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# Solution of a Singular Class of Boundary Value Problems by Variation Iteration Method

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**Abstract**

In this paper, an effective methodology for finding solution to a general class of singular second order linear as well as nonlinear boundary value problems is proposed. These types of problems commonly occur in physical problems. The solution is developed by constructing a sequence of correctional functional via variation Iteration theory. The analytical convergence of such occurring sequences befitting to the context of the class of such existing problems is also discussed. The efficacy of the proposed method is tested on various problems. It is also observed that execution of only few successive iterations of correction functionals may lead to a solution that is either exact solution or very close to the exact solution.

**Keywords:** Variation iteration method, sequence, linearization, discretization, transformation Convergence, Lagrange multiplier, smooth function, B-Spline, projection method, Lie group

## 1. Introduction

A wide spectrum of well defined properties and behavior systematically associated to a class of events/situations occurring on varied fronts in celestial bodies or multidisciplinary sciences either internally or externally or both ways simultaneously are realized or discerned in real or abstract sense. When these problems are modeled mathematically in order to envisage or acknowledge the endowed and all inherent characteristics in and around thereof, a class of second order singular differential equations along with two boundary conditions comes into coherent consideration. Therefore, for such class a suitable and sustainable solution either numerically appropriate or analytically in the exact form, is must and equally important in whatsoever manner it is made possible by applying so any feasible proposed variant.

Consider a general class of boundary value problems as follows

$$x^{-\alpha}\,(x^{\alpha}\,y^{/}\,)^{/} = f\,(x,y) \qquad 0< x \leq 1 \tag{1.1}$$

$$y(0) = A \quad , \quad y(1) = B$$

A, B are constants and $\alpha \in \mathbb{R} -$ set of real numbers. The function $f(x,y)$ is a real valued continuous function of two variables $x$ and $y$ such that $(x,y) \in \mathbb{R} \times \mathbb{R}$ and that $\frac{\partial f}{\partial y}$ is a nonnegative and continuous function in a domain $R = \{(x,y) : (x,y) \in [0\ 1] \times \mathbb{R}\}$. Solution to such class of problems exists [7-8]. Out of such class it plausible to consider a sub-class formed when $\alpha \in (0\ 1) \subseteq \mathbb{R}$ for elaborated analysis and discussion of facts. The class of problems (1.1) from a specific area of the field of differential equation has been a matter of immense research and keen interest to researchers in recent past. Several methods like B-Spline, homotopy method, Lie group analysis, power series method, projection method, Adomian method, multi- integral method, finite difference method [9- 15] have been applied on to justify an immaculate importance of such class of problems. Variation iteration method, a modified Lagrange method [16] originally proposed by He [17-21], stands recognized as promising and profusely used method of research in almost all disciplines of science and technology as an alternative method which is different from other methods of linearization, transformation and discretization used to solve such type of problems in some way or other way round. It is pertinent to note that the proposed method has fared well, over a large class of mathematically modeled problems whenever or wheresoever's such a suitable situation has have aroused and it is demanded to be applied so. Eventually, credit accrue to variation iteration method for solving a class of distinguished and challenging problems like, nonlinear coagulation problem with mass loss ,nonlinear fluid flow in pipe-like domain, nonlinear heat transfer, an approximate solution for one dimensional weakly nonlinear oscillations, nonlinear relaxation phenomenon in polycrystalline solids, nonlinear thermo elasticity, cubic nonlinear Schrodinger equation, semi-linear inverse parabolic equation, ion acoustic plasma wave, nonlinear oscillators with discontinuities ,non-Newtonian flows, Burger's and coupled Burger's equation, multispecies Lotaka –Volterra equations, rational solution of Toda lattice equation, Helmholtz equation, generalized KdV equation[17-34].

## 2. Variation Iteration Method (VIM)

The basic virtue and fundamental principle associated to variation iteration method may be expressed in brief by considering a general differential equation involving a differential operator D as follows.

$$\text{Let} \quad Dy(x) = g(x) \qquad x \in I \subseteq \mathbb{R} \tag{2.1}$$

$y(x)$ is sufficiently smooth function on some domain $\Omega$ and $g(x)$ an inhomogeneous real valued function. (2.1) can be rewritten as,

$$L(y(x)) + N(y(x)) = g(x) \qquad x \in I \subseteq \mathbb{R} \tag{2.2}$$

where L and N are linear and nonlinear differential operators, respectively.

Ostensibly, the privileged variation iteration method has natural aptness and basic tendency to generate a recursive sequence of correction functionals that commands and allows to conserve a real power and absolute potential for finding a just and acceptable solution to the given class of problems (1.1) and the sequence of correctional functional over(2.2) is

$$y_{n+1}(x) = y_n(x) + \int_0^x \mu(s)\,((L(y_n(s)) + N(\widetilde{y_n(s)})) - g(s))\,ds \qquad , n\geq 0 \tag{2.3}$$

where $\mu$ stands for Lagrange multiplier determined optimally satisfying all stationary conditions after the variation method is applied to (2.3). The importance and therefore utility of method all over lies with the assumption and choice of considering the concerned inconvenient highly nonlinear and complicated dependent variables as restricted variables thereby minimizing its magnitude, the accruing error that might have crept into the error prone process while finding a solution to (1.1). As aforementioned, $\widetilde{y_n}$ is the restricted variation, which means $\delta\widetilde{y_n}=0$. Eventually, after desired $\mu$ is determined, a proper and suitable selective function (linear or nonlinear) with respect to (2.2) is assumed as an initial approximation for finding next successive iterative function by recursive sequence of correction functional. Thereafter boundary conditions are imposed on the final or preferably on limiting value (as $n \to \infty$) of sequential approximations incurred after due process of iteration.

## 3. Variational Method and Lagrange Multiplier

The variational method and Lagrange multiplier are convoluted corresponding to (1.1) by the iterative and successive correction functional relation as

$$y_{n+1}(x) = y_n(x) + \int_0^x \mu(s)\,\left(s^\alpha y_n^{/}(s)\right)^{/} - \widetilde{x^\alpha} f(s, y_n(s)))\,ds \qquad n\geq 0 \tag{3.1}$$

where $y_n(x)$ is $n^{th}$ approximated iterative solution of (1.1). Suppose optimal value of $\mu(s)$ is identified naturally by taking variation with respect to $y_n(x)$ and subject to restricted variation $\delta\widetilde{y_n}(x)=0$. Then from (3.1) we have

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta\int_0^x \mu(s)((s^\alpha y_n^{/})^{/} - \widetilde{s^\alpha} f(s, y_n(s))\,ds \qquad n\geq 0 \tag{3.2}$$

Integrating by parts and considering the restricted variation of $y_n$ (i.e. $\delta y_n=0$) as well relation (3.2) gives

$$\delta y_n(x) = (1-\mu^{/}(s))\,\delta y_n(x) + \delta(\mu(s)\,s^\alpha y_n^{/}(s))\,|_{s=x} + \int_0^x (\mu^{/}(s)\,s^\alpha)^{/}\delta y_n(s)ds, \quad n \geq 0$$

Therefore, the stationary conditions are

$$\mu^{/}(s)\,s^\alpha = 0, \quad \mu(x) = 0, \quad (\mu^{/}(s)s^\alpha)^{/} = 0$$

It gives

$$\mu(s) = \frac{s^{1-\alpha} - x^{1-\alpha}}{1-\alpha}$$

From (3.1), the sequence of correction functionals is given by

$$y_{n+1}(x) = y_n(x) + \frac{1}{1-\alpha}\int_0^x (s^\alpha - x^\alpha)\,((s^\alpha y_n(s))^{/} - s^\alpha \widetilde{f(s, y_n(s))})ds \qquad n\geq 0 \tag{3.4}$$

It may be deduced from (3.4) that the limit of the convergent iterative sequence $\{y_n\}_{n=1}^{\infty}$, if it converges on satisfying given boundary conditions, is the exact solution to (1.1).

## 4. Convergence of Iterative Sequence

In order to carry out convergence analysis of the sequence of correctional functionals generated by execution of VIM with respect to given class (1.1) in view of (3.1), we consider

$y_{n+1(x)} = y_n(x) + \sum_{k=0}^{n-1}(y_{k+1}(x) - y_k(x))$ is the $n^{th}$ partial sum of the infinite series

$$y_0(x) + \sum_{k=0}^{\infty}(y_{k+1}(x) - y_k(x)) \tag{4.1}$$

And that convergence of auxiliary series (4.1) necessarily implies the convergence of iterative sequence $\{y_n(x)\}_{n=1}^{\infty}$ of partial sums of the series (4.1).

Let $y_0(x)$ be the assumed initial selective function. The first successive variation iterate is given by

$$y_1(x)=\int_0^x \mu(s)((s^{\alpha}y_0^{/}(s))^{/}\_s^{\alpha} f(s,y_0(s)))ds \tag{4.2}$$

Integrating by parts and in sequel applying the existing stationary conditions, we have

$$|y_1(x)- y_0(x)|=|\int_0^x(y_0^{/}(s)+\mu(s)s^{\alpha}f(s,y_0(s))ds| \tag{4.3}$$

Or $|y_1(x)-y_0(x)| \le \int_0^x(|y_0^1(s)|+|s^{\alpha}||\mu(s)||f(s,y_0(s)|)ds$

Or $$|y_1(x)-y_0(x)| \le \int_0^x(|y_0^1(s)| + |\mu(s)||f(s,y_0(s)|) ds \tag{4.4}$$

Again pursuing similar steps as in (4.2) and adopting usual stationary conditions likewise, relation (3.4) gives

$$|y_2(x) - y_1(x)|=|\int_0^x \mu(s)\, s^{\alpha}(f(s),y_1(s))-f(s,y_0(s))ds| \tag{4.5}$$

Or, $|y_2(x) - y_1(x)| \le \int_0^x |\mu(s)||s^{\alpha}|(f(s),y_1(s))-f(s,y_0(s))|ds$

Or, $$|y_2(x) - y_1(x)| \le \int_0^x |\mu(s)| (f(s), y_1(s))- f(s, y_0(s)) |ds \tag{4.6}$$

In general, we have

$$|y_{n+1}(x)-y_n(x)|=|\int_0^x \mu(s)s^{\alpha}(f(s,y_n(s))-f(s,y_{n-1}(s)))ds| \tag{4.7}$$

Or, $|y_{n+1}(x) - y_n(x)| \le \int_0^x |\mu(s)||s^{\alpha}|| (f(s,y_n(s)) - f(s,y_{n-1}(s))) |ds \quad \forall \quad n \quad \ge \quad 2$

Or, $$|y_{n+1}(x) - y_n(x)| \le \int_0^x |\mu(s)|| (f(s,y_n(s)) - f(s,y_{n-1}(s))) |ds \quad \forall \quad n \quad \ge 2 \tag{4.8}$$

Since f(x, y) and $\frac{\partial f(x,y)}{\partial y}$ are continuous on R, therefore for fix s$\epsilon$ [0 1] and by virtue of mean value theorem $\exists$ $(s,\theta_n^0(s)) \in R$ satisfying (say, $y_{n-1}(s) < \theta_n^0(s) < y_n(s)$),

$\forall n \in IN$ , $s \le x \le 1$ , such that

$$|f(s, y_n(s)) - f(s, y_{n-1}(s))| = |\frac{\partial f(s,\theta_{n+1}^0(s))}{\partial y}||y_n(s) - y_{n-1}(s)| \quad \forall\ n \quad \ge \quad 2 \tag{4.9}$$

Now, suppose

$$M_{\infty}^1 = \sup(|y_0^{/}(s)|+|\mu(s)||f(s,y_0(s))|, \quad s \le x \le 1 \tag{4.10}$$

and $$M_2^{\infty}=\sup(|\mu(s)||\frac{\partial f(s,\theta_n^0(s))}{\partial y}|), \quad s \le x \le 1,\ n\in IN \tag{4.11}$$

Again to begin with assume

$$M=\sup(M_{\infty}^1,M_{\infty}^2) \tag{4.12}$$

We observe and proceed to establish the truthfulness of the inequality

$$|y_{n+1}(s) - y_n(s)| \le \frac{M^{n+1}x^{n+1}}{n+1!} \quad \forall n \in IN \tag{4.13}$$

Relations (4.4), (4.10), (4.9) and (4.12) give

$$|y_1(x)-y_0(x)| \le \int_0^x M_1\, ds \le \int_0^x M\, ds\, ds = Mx \tag{4.14}$$

As well as, $|y_2(x) - y_1(x)| \le \sup|\mu(s)||\frac{\partial f(s,\theta_1^0(s))}{\partial y}||\int_0^x|(y_1(s)) - y_0(s))|ds$

or $|y_2(x) - y_1(x)| \le \sup(|\mu(s)||\frac{\partial f(s,\theta_1^0(s))}{\partial y}|\int_0^x|(y_1(s)) - y_0(s))|ds=M\int_0^x M\, ds=\frac{M^2x^2}{2}$

$s \le x \le 1$ ,$n\in IN$

Thus, the statement (4.13) is true for natural number n=1

Suppose that $|y_n(s) - y_{n-1}(s)| \le \frac{M^n x^n}{n!}$ holds for some, $n \in IN$

Then, relations (4.8), (4.9) and (4.12) imply

$$|y_{n+1}(x) - y_n(x)| \le \int_0^x |\mu(s)| \left|\frac{\partial f(s,\theta_n^0(s))}{\partial y}\right| |y_n(s) - y_{n-1}(s)| ds$$

i.e. $$|y_{n+1}(x) - y_n(x)| \le \int_0^x |\sup(\mu(s))| \left(\sup\left|\frac{\partial f(s,\theta_n^0(s))}{\partial y}\right|\right) |y_n(s) - y_{n-1}(s)| ds$$
$s \le x \le 1$ $n \in IN$

or , $$|y_{n+1}(x) - y_n(x)| \le \sup(|\mu(s)|\left|\frac{\partial f(s,\theta_{n+1}^0(s))}{\partial y}\right| \int_0^x |y_n(s) - y_{n-1}(s)| ds$$

$$\le M \int_0^x \frac{M^n s^n}{n!} ds = \frac{M^{n+1} x^{n+1}}{n+1!}$$

Therefore, by Principle of Induction

$|y_{n+1}(x) - y_n(x)| \le \frac{M^{n+1} x^{n+1}}{n+1!}$ holds $\forall x \in [0\ 1]$ and $\forall n \in IN$.

So the series (4.1) converges both absolutely and uniformly for all $x \in [0\ 1]$

Since, $|y_0(x)| + \sum_{n=0}^{\infty} |y_{n+1}(x) - y_n(x)| \le |y_0(x)| + \sum_{n=0}^{\infty} \frac{M^{n+1} x^{n+1}}{n+1!} = |y_0(x)| + (e^{Mx} - 1)$, $\forall x \in [01]$

Asserting that the series $y_0(x) + \sum_{k=0}^{\infty} (y_{k+1}(x) - y_k(x))$ converges uniformly $\forall x \in [01]$ and hence the sequence of its partial sums $\{y_n(x)\}_{n=0}^{\infty}$ converges to a limit function as the solution.

## 5. Numerical Problem

To begin with implementation and analyze scope of VIM, we apply this very method to find the solution of linear and nonlinear problems that have been solved by different methods in literature. Specifically to mention is the method to solve it numerically and via numerical finite difference technique of solution.

Example 1: Consider the following boundary value problem [12]

$$y^{(2)}(x) + \frac{\alpha}{x} y^{(1)}(x) = -x^{1-\alpha} \cos x - (2-\alpha) x^{1-\alpha} \sin x \quad (5.1)$$
$$y(0) = 0 \ , \quad y(1) = \cos 1$$

Solution: To solve this we construct correction functional as follows

$$y_{n+1}(x) = y_n(x) + \int_0^x \mu(s) \left((-s^\alpha y_n^{/}(s))^{/} - s\cos s - (2-\alpha)\sin s\right) ds \ , \quad n \ge 0$$

where $\mu(s)$ Is optimally identified Lagrange multiplier similar to (3.3). The first iterative solution is given by

$$y_1(x) = y_0(x) + \int_0^x \mu(s) \left((-s^\alpha y_0^{/}(s))^{/} - s\cos s - (2-\alpha)\sin s\right) ds$$

Since the selective function $y_0(x)$ is arbitrary for simplicity and easiness we may choose

$$y_0(x) = a_0 x^{1-\alpha} \ , \quad \text{so that } (-s^\alpha y_0^{/})^{/} = 0$$

Thus, $$y_1(x) = a_0 x^{1-\alpha} + \int_0^x \mu(s) (-s\cos s - (2-\alpha)\sin s) ds$$

Now performing usual simplifications and applying term by term series integration, we get

$$y_1(x) = a_0 x^{1-\alpha} - \left[\sum_0^{\infty} (-1)^n \frac{x^{2n+3}}{(2n+3-\alpha)(2n+1)!} + (1-\alpha) \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1-\alpha}}{(2n+1-\alpha)(2n)!}\right]$$

or $$y_1(x) = a_0 x^{1-\alpha} + x^{1-\alpha} \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

or $$y_1(x) = a_0 x^{1-\alpha} + x^{1-\alpha} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} - x^{1-\alpha}$$

i.e. $$y_1(x) == (a_0 - 1) x^{1-\alpha} + x^{1-\alpha} \cos x \quad (5.2)$$

In order to match the boundary condition $y(1) = \cos(1)$ taking limit as $(x \to 1)$ we find $a_0 = 1$ , only the first iterate giving the exact solution as $y(x) = y_1(x) = x^{1-\alpha} \cos x$

Example-2: Consider the boundary value problem [12]

$$(x^\alpha y^{/})^{/} = \beta x^{\alpha+\beta-2} ((\alpha + \beta - 1) + \beta x^\beta) y$$

$$y(0)=1 \quad , \quad y(1)=\exp(1) \tag{5.3}$$

Solution**:** The correction functional for the problem (5.3) is

$$y_{n+1}(x)=y_n(x)+\int_0^x \mu(s)\left((s^\alpha y_n^{/})^{/}-\beta(\alpha+\beta-1)s^{\alpha+\beta-2}-\beta^2 s^{\alpha+\beta-2}\right) y_n(s) \tag{5.4}$$

$\mu(s)$ Is optimally identified Lagrange multiplier similar as (3.2)

Inserting, $y(0)=y_0(x)=1$ to (5.4) when n=1, as selective initial approximation function we process out following induced successive iterative approximate solutions as

$$y_1(x)=1+x^\beta+\beta\frac{x^{2\beta}}{2(\alpha+\beta-1)}$$

$$y_2(x)=1+x^\beta+\frac{x^{2\beta}}{2.1}+\beta\frac{x^{3\beta}}{3(\alpha+3\beta-1)}$$

$$y_3(x)=1+x^\beta+\frac{x^{2\beta}}{2.1}+\frac{x^{3\beta}}{3.2.1}+\beta\frac{x^{4\beta}}{4.2(\alpha+4\beta-1)}$$

$$y_4(x)=1+x^\beta+\frac{x^{2\beta}}{2.1}+\frac{x^{3\beta}}{3.2.1}+\frac{x^{4\beta}}{4.3.2.1}+\beta\frac{x^{5\beta}}{5.3.2(\alpha+5\beta-1)}$$

$$y_5(x)=1+x^\beta+\frac{x^{2\beta}}{2.1}+\frac{x^{3\beta}}{3.2.1}+\frac{x^{4\beta}}{4.3.2.1}+\frac{x^{5\beta}}{5.4.3.2.1}+\beta\frac{x^{6\beta}}{6.4.3.2(\alpha+6\beta-1)}$$

Similarly, continuing in like manner inductively we find the general term of the sequence

$$y_n(x)=1+x^\beta+\frac{x^{2\beta}}{2.1}+\frac{x^{3\beta}}{3.2.1}+\frac{x^{4\beta}}{4.3.2.1}+\frac{x^{5\beta}}{5.4.3.2.1}+\frac{x^{6\beta}}{6.5.4.3.2.1}+\ldots\ldots\ldots+\frac{x^{n\beta}}{n!}+\frac{n\beta x^{(n+1)\beta}}{n+1!(\alpha+(n+1)\beta-1)}$$

i.e. $$y_n(x)=\sum_{k=0}^{n}\frac{x^{k\beta}}{k!}+\frac{n\beta x^{(n+1)\beta}}{n+1!(\alpha+(n+1)\beta-1)} \tag{5.5}$$

Now, we observe that $T_n=\frac{n\beta x^{(n+1)\beta}}{n+1!(\alpha+(n+1)\beta-1)}$ (say), is the general term of a convergent

Series $\sum_{n=0}^{\infty}\frac{n\beta x^{(n+1)\beta}}{n+1!(\alpha+(n+1)\beta-1)}$ .

Therefore, $\lim(n\to\infty)\frac{n\beta x^{(n+1)\beta}}{n+1!(\alpha+(n+1)\beta-1)}=0$ and (5.5) facilitates the exact solution to (5.3) as

$y(x)=\lim(n\to\infty)\left(\sum_{k=0}^{n}\frac{x^{k\beta}}{k!}\right)=\exp(x^\beta)$.

Example-3: Consider the boundary value problem [9]

$$(x^\alpha y^{/})^{/}=\frac{\beta x^\alpha}{4+x^\beta}(\beta x^\beta e^y-(\alpha+\beta-1))$$
$$y(0)=\ln\frac{1}{4} \quad , \quad y(1)=\ln\frac{1}{5} \tag{5.6}$$

Solution: Let, $y_0=y(0)=\ln\frac{1}{4}$ , be the selective initial approximation function .Then by VIM

First iterative approximate solution to (5.6) simplifies to

$$y_1(x)=\ln\frac{1}{4}+\int_0^x\frac{\mu(s)}{4+x^\beta}\left(\frac{\beta^2 s^{\alpha+2\beta-2}}{4}-(\alpha+\beta-1)\beta^\alpha s^{\alpha+\beta-2}\right)ds \tag{5.7}$$

where as $\mu(s)$ is optimally identified Lagrange multiplier as existing in (3.3) and after simplifying (5.4) the required first approximate solution to (5.6) satisfying the given boundary condition $y(0)=\ln\frac{1}{4}$ is as follows

$$y_1(x)=\ln\frac{1}{4}-\frac{x^\beta}{4}+\frac{1}{2}\left(\frac{x^\beta}{4}\right)^2+\sum_{n=3}^{\infty}\left(\frac{\alpha+2\beta-1}{\alpha+n\beta-1}\right)\left(\frac{(-1)^n}{n}\right)\left(\frac{x^\beta}{4}\right)^n \tag{5.8}$$

Now, we observe in (5.8) that the first three terms of the first approximate iterative solution of (5.6) match the first three terms of the expanded Taylor's series solution even though only first boundary condition is being used so for. However, if we allow $\beta$ to tend to zero in (5.8) as $\beta$ is arbitrary, $y_1(x)$ violates the condition $y(0) = \ln\ \frac{1}{4}$ . But if the terms ( $\frac{\alpha+2\beta-1}{\alpha+n\beta-1}$) and $\left(\frac{(-1)^n}{n}\right)(\frac{x^\beta}{4})^n$ are treated independent to each other and arbitrarily parameter $\beta$ is allowed to approach to zero only in the coefficient ($\frac{\alpha+2\beta-1}{\alpha+n\beta-1}$) of $\left(\frac{(-1)^n}{n}\right)(\frac{x^\beta}{4})^n$ independently, the boundary condition $y(1) = \ln\frac{1}{5}$ expressed in expanded series form matches the prescribed value if it is imposed on $y_1(x)$. Thus improvisation on $y_1(x)$in this way not only shoots to satisfy the other boundary condition but also exculpate to procures the exact solution. Therefore, allowing the process to do so and let the first iterate mend its way to produce exact solution $y(x) = y_1(x)$ to the problem (5.3). Therefore the exact solution to (5.6) is given by

$$y(x) = y_1(x) = \ln\frac{1}{4} - \frac{x^\beta}{4} + \frac{1}{2}(\frac{x^\beta}{4})^2\ )^n + \sum_{n=3}^{\infty}(\frac{(-1)^n}{n})(\frac{x^\beta}{4})^n = \ln\frac{1}{4+x^\beta}$$

## 6. Conclusion

In this paper, we have applied the He's variation iteration method successfully to a linear as well as to a nonlinear class of boundary value problems. The convergence analysis of the proposed method with reference to considered class has also been presented in exhaustive manner. A proper selection of selective function and careful imposition of boundary condition on iterative function may lead to an exact solution or any other solution of high accuracy even to a non-linear problem in just only some maneuvered simplifications.

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