. We consider the function $\psi(x)=R^x\exp\{-(\frac{\rho-1}{\rho})\,x\,F[\frac{x}{\rho};\frac{1}{\overline{\sigma}},\rho-1]\}$. Let

$$\frac{\psi'(x)}{\psi(x)}=\ln R-(\frac{\rho-1}{\rho})\;F[\frac{x}{\rho};\frac{1}{\overline{\sigma}},\rho-1]-\frac{dF[\frac{x}{\rho};\frac{1}{\overline{\sigma}},\rho-1]}{d\ln x}=0. \tag{2.8}$$

As $x\to\infty$, by the assumption of the theorem, for finite $\sigma(0<\sigma<\infty)$,
$dF[x;\overline{\sigma},\rho-1]/d\ln x$ is bounded. So there is a $A>0$ such that for $x>x_1$ we have

$$\left|\frac{dF[\frac{x}{\rho};\frac{1}{\overline{\sigma}},\rho-1]}{d\ln x}\right| \le A. \tag{2.9}$$

We can take $A>\ln 2$. It is then obvious that inequalities (2.6) and (2.7) hold for $\|t\|\ge m_1(R)=E[\rho\alpha^{-1}\{\overline{\sigma}(\alpha(\ln R+\ln 2))^{(\rho-1)}\}]+1$. We let m_0 designate the number $\max(N(\varepsilon),E[x_1]+1)$. For $R>R_1(m_0)$ we have $\psi'(m_0)/\psi(m_0)>0$. From (2.9) and (2.8) it follows that $\psi'(m_1(R))/\psi(m_1(R))<0$. We hence obtain that if for $R>R_1(m_0)$ we let $x^*(R)$ designate the point where $\psi(x^*(R))=\max\limits_{m_0\le x\le m_1(R)}\psi(x)$, then

$$m_0<x^*(R)<m_1(R) \text{ and } x^*(R)=\rho\alpha^{-1}\{\overline{\sigma}(\alpha(\ln R-a(R)))^{\rho-1}\}.$$

Where

$$-A<a(R)=\frac{dF[\frac{x}{\rho};\frac{1}{\overline{\sigma}},\rho-1]}{d\ln x}\Big|_{x=x^*(R)}<A.$$

Further

$$\max_{m_0<\|t\|<m_1(R)}(|a_t|d_t(G)R^{\|t\|} \le \max_{m_0<x<m_1(R)}\psi(x)=\frac{R^{\rho\alpha^{-1}\{\overline{\sigma}(\alpha(\ln R-a(R)))^{\rho-1}\}}}{e^{\rho\alpha^{-1}\{\overline{\sigma}(\alpha(\ln R-a(R)))^{\rho-1}\}(\ln R-a(R))}}=$$

$$=\exp\{a(R)\,\rho\alpha^{-1}\{\overline{\sigma}(\alpha(\ln R-a(R)))^{\rho-1}\}\}\le$$

$$\le\exp\{A\rho\alpha^{-1}\{\overline{\sigma}(\alpha(\ln R+A))^{\rho-1}\}\}.$$

It is obvious that (for $R>R_1(m_0)$)

$$M_G(R,f)\le\sum_{\|t\|=0}^{\infty}|a_t|\,d_t(G)\,R^{\|t\|}=$$

$$= \sum_{\|t\|=0}^{m_0} |a_t| d_t(G) R^{\|t\|} + \sum_{\|t\|=m_0+1}^{m_1(R)} |a_t| d_t(G) R^{\|t\|} + \sum_{\|t\|=m_1(R)+1}^{\infty} |a_t| d_t(G) R^{\|t\|}$$

$$\leq O(R^{m_0}) + m_1(R) \max_{m_0 < \|t\| < m_1(R)} (|a_t| d_t(G) R^{\|t\|}) + 1$$

$$M_G(R, f)(1+o(1)) \leq \exp\{(A\rho + o(1))\alpha^{-1}[\overline{\sigma}(\alpha(\ln R + A))^{\rho-1}]\}$$

$$\alpha(\ln M_G(R, f)) \leq \overline{\sigma}[\alpha(\ln R + A)]^{\rho-1} \leq \overline{\sigma}[\alpha(\ln R + A)]^{\rho}.$$

We then have

$$\frac{\alpha[(A\rho + o(1))^{-1} \ln M_G(R, f)]}{[\alpha(\ln R + A)]^{\rho}} \leq \overline{\sigma} = \sigma + \varepsilon.$$

Since $\alpha(x) \in \overline{\Omega} \subseteq \Lambda$, now proceeding to limits we obtain

$$\limsup_{R\to\infty} \frac{\alpha(\ln M_G(R, f))}{[\alpha(\ln R)]^{\rho}} \leq \sigma. \tag{2.10}$$

From inequality (2.4) and (2.10), we get the required the result.

Now we prove

Theorem 2.2: Let $\alpha(x) \in \overline{\Omega}$, then a necessary and sufficient condition for an entire function $f(z) \in B(p,q,k)$ to be of generalized type τ having finite generalized order ρ, $1 < \rho < \infty$ is

$$\tau = \limsup_{\|t\|\to\infty} \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1} \ln((E_{\|t\|}(B(p,q,k))\, d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}}. \tag{2.11}$$

Proof. First we consider the space $B(p,q,k)$, $q = 2$, $0 < \rho < \infty$ and $k \geq 1$. Let $f(z) \in B(p,q,k)$ be of generalized type τ with generalized order ρ. Then from the Theorem 2.1, we have

$$\limsup_{\|t\|\to\infty} \frac{\alpha(\frac{\|t\|}{\rho})}{\{\alpha[\frac{\rho}{\rho-1} \ln(|a_t| d_t(G))^{-1/\|t\|}]\}^{\rho-1}} = \tau. \tag{2.12}$$

For a given ε > 0, and all $\|t\| > m = m(\varepsilon)$, we have

$$|a_t| d_t(G) \leq \frac{1}{\exp\{(\rho-1)\frac{\|t\|}{\rho} F[\frac{\|t\|}{\rho}; \frac{1}{\tau}, \rho-1]\}}. \tag{2.13}$$

Let $g_t(f,z) = \sum_{j=0}^{\|t\|} a_j z^j$ be the t^{th} partial sum of the Taylor series of the function $f(z)$. Following [5, p.1396], we get

$$E_{\|t\|}(B(p,2,k); f) \leq B^{1/k}[(\|t\|+1)k+1; k(1/p - 1/2)]\{\sum_{\|j\|=\|t\|+1}^{\infty} (|a_j| d_j(G))^2\}^{1/2} \tag{2.14}$$

where $B(a,b)$ $(a,b > 0)$ denotes the beta function. By using (2.13), we have

$$E_{\|t\|}(B(p,2,k); f) \leq \frac{B^{1/k}[(\|t\|+1)k+1; k(1/p-1/2)]}{\exp\{(\rho-1)\frac{\|t\|+1}{\rho} F[\frac{\|t\|+1}{\rho}; \frac{1}{\tau}, \rho-1]\}} \{\sum_{\|j\|=\|t\|+1}^{\infty} \psi_j^2(\alpha)\}^{1/2}, \tag{2.15}$$

where

$$\psi_j(\alpha) \cong \frac{\exp\{\frac{\|t\|+1}{\rho}(\rho-1)[\alpha^{-1}\{(\frac{\alpha(\frac{\|t\|+1}{\rho})}{\tau+\varepsilon})^{1/(\rho-1)}\}]\}}{\exp\{\frac{\|j\|}{\rho}(\rho-1)[\alpha^{-1}\{(\frac{\alpha(\frac{\|j\|}{\rho})}{\tau+\varepsilon})^{1/(\rho-1)}\}]\}}.$$

Set

$$\psi(\alpha) \cong \exp\{-\frac{(\rho-1)}{\rho}[\alpha^{-1}\{(\frac{\alpha(\frac{1}{\rho})}{\tau+\varepsilon})^{1/(\rho-1)}\}]\}.$$

Since $\alpha(x)$ is increasing and $\|j\| \geq \|t\|+1$, we get

$$\psi_j(\alpha) \leq \exp\{\frac{((\|t\|+1)-\|j\|}{\rho}(\rho-1)[\alpha^{-1}\{(\frac{\alpha(\frac{\|t\|+1}{\rho})}{\tau+\varepsilon})^{1/(\rho-1)}\}]\} \leq \psi^{\|j\|-(\|t\|+1)}(\alpha). \tag{2.16}$$

Since $\psi(\alpha) < 1$, we get from (2.15) and (2.16),

$$E_{\|t\|}(B(p,2,k);f) \leq \frac{B^{1/k}[(\|t\|+1)k+1;k(1/p-1/2)]}{(1-\psi^2(\alpha))^{1/2}[\exp\{(\rho-1)\frac{\|t\|+1}{\rho}[\alpha^{-1}\{(\frac{\alpha(\frac{\|t\|+1}{\rho})}{\tau+\varepsilon})^{1/(\rho-1)}\}]\}]}. \tag{2.17}$$

For $\|t\| > m$, (2.17) yields

$$\tau+\varepsilon \geq \frac{\alpha(\frac{\|t\|+1}{\rho})}{\{\alpha(\frac{\rho}{(1+\frac{1}{\|t\|})(\rho-1)}\{\ln((E_{\|t\|}\,d_t)^{-1/\|t\|})+\ln(\frac{B^{1/k}[(\|t\|+1)k+1;k(1/p-1/2)]}{(1-\psi^2(\alpha))^{1/2}})^{1/\|t\|}\})\}^{(\rho-1)}}$$

Now

$$B[(\|t\|+1)k+1;k(1/p-1/2)] = \frac{\Gamma((\|t\|+1)k+1)\Gamma(k(1/p-1/2))}{\Gamma((\|t\|+1/2+1/p)k+1)}.$$

Hence

$$B[(\|t\|+1)k+1;k(1/p-1/2)] \simeq \frac{e^{-[(\|t\|+1)k+1]}[(\|t\|+1)k+1]^{(\|t\|+1)k+3/2}\Gamma(1/p-1/2)}{e^{[(\|t\|+1/2+1/p)k+1]}[(\|t\|+1/2+1/p)k+1]^{(\|t\|+1/2+1/p)k+3/2}}.$$

Thus

$$\{B[(\|t\|+1)k+1;k(1/p-1/2)]\}^{1/(\|t\|+1)} \cong 1. \tag{2.18}$$

Now proceeding to limits, we obtain

$$\tau \geq \limsup_{\|t\|\to\infty} \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln((E_{\|t\|}\,d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}} \tag{2.19}$$

For reverse inequality, by [5, p.1398], we have

$$|a_{t+1}|B^{1/k}[(\|t\|+1)k+1;k(1/p-1/2)] \leq E_{\|t\|}(B(p,2,k);f). \tag{2.20}$$

Then for sufficiently large $\|t\|$, we have

$$\frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln((E_{\|t\|}\,d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}} \geq$$

$$\frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\{\ln((|a_{t+1}|\,d_t(G))^{-1/\|t\|})+\ln(B^{-\rho/\|t\|k}[(\|t\|+1)k+1;(1/p-1/2)])\}\}]^{(\rho-1)}}$$

$$\geq \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\{\ln((|a_t|\,d_t(G))^{-1/\|t\|})+\ln(B^{-\rho/\|t\|k}[(\|t\|+1)k+1;(1/p-1/2)])\}\}]^{(\rho-1)}}.$$

By applying limits and from (2.12), we obtain

$$\limsup_{\|t\|\to\infty} \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln((E_{\|t\|}\,d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}} \geq \tau. \tag{2.21}$$

From (2.19) and (2.21), we obtain the required relation

$$\limsup_{\|t\|\to\infty} \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln((E_{\|t\|}\, d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}} = \tau. \tag{2.22}$$

In the second step, we consider the spaces $B(p,q,k)$ for $0 < p < q, q \neq 2,$ and $q,k \geq 1.$ Gvaradze [1] showed that, for $p \geq p_1, q \geq q_1$ and $k \leq k_1$, if at least one of the inequalities is strict, then the strict inclusion $B(p,q,k) \subset B(p_1,q_1,k_1)$ holds and the following relation is true:

$$\|f\|_{p_1,q_1,k_1} \leq 2^{1/q-1/q_1}[k(1/p-1/q)]^{1/k-1/k_1}\|f\|_{p,q,k}.$$

For any function $f(z) \in B(p,q,k)$, the last relation yields

$$E_{\|t\|}(B(p_1,q_1,k_1); f) \leq 2^{1/q-1/q_1}[k(1/p-1/q)]^{1/k-1/k_1} E_{\|t\|}(B(p,q,k); f). \tag{2.23}$$

For the general case $B(p,q,k), \quad q \neq 2,$ we prove the necessity of condition (2.13).

Let $f(z) \in B(p,q,k)$ be an entire transcendental function having finite generalized order $\rho(\alpha; f)$ whose generalized type is defined by (2.12). Using the relation (2.13), for $\|t\| > m$ we estimate the value of the best polynomial approximation as follows

$$E_{\|t\|}(B(p,q,k); f) = \|f - g_t(f)\|_{p,q,k} \leq (\int_0^1 (1-R)^{(k(1/p-1/q)-1}M_{q,G}^k\, dR)^{1/k}.$$

Now

$$|f|^q = \left|\sum a_t z^t\right|^q \leq (\sum |a_t r^t|)^q \leq (r^{\|t\|+1}\sum_{\|k\|=\|t\|+1}^{\infty}|a_k|)^q.$$

Hence

$$E_{\|t\|}(B(p,q,k); f)\, d_t(G) \leq B^{1/k}[(\|t\|+1)k+1; k(1/p-1/2)] \sum_{\|k\|=\|t\|+1}^{\infty}|a_k|$$

$$\leq \frac{B^{1/k}[(\|t\|+1)k+1; k(1/p-1/q)]}{(1-\psi(\alpha))[\exp\{\frac{\|t\|+1}{\rho}(\rho-1)[\alpha^{-1}\{(\frac{\alpha(\frac{\|t\|+1}{\rho})}{\tau+\varepsilon})^{1/(\rho-1)}\}]\}]}. \quad (2.24)$$

For $\|t\| < m$, (2.24) yields

$$\tau + \varepsilon \geq \frac{\alpha(\frac{\|t\|+1}{\rho})}{\{\alpha(\frac{\rho}{(1+\frac{1}{\|t\|})(\rho-1)}\{\ln((E_{\|t\|}\, d_t(G))^{-1/\|t\|}) + \ln(\frac{B^{1/k}[(\|t\|+1)k+1; k(1/p-1/2)]}{(1-\psi(\alpha))})^{1/\|t\|}\})\}^{(\rho-1)}}$$

Since $\psi(\alpha) < 1$, and $\alpha \in \overline{\Omega}$, proceeding to limits and using (2.18), we obtain

$$\tau \geq \limsup_{\|t\|\to\infty} \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln((E_{\|t\|}\, d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}}.$$

For the reverse inequality, let $0 < p < q < 2$ and $k, q \geq 1$. By (2.23), where $p_1 = p, q_1 = 2,$ and $k_1 = k,$ and the condition (2.13) is already proved for the space $B(p,2,k)$, we get

$$\limsup_{\|t\|\to\infty} \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln((E_{\|t\|}(B(p,q,k); f)\, d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}}$$

$$\geq \limsup_{\|t\|\to\infty} \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln((E_{\|t\|}(B(p,2,k); f)\, d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}} = \tau.$$

Now let $0 < p \leq 2 < q$. Since we have

$$M_{2,G}(R,f)\le M_{q,G}(R,f),\quad 0<R<1,$$

therefore

$$E_{\|t\|}(B(p,q,k);f)\ge \left|a_{\|t\|+1}\right| B^{1/k}[(\|t\|+1)k+1;k(1/p-1/q)]. \tag{2.25}$$

Then for sufficiently large $\|t\|$, we have

$$\frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln((E_{\|t\|}\,d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}}\ge$$

$$\frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\{\ln((\left|a_{\|t\|+1}\right| d_t(G))^{-1/\|t\|})+\ln(B^{-\rho/\|t\|k}[(\|t\|+1)k+1;k(1/p-1/q)])\}\}]^{(\rho-1)}}$$

$$\ge \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\{\ln((\left|a_{\|t\|}\right| d_t(G))^{-1/\|t\|})+\ln(B^{-\rho/\|t\|k}[(\|t\|+1)k+1;k(1/p-1/q)])\}\}]^{(\rho-1)}}.$$

By applying limits and from (2.12), we obtain

$$\limsup_{\|t\|\to\infty}\frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln((E_{\|t\|}\,d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}}\ge \limsup_{\|t\|\to\infty}\frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln((\left|a_{\|t\|}\right| d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}}=\tau.$$

Now we assume that $2\le p<q$. Set $q_1=q, k_1=k$, and $0<p_1<2$ in the inequality (2.23), where p_1 is an arbitrary fixed number. Substituting p_1 for p in (2.25), we get

$$E_{\|t\|}(B(p,q,k);f)\ge \left|a_{\|t\|+1}\right| B^{1/k}[(\|t\|+1)k+1;k(1/p_1-1/q)]. \tag{2.26}$$

Using (2.26) and applying the same analogy as in the previous case $0<p\le 2<q$, for sufficiently large $\|t\|$, we have

$$\frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln((E_{\|t\|}\,d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}}\ge$$

$$\frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\{\ln((\left|a_{\|t\|+1}\right| d_t(G))^{-1/\|t\|})+\ln(B^{-\rho/\|t\|k}[(\|t\|+1)k+1;k(1/p_1-1/q)])\}\}]^{(\rho-1)}}$$

$$\ge \frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\{\ln((\left|a_{\|t\|}\right| d_t(G))^{-1/\|t\|})+\ln(B^{-\rho/\|t\|k}[(\|t\|+1)k+1;k(1/p-1/q)])\}\}]^{(\rho-1)}}$$

By applying limits and using (2.12), we obtain

$$\limsup_{\|t\|\to\infty}\frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln((E_{\|t\|}\,d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}}\ge\tau.$$

From relation (2.19) and (2.21), and the above inequality, we obtain the required relation (2.22).

Theorem 2.3: Assuming that the condition of Theorem 2.2 are satisfied and $\xi(\alpha)$ is a positive number, a necessary and sufficient condition for a function $f(z)\in H_q$ to be an entire function of generalized type $\xi(\alpha)$ having finite generalized order ρ is that

$$\limsup_{\|t\|\to\infty}\frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln((E_{\|t\|}(H_q;f)\,d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}}=\xi(\alpha). \tag{2.27}$$

Proof. Let $f(z)=\sum_{\|t\|=0}^{\infty} a_t z^t$ be an entire transcendental function having finite generalized order ρ and generalized type τ. Since

$$\lim_{\|t\|\to\infty} \sqrt[\|t\|]{|a_t|}=0 \tag{2.28}$$

$f(z)\in B(p,q,k)$, where $0<p<q\le\infty$ and $q,k\ge 1$. From relation (1.8), we get

$$E_{\|t\|}(B(q/2,q,q);f)\le \varsigma_q E_{\|t\|}(H_q;f), \quad 1\le q<\infty. \tag{2.29}$$

where ς_q is a constant independent of $\|t\|$ and f. In the case of Hardy space H_∞,

$$E_{\|t\|}(B(p,\infty,\infty);f)\le E_{\|t\|}(H_\infty;f), \quad 1<p<\infty. \tag{2.30}$$

Since

$$\xi(\alpha;f)=\limsup_{\|t\|\to\infty}\frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln((E_{\|t\|}(H_q;f)\,d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}}$$

$$\ge \limsup_{\|t\|\to\infty}\frac{\alpha(\frac{\|t\|}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}\ln((E_{\|t\|}((B(q/2,q,q);f)\,d_t(G))^{-1/\|t\|})\}]^{(\rho-1)}}$$

$$\ge\tau, \quad 1\le q<\infty. \tag{2.31}$$

Using estimate (2.30) we prove inequality (2.31) in the case $q=\infty$.
For the reverse inequality

$$\xi(\alpha;f)\le\tau, \tag{2.32}$$

we use the relation (2.13), which is valid for $\|t\|>m$, and estimate from above, the generalized type τ of an entire transcendental function $f(z)$ having finite generalized order ρ, as follows. We have

$$E_{\|t\|}(H_q;f)\le\|f-g_t\|_{H_q}$$

$$\le\sum_{\|j\|=\|t\|+1}^{\infty}|a_j|$$

$$\le\frac{1}{[\exp\{(\rho-1)\frac{\|t\|+1}{\rho}[\alpha^{-1}\{(\frac{\alpha(\frac{\|t\|+1}{\rho})}{\tau+\varepsilon})^{1/(\rho-1)}\}]\}]}\sum_{\|j\|=\|t\|+1}^{\infty}\psi_j(\alpha)$$

Using (2.16),

$$E_{\|t\|}(H_q;f)\le\|f-g_t\|_{H_q}$$

$$\le\frac{1}{d_t(G)(1-\psi(\alpha))[\exp\{(\rho-1)\frac{\|t\|+1}{\rho}[\alpha^{-1}\{(\frac{\alpha(\frac{\|t\|+1}{\rho})}{\tau+\varepsilon})^{1/(\rho-1)}\}]\}]}.$$

$$\frac{1}{E_{\|t\|}(H_q;f)\,d_t(G)}\ge(1-\psi(\alpha))[\exp\{(\rho-1)\frac{\|t\|+1}{\rho}[\alpha^{-1}\{(\frac{\alpha(\frac{\|t\|+1}{\rho})}{\tau+\varepsilon})^{1/(\rho-1)}\}]\}]$$

This yields

$$\tau+\varepsilon\ge\frac{\alpha(\frac{\|t\|+1}{\rho})}{[\alpha\{\frac{\rho}{\rho-1}[\ln((E_{\|t\|}(H_q;f)\,d_t(G))^{-1/\|t\|+1})+\ln((1-\psi(\alpha))^{-1/\|t\|+1})]\}]^{(\rho-1)}}. \tag{2.33}$$

Since $\psi(\alpha)<1$ and by applying the properties of the function α, passing to the limit as $\|t\|\to\infty$ in (2.33), we obtain inequality (2.32). thus we have finally

$$\xi(\alpha)=\tau. \tag{2.34}$$

This prove Theorem 2.2.

Remark : An analog of Theorem 2.3 for the Bergman space follows from (1.8) for $1 \le q < \infty$ and from Theorem 2.2 for $q = \infty$.

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