Approximation Method for the Heat Equation with Derivative Boundary Conditions

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Abstract

In this paper, modification of Adomian decomposition method is introduced for solving heat equation with derivative boundary conditions. Some examples and the obtained results demonstrate efficiency of the proposed method.

Keywords: Modified decomposition method, Heat equation, Derivative boundary conditions.

1. Introduction

Many classes of linear and nonlinear differential equations can solve using Adomian decomposition method and in computation and faster in convergence it is much simpler than any other method available in the open literature.

Many Authors have proposed numerical methods for solving problems [1-10]. Later A. Cheniguel [11], present Adomian decomposition for solving non-homogeneous heat equation with derivative boundary conditions. In this work we propose a new technique based on the modification of Adomian decomposition method.

2. Solution Heat Equation by Modified Adomian's Decomposition Method

In this section, we present modified decomposition method for solving the heat equation with derivative boundary conditions:

$$D_t u(x,t) = D_{xx} u(x,t) + q(x,t) \tag{1}$$

$$u(x,0) = f(x), 0 \le x \le 1 \tag{2}$$

$$u(0,t) = g_1(t), 0 < t \le T \tag{3}$$

$$u(1,t) = g_2(t), 0 < t \le T \tag{4}$$

Where f, g_1, g_2 and q are known functions, T is given constant.

We start with Adomian decomposition method

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

Now, Equation (1) can be rewritten as

$$L_t u(x,t) = L_{xx} u(x,t) + q(x,t) \tag{5}$$

Where the differential operators $L_t(\cdot) = \frac{\partial}{\partial t}(\cdot)$ and $L_{xx} = \frac{\partial^2}{\partial x^2}$, the inverse L^{-1} is assumed an integral operator given by

$$L^{-1} = \int_0^t (\cdot)dt \tag{6}$$

The operating with the operator L^{-1} on both sides of Equation (5) we have

$$L^{-1}(L_t u((x,t))) = L^{-1}(L_{xx}(u(x,t))) + L^{-1}(q(x,t))$$

Therefore, we can write,

$$u(x,t) = u(x,0) + L_t^{-1}\left(L_{xx}\left(\sum_{n=0}^{\infty} u_n \right)\right) + L^{-1}\left(q(x,t)\right) \tag{7}$$

By Wazwaz [12], the modified decomposition method is based on the assumption that the function $K(x)$ can be divided into two parts, namely $K_1(x)$ and $K_2(x)$. Under this assumption we set

$$K(x) = K_1(x) + K_2(x)$$

we suggest the following modification

$$u_0 = K_1$$

$$u_1 = K_2 + L_t^{-1}\left(L_{xx} u_0\right)$$

$$u_{n+1} = L_t^{-1}\left(L_{xx}\left(\sum_{n=0}^{\infty} u_n \right)\right), \text{ n} \geq 1$$

3. Numerical Illustration:

Example 1: Consider the problem (1) with the following conditions, as taken in [11]

$$D_t u(x,t) = D_x^2 u(x,t) - 2e^{x-t}$$

$$u(x,0) = e^x, 0 \leq x \leq 1$$

$$u_x(0,t) = e^{-t}, 0 < t \leq T$$

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$$u(1,t) = e^{1-t}, 0 < t \le T$$

Now after modified decomposition method, we obtain:

$$u_0(x,t) = e^{x-t}$$
$$u_1(x,t) = 0$$
$$u_2(x,t) = 0$$
$$u_3(x,t) = 0$$

Then the series form is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t)$$

$$= e^{x-t}$$

This is the exact solution $u(x,t) = e^{x-t}$.

The plot of the exact solution surface is shown in Figure 1 and the numerical solution surface is shown in Figure 2 for heat equation

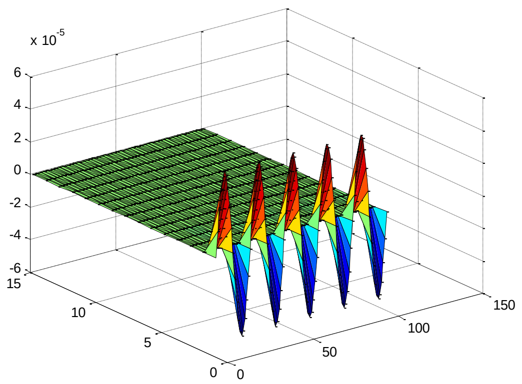

Figure 1: Exact solution

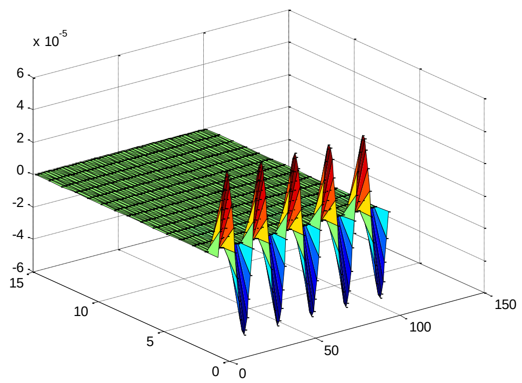

Figure 2: Numerical solution

Example 2: Consider the problem (1) with the following derivative boundary and initial conditions, as taken in [11]

$$D_t u(x,t) = D_x^2 u(x,t) + xt^2$$

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$$u(x,0) = \sin(x), 0 \le x \le 1$$

$$u_x(0,t) = 1, 0 < t < T$$

$$u_x(1,t) = \sin(t), 0 < t < T$$

We apply the above modified decomposition method; we obtain:

$$u_0(x,t) = e^{-t}\sin(x) + \frac{1}{3}t^3 x$$
$$u_1(x,t) = 0$$
$$u_2(x,t) = 0$$
$$u_3(x,t) = 0$$

Then the series form is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t)$$

$$= e^{-t}\sin(x) + \frac{1}{3}t^3 x$$

Which gives the exact solution $u(x,t) = e^{-t}\sin(x) + \frac{1}{3}t^3 x$.

Figure 3 and Figure 4 show the plot of the exact solution surface and the numerical solution surface for heat equation respectively.

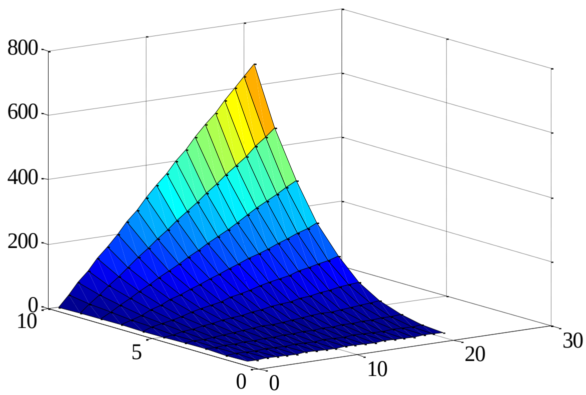

Figure 3: Exact solution

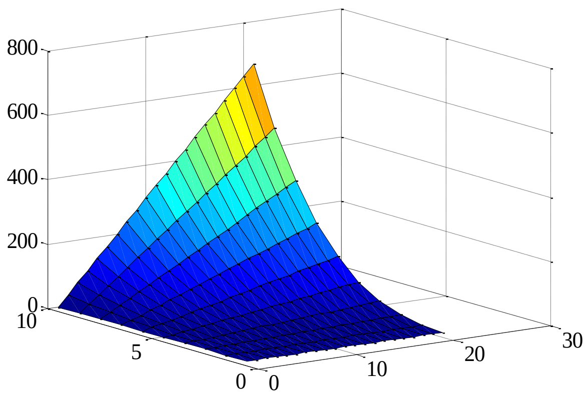

Figure 4: Numerical solution

Example 3: Consider heat equation with derivative boundary conditions for the equation (1), as taken in [11]

$$D_t u(x,t) = D_x^2 u(x,t)$$

$$u(x,0) = \sin(\pi x), 0 \le x \le 1$$

$$u_x(0,t) = \pi e^{-\pi^2 t}$$

$$u(1,t) = -\pi e^{\pi^2 t}$$

Now we apply the above modified decomposition method, we obtain:

$$u_0(x,t) = e^{-\pi^2 t}\sin(\pi x)$$
$$u_1(x,t) = 0$$
$$u_2(x,t) = 0$$
$$u_3(x,t) = 0$$

Then the series form is given by:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t)$$

$$= e^{-\pi^2 t}\sin(\pi x)$$

This is the exact solution $u(x,t) = e^{-\pi^2 t}\sin(\pi x)$.

Figure 5 and Figure 6 show the plot of the exact and the numerical solution surface for heat equation respectively.

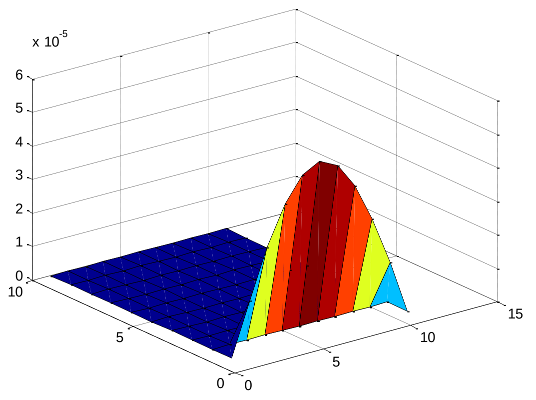

Figure 5: Exact solution

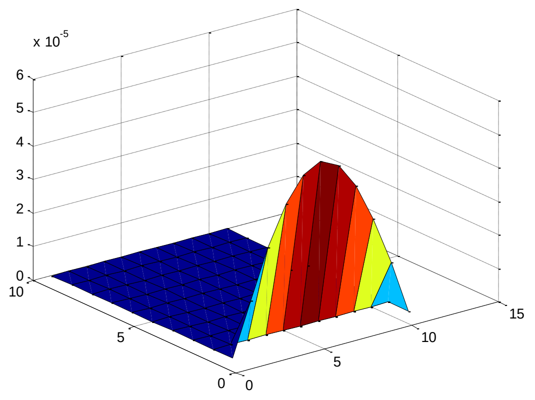

Figure 6: Numerical solution

4. Conclusion

In this paper, we have applied the modified decomposition method for the solution of the heat equation with derivative boundary conditions. This algorithm is simple and easy to implement. The obtained results confirmed a good accuracy of the method. On the other hand, the calculations are simpler and faster than in traditional techniques.

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