Mathematical Theory and Modeling www.iiste.org
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.1, No.2, 2011

A Unique Common Fixed Point Theorem for Two Maps Under ψ - φ Contractive Condition In Ultra Metric Spaces

K.P.R.RAO
Department of Applied Mathematics,
AcharyaNagarjunaUnivertsity-Dr.M.R.Appa Row Campus,
Nuzvid- 521 201, Krishna District, Andhra Pradesh, India
E-mail address: kprrao2004@yahoo.com

G.N.V.KISHORE
Department of Mathematics,
Swarnandhra Institute of Engineering and Technology,Seetharampuram,
Narspur- 534 280 , West Godavari District, Andhra Pradesh, India
E-mail address: kishore.apr2@gmail.com

Abstract

The purpose of this paper is to prove some common fixed point theorems for a pair of Jungck type self maps satisfying $\psi - \varphi$ contractive condition on a spherically complete ultra metric space.

Keywords: Ultra metric space, Spherically complete, Common fixed point.

1. Introduction

Generally to prove fixed point theorems for maps satisfying strictly contractive conditions, one has to assume the continuity of maps and compact metric spaces. In spherically complete ultra metric spaces, the continuity of maps are not necessary to obtain fixed points.

First we state some known definitions.

Definition 1.1 [3]. Let (X, d) be a metric space. If the metric d satisfies strong triangle inequality:

$d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$

then d is called an ultra metric on X and the pair (X, d) is called an ultra metric space.

Definition 1.2 [3]. An ultra metric space (X, d) is said to be spherically complete if every shrinking collection of balls in X has a non empty intersection.

Recently Gajic [1] proved the following

Theorem 1.3 (Theorem1, [1]): Let (X, d) be a spherically complete ultra metric space. If $T : X \to X$ is a mapping such that

$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty)\}$ for all $x, y \in X, x \neq y$

then T has a unique fixed point in X.

Now we extend this theorem for a pair of maps of Jungck type, by using $\psi - \varphi$ contractions.

2 Main Result

Theorem 2.1 : Let (X, d) be a spherically complete ultra metric space. Let T, f : $X \to X$ be mappings satisfying

(i) $\psi(d(Tx, Ty)) < \psi(\max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\}) - \varphi(\max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\})$
for all $x, y \in X$ such that $x \neq y$, where $\psi:[0, \infty) \to [0, \infty)$ is non – decreasing function and
$\varphi: [0, \infty) \to [0, \infty)$ is function with $\varphi(t) > 0$ if $t > 0$,

(ii) $T(X) \subseteq f(X)$.

Then there exists $z \in X$ such that $fz = Tz$.
Further if f and T are coincidentally commuting at z, then z is the unique common fixed point of f and T.
Proof: Let $B_a = (fa, d(fa, Ta))$ denote the closed sphere centered at fa with the radius d(fa, Ta) and let A be the collection of these spheres for all $a \in X$. Then the relation $B_a \preceq B_b$ iff $B_b \subseteq B_a$ is partial order on A. Let A_1 be a totally order sub family of A. Since (X, d) is spherically complete, we have

$$\bigcap_{B_a \in A} B_a = B \neq \phi.$$

Let $fb \in B$ and $B_a \in A_1$ then $fb \in B_a$.
Hence $d(fa, fb) \leq d(fa, Ta)$. (2.1)
If $a = b$, then $B_a = B_b$.
Assume $a \neq b$.
Let $x \in B_b$ then

$$\begin{aligned} d(x, fb) &\leq d(fb, Tb) \\ &\leq \max\{d(fb, fa), d(fa, Ta), d(Ta, Tb)\} \\ &= \max\{ d(fa, Ta), d(Ta, Tb)\}. \end{aligned}$$

Case(i): If d(fa, Ta) is maximum, then $d(x, fb) \leq d(fa, Ta)$.
Case(ii): If d(Ta, Tb) is maximum, then

$$d(x, fb) \leq d(fb, Tb) \leq d(Ta, Tb)$$

$$\begin{aligned} \psi(d(x, fb)) &\leq \psi(d(fb, Tb)) \\ &\leq \psi(d(Ta, Tb)) \\ &< \psi(\max\{d(fa, fb), d(fa, Ta), d(fb, Tb)\}) - \varphi(\max\{d(fa, fb), d(fa, Ta), d(fb, Tb)\}) \\ &= \psi(\max\{ d(fa, Ta), d(fb, Tb)\}) - \varphi(\max\{d(fa, Ta), d(fb, Tb)\}), \text{ from (2.1)} \end{aligned}$$

If d(fb, Tb) is maximum, then from the above we have

$$\begin{aligned} \psi(d(fb, Tb)) &< \psi(d(fb, Tb)) - \varphi(d(fb, Tb)) \\ &< \psi(d(fb, Tb)). \end{aligned}$$

It is a contradiction. Hence d(fa, Ta) is maximum.
Thus

$$\begin{aligned} \psi(d(x, fb)) &< \psi(d(fa, Ta)) - \varphi(d(fa, Ta)) \\ &< \psi(d(fa, Ta)). \end{aligned}$$

By definition of ψ, $d(x, fb) \leq d(fa, Ta)$.

Thus in both cases we have that $d(x, fb) \leq d(fa, Ta)$. (2.2)

Now $d(x, fa) \leq \max\{d(x, fb), d(fb, fa)\}$

$\leq d(fa, Ta)$, from (2.1) and (2.2).

It follows that $x \in B_a$.

Hence $B_b \subseteq B_a$ for any $B_a \in A_1$.

Thus B_b is an upper bound in A for the family of A_1 and hence by Zorn's lemma, A has a maximal element B_z, $z \in X$.

Suppose $fz \neq Tz$.

Since $Tz \in T(X) \subseteq f(X)$. Then there exists $v \in X$ such that $Tz = fv$. Clearly $z \neq v$.

$$\begin{aligned}\psi(d(fv, Tv)) &= \psi(d(Tz, Tv)) \\ &< \psi(\max\{d(fz, fv), d(fz, Tz), d(fv, Tv)\}) - \varphi(\max\{d(fz, fv), d(fz, Tz), d(fv, Tv)\}) \\ &= \psi(\max\{d(fz, fv), d(fv, Tv)\}) - \varphi(\max\{d(fz, fv), d(fv, Tv)\}).\end{aligned}$$

If $d(fv, Tv)$ is maximum, then from the above , we have

$$\begin{aligned}\psi(d(fv, Tv)) &< \psi(d(fv, Tv)) - \varphi(d(fv, Tv)) \\ &< \psi(d(fv, Tv)).\end{aligned}$$

It is a contradiction. Hence $d(fv, fz)$ is maximum.

Thus

$$\begin{aligned}\psi(d(fv, Tv)) &< \psi(d(fv, fz)) - \Phi(d(fv, fz)) \\ &< \psi(d(fv, fz)). \qquad (2.3)\end{aligned}$$

If $fz \in B_v$, then $d(fz, fv) \leq d(fv, Tv)$.

By definition of ψ, $\psi(d(fz, fv)) \leq \psi(d(fv, Tv))$.

It is a contradiction to (2.3). Thus $fz \notin B_v$.

Hence $B_z \not\subset B_v$. It is contradiction to maximality of B_z.

Hence $fz = Tz$.

Suppose f and T are coincidentally commuting at $z \in X$.

Then $f^2z = f(fz) = f(Tz) = T(fz) = T(Tz) = T^2z$.

Suppose $fz \neq z$.

$$\begin{aligned}\psi(d(Tfz, Tz)) &< \psi(\max\{d(f^2z, fz), d(f^2z,, Tfz), d(fz, Tz)\}) - \varphi(\max\{d(f^2z, fz), d(f^2z,, Tfz), d(fz, Tz)\}) \\ &= \psi(\max\{d(Tfz, Tz), d(Tfz,, Tfz), d(Tz, Tz)\}) - \varphi(\max\{d(Tfz, Tz), d(Tfz,, Tfz), d(Tz, Tz)\}) \\ &= \psi(d(Tfz, Tz)) - \varphi(d(Tfz, Tz)) \\ &< \psi(d(Tfz, Tz)).\end{aligned}$$

It is a contradiction.

Hence $fz = z = Tz$.

Therefore z is common fixed point of f and T.

Suppose w is another common fixed point of f and T such that $z \neq w$.

$$\begin{aligned}\psi(d(z, w)) &= \psi(d(Tz, Tw)) \\ &< \psi(\max\{d(fz, fw), d(fz, Tz), d(fw, Tw)\}) - \varphi(\max\{d(fz, fw), d(fz, Tz), d(fw, Tw)\}) \\ &= \psi(\max\{d(z, w), d(z, z), d(w, w)\}) - \varphi(\max\{d(z, w), d(z, z), d(w, w)\}) \\ &= \psi(d(z, w)) - \varphi(d(z, w)) < \psi(d(z, w)).\end{aligned}$$

It is a contradiction.

Therefore z = w.

Hence z is unique common fixed point of f and T.

References

Gajic.Lj, (2001), “On ultra metric spaces”, Novi Sad J.Math.31, 2, 69 - 71.

Gajic.Lj, (2002, “A multi valued fixed point theorem in ultra metric spaces”, Math.vesnik, 54,(3-4)), 89-91.

Van Roovij.A.C.M, (1978), Non - Archimedean Functional Analysis, Marcel Dekker, New York.