Solution of a Subclass of Singular Second Order Differential Equation of Lane Emden type by Taylor Series Method

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Abstract

In this paper, a subclass of second order differential equation of Lane Emden type followed by imposed initial or boundary condition is identified for solving by Taylor's series method. Such types of problems come into existence while modeling upon some of peer physical phenomenon. The proposed method eventually produces an analytic solution in the form of a polynomial function. Again some of the problems available in literature are also considered and tested for solution to justify the suitability and viability of the method.

Keywords: Variation iteration method, boundary value problems, analytic solution, Lane-Emden equation, He's Polynomial, Taylor series.

1. Introduction

In order to acknowledge and study the inherent properties and systematic behaviour associated to a class of phenomenon occurring on various fronts pertaining to astrophysics, electro hydrodynamics and human physiology in multidisciplinary sciences, mathematical modeling is must and that has proved an efficacious way to mathematicians by and large now. Thereupon, when modeled mathematically such observations indeed give rise to a class of second order singular differential equations along with coherent boundary (or initial) conditions . Let the general class of singular second order differential equations as

$$\frac{1}{p(x)}\, y'' + \frac{1}{q(x)}\, y' + \frac{1}{r(x)}\, y(x) = f(x) \qquad 0<x\leq 1 \tag{1.1}$$

Subject to boundary condition $y(0) = A$, $y(1) = B$ (1.2)

or

Initial conditions $y(0) = C$, $y'(0) = D$ (1.3)

A, B, C and D are real constants. The function f (x), p(x), q(x) and r(x) are real valued continuous functions defined on [0 1]. Interestingly enough the universe is full of various phenomenon like temperature variation of a self-gravitating star, thermal behavior of a spherical cloud of a gas, the situation celestial scientific observation where distance of center of cloud is taken as proportional to gravitational potential under mutual attraction of gravitational force, kinetics of combustion along with concentration of a reactant, isothermal processes associated to a gas spheres and many more like universal processes induce (1.1)to Lane-Emden and Emden-Fowler type of special and most important equations[1,2,4,8,10,11,12,33] follows

$$y''(x) + \frac{k}{x}\, y'(x) + f(x, y) = g(x) \qquad 0<x\leq 1, \quad k\geq 0 \tag{1.4}$$

subject to conditions (1.2) or (1.3).

If in (1.4), f(x,y) annihilates to become purely a function of real variable 'y' say h(y) in some defined and appropriate domain then such class of differential equation is known as Lane Emden equation type of problem[9,13,15,18,23,24,25,34] . Out of such class if we again consider a sub class of (1.4) when g(x) is any polynomial function giving rise to a special class of singular differential equations in the following mathematically modeled form

$$y''(x) + \frac{k}{x}\, y'(x) + a\,P(x)\; y^{m}(x) = Q(x) \qquad 0<x\leq 1 \qquad , m\in I^{+} \tag{1.5}$$

subject to conditions (1.2) or (1.3).

where 'a' is a real constant. P(x) and Q(x) are polynomials of such that degree of P(x) is less than degree of Q(x).Occurrence of such physical events guarantees the existence of solution to (1.5) subject to conditions (1.2) or (1.4). Existence and uniqueness of solutions can also be ascertained from [14,28,32].As a matter of fact such type a of class of problems are related to a specific area of the field of differential equation that has been a matter of immense research and keen interest to researchers in recent past. So for as several other methods like B-Spline, Homotopy method, Lie group analysis, variation iteration method, Adomian method, multi- integral method, finite difference method [6,7,9,13,16,18,19,20,21,23,24,25,29,30,31,34] have been applied on forth with to elicit out the much needed solution which justify and speaks the volume of immaculate importance of such class of problems. Among other methods Taylor series method can also be successfully applied to such class of problems under consideration producing exact solution in the compact polynomial form. This classical method suits well to such important problems, stands recognized as promising and profusely used method of research in almost all disciplines of science and technology as an alternative to other different methods of linearization, transformation and discretization generally used to solve such type of problems in some way or other way round. It is pertinent to note that the proposed method has fared well, over a large class of mathematically modeled problems whenever or upon wheresoever such a suitable situation has have aroused and it is demanded to be applied so. Eventually, credit accrues to Taylor series method for solving a class of distinguished and challenging problems.

2. Method of solution and description of the variant

Consider a subclass of differential equation

$$y''(x) + \frac{k}{x}\, y'(x) + a\,P(x) y^{m}(x) = Q(x) \qquad 0<x\leq 1 \quad , \quad m\in I^{+} \tag{2.1}$$

Subject to condition (1.2) or (1.3)

Since x=0 is a singular point of equation (2.1). In order to avert such an inconvenient situation we modify and rewrite equation (2.1) as follows

$$x y''(x) + k\; y'(x) + a.x.p(x) y^{m}(x) = x\,Q(x) \qquad 0\leq x\leq 1 \quad , \quad m\in I^{+} \tag{2. 2}$$

Obviously equation (2.1) is modified equivalently to be defined for all x in [o 1].Let y(x) be the unique existing analytic solution in an appropriate domain [14, 28, 32].Considering (x→0) by L Hospital rule gives$y'(0) = 0$ as one of the self-implied basic conditions that help generate recursive inter-relations between derivatives at some point x=x_0 ∈[0 1].Because of simplicity and without loss of generality, we take x_0 as zero. Consequently the Taylor series solution of (2.1) along with (1.2) or (1.3) is

$$y(x) = (0) + \frac{1}{1!} y'(0)\, x^2 + \frac{1}{3!} y'''(0)\, x^3 + \cdots + \frac{1}{n!} y^{(n)}(0)\, x^n + \cdots \tag{2.3}$$

In (2.3), some derivatives i.e. $y^{i}(0), i = 1,2, \ldots,$ may be functions of y(0). Using other boundary condition, we obtain the value of unknown y(0). The prevalent salient derived features about finding the solution to the subclass (2.2) (along with the given conditions) may be observed and thereby accordingly adopted which are given below.

- Solution exist and is unique on the domain [0 1].
- Solution is an analytic polynomial function.
- The degree of solution polynomial will not exceed, μ∈IN

Where μ = (degree of Q(x)-degree of P(x))/m, .i.e., telescoping of the Tailor series is proposed suitably by truncation of terms of the series carried out after definite stage. The method is illustrated explicitly by treating some examples from available literature.

3. Numerical Illustrations

In this section, we intend to put forward and show the power and potential of the proposed method by applying it on to some of the examples present in the literature and so for that had been treated with by some other alternative methods

3.1 Example

Consider the boundary value problem [30]

$$-y''(x) + 2\frac{y'(x)}{x} + (1-x^2)y(x) = x^4 - 2x^2 + 7 \qquad x>0 \tag{3.1.1}$$

Subject to boundary condition $y'(0) = 0,\ y(1) = 0$

Solution: we rewrite (3.11) to handle singularity at point zero as follows

$$xy''(x) + 2y'(x) - x(1-x^2)y(x) = -x^5 + 2x^3 - 7x \tag{3.1.2}$$

Subject to boundary condition $y'(0) = 0,\ y(1) = 0$ for every x in [0 1]

Differentiating (3.1.2) successively and then taking limit as (x→0) we get

$$y^2(0) = \frac{y(0)-7}{3} \tag{3.1.3}$$

$$y^3(0)=0 \tag{3.1.4}$$

$$y^4(0) = \frac{3y''(0)-6y(0)+12}{5} = 1 - y(0) \tag{3.1.5}$$

$$y^5(0) = 0 \tag{3.1.6}$$

$$y^{(6)}(0) = \frac{25(1-y(0))}{7} \tag{3.1.7}$$

Obviously all the odd derivatives are zero at the point zero on the real line and even derivatives are given by the following recurrence relations among derivatives as

$$y^{(k)}(0) = \frac{1}{k+1}\ [\ (k-1)y^{k-2}(0) - (k-1(k-2)(k-3)y^{k-4}(0)].,\ k \text{ is even integer} \geq 8 \tag{3.1.8}$$

Evidently, all even derivatives except the second order are multiples of $(y(0)-1)$.

Therefore, the solution to the given problem by Taylor series method is as

$$y(x) = y(0) + \frac{y(0)\_7}{2!3}x^2 + \frac{5(1-y(0))}{4!5}x^4 + \frac{25(1-y(0))}{6!7}x^6 + \frac{185(1-y(0))}{8!9}x^8 + \cdots$$

$$\text{or, } y(x) = y(0) + \frac{y(0)-7}{2!3}x^2 + (1-y(0))[\frac{5}{4!5}x^4 + \frac{25}{6!7}x^6 + \frac{185}{8!9}x^8 + \cdots] \tag{3.1.9}$$

Imposing the given next boundary condition y (1) =0 on (3.1.9) we have

$$0 = y(0) + \frac{y(0)-7}{2!3} + (1-y(0))[\frac{5}{4!5} + \frac{25}{6!7} + \frac{185}{8!9} + \cdots]$$

That gives , $y(0) = 1$

Substituting the value of y(0) in (3.1.9) ,we have the solution

$$y(x) = 1 - x^2$$

Interestingly, if we apply (2.3.2) to this problem we straightway have $y^n(0) = 0$, whenever n≥ 4, implying thereby $y^4(0) = \frac{3y''(0)-6y(0)+12}{5} = 1 - y(0) = 0$ implying that y(0)=1 and $y(x) = 1 - x^2$ is a solution. Evidently introduce and involvement of telescoping process produces required solution quite comfortably in an easy manner.

3.2 Example

Consider the nonhomogeneous boundary value problem [29]

$y''(x)+\frac{8}{x}\ y'(x)+xy(x)=x^5-\ x^4+44x^2-30x$ x> 0 (3.2.1)

Subject to initial condition $y(0)=0$, $y'(0)=0$ (3.2.2)

or

subject to boundary condition $y(0)=0$, $y(1)=0$ (3.2.3)

for applying the proposed method of solution (63.2.1) is required to be modified as

$xy''(x)+8y(x)+x^2y(x)=x^6-x^5+44x^3-30x^2$ $0\le x\le 1$ (3.2.4)

In view of (3.2.2) if modified differential equation is differentiated step by step and (x→0) performed then we have

$y^2(0)=0$
$y^3(0)=-3!$
$y^n(0)=0$, whenever n >4

Therefore the solution of the initial value problem by proposed method is $y(x)=x^4-x^3$.

3.3 *Example*

Consider the nonlinear differential equation [30]

$y''(x)+\frac{2}{x}\quad y'(x)+y^3(x)=x^6+6$ (3.3.1)

Subject to boundary condition $y'(0)=0$, $y(1)=1$ (3.3.2)

The proposed method for finding the solution of (3.3.1) along with (3.3.2) is applied on the modified differential equation

$xy''(x)+2y'(x)+xy^3(x)=x^7+6x$ $0\le x\le 1$ (3.3.3)

with boundary conditions (3.3.2)

Differentiate (3.3.3) step by step to get various derivatives of dependent variable at point zero as

$y_2(0)=2!-y^3(0)$ (3.3.4)

$y_3(o)=0$ (3.3.5)

$y_4(0)=\left(-\frac{9}{5}\right)y^2(0)\,y_2(0)$ (3.3.6)

$y_5(0)=0$ (3.3.7)

$y_6(0)=\frac{9}{7}\left(3y^3(0)-8y_2(0)\right)y(0)y_2(0)$ (3.3.8)

$y_7(0)=0$

Then the solution to the given problem is given by Tailors series as

$$y(x)=y(0)+y_1(0)x+\frac{1}{2!}y_2(0)x^2+\frac{1}{3!}y_3(0)x^3+\frac{1}{4!}y_4(0)x^4+\frac{1}{5!}y_5(0)x^5+\frac{1}{6!}y_6(0)x^6+\ldots\ldots$$

or, $y(x)=y(0)+\frac{1}{2!}y_2(0)x^2+\frac{1}{4!}y_4(0)x^4+\frac{1}{6!}y_6(0)x^6$ + ------

or, $y(x)=y(0)+\frac{1}{2!}(2-y^3(0))x^2+\frac{1}{4!}y_4(0)x^4+\frac{1}{6!}y_6(0)x^6$ +------

Obviously on taking limit (x→ 1) we get

$1=y(0)+\frac{1}{2!}(2-y^3(0))+\frac{1}{4!}y_4(0)+\frac{1}{6!}y_6(0)$ +-------

Clearly y(0) =0 is root of the above equation yielding y(x) =x^2 as the solution to the problem (3.3.1) with its imposed condition. Moreover, adopting (2.3.2) instantly on (3.3.1) implies $y_n(0) = 0$ for all (n ≥ 3) thereby asserting immediately and immaculately that y(0)=0 to have y=x^2 as the required solution to the problem (3.3.1).

3.4 Example

Consider the non-linear differential equation [35]

$$y''(x) + \frac{4}{x}y'(x) + y^2(x) = x^6 + 4x^3 + 18x + 4 \tag{3.4.1}$$

Subject to $y(0) = 2,$ $y'(0) = 0$ (3.4.2)

Now we modify differential equation (3.4.1) at singular point $x = o$ as follows

$$xy''(x) + 4\,y'(x) + xy^2(x) = x^7 + 4x^4 + 18x^2 + 4x, \quad \text{for every } x \in [0\ 1] \tag{3.4.3}$$

Now for finding appropriate solution of the given problem we differentiate (3.4.3) repeatedly and stepwise.

$$y_2(0) = \frac{1}{5}(4 - y^2(0)) \ = \ \frac{1}{5}\big(2 + y(0)\big)(2 - y(0)) \tag{3.4.4}$$

$$y_3(0) = 6 \tag{3.4.5}$$

$$y_4(0) = (-1)\frac{6}{7}y(0)\,y_2(0) \tag{3.4.6}$$

$$y_5(0) = 6(-y(0) + 2 \tag{3.4.7}$$

$$y_6(0) = \frac{2}{21}(2y^2(0) - 70y_2(0) - 16)\ \ y_2(o) \tag{3.4.8}$$

Now by the use of condition y (0) =2 we get $y_3(0) = 6,$ $y_5(0)$ =0 and $y_2(0)$=$y_4(0)$=$y_6(0)$=0

Now vanishing of fourth, fifth and sixth derivatives at zero simultaneously implies
$y_n(0) = 0$ for all n≥ 7. So required analytical solution is given by $y(x) = x^3 + 2$.

Again if we consider the differential equation (3.4.1) under imposed boundary condition $y^{/}(0)$=0 and $y(1)$=3.Surprisingly, though the solution process involved inquest of solution to boundary value problem (3.4.1) and (3.4.2) appears to be quite tedious and cumbersome. But indeed, by use of (2.3.2) as an essential fact implies straight way $y_n(0) = 0$ for all (n≥ 4) and that upon induction into deduced relations (3.4.5),(3.4.7) and (3.4.6) produces $y(x) = x^3 + 2$ as the required solution satisfactorily.

3.5 Example

Consider the differential equation [17,21]

$$y''(x) + \frac{y'(x)}{x} + y(x) = 4 - 9x + x^2 - x^3 \tag{3.5.1}$$

Subject to the boundary conditions

$$y'(0) = 0\ ,\ y(1) = 0 \tag{3.5.2}$$

Clearly x=0 is the singular point of the differential equation (3.5.1) consequently, we modify the differential equation (3.5.1) at singular point $x = o$ as follows

$$xy''(x) + y(x) + xy(x) = 4x - 9x^2 + x^3 - x^4 \quad \text{for every} \quad x \in [0\ 1] \tag{3.5.3}$$

As of now, differentiating (3.5.3) recursively and taking (x$\rightarrow 0$) we get

$$y_2(0) = \frac{1}{2}(4 - y(0)) \tag{3.5.4}$$
$$y_3(0) = -6 \tag{3.5.5}$$

$$y_4(0) = \left(-\frac{3}{4}\right)(y_2(0) - 2) \tag{3.5.6}$$

$$y_5(0) = \left(-\frac{4}{5}\right)(y_3(0) + 6) = 0 \tag{3.5.7}$$

$$y_6(0) = \left(-\frac{3}{4}\right)\left(-\frac{5}{6}\right)(y_2(0) - 2) \tag{3.5.8}$$

$$y_7(0) = \left(\frac{6}{7}\right) y_5(0) = 0 \tag{3.5.9}$$

$$y_8(0) = \left(-\frac{7}{8}\right) y_6(0) = \left(-\frac{7}{8}\right)\left(-\frac{5}{6}\right)\left(-\frac{3}{4}\right)(y_2(0) - 2) \tag{3.5.10}$$

It is apparent that all odd derivatives are vanishing except n=3 and

$y_{n+7}(0) = -\left(\frac{n+6}{n+7}\right) y_{n+5}(0)$ for 'n' odd for (n≥1)

Then the Taylors series solution is as

$$y(x) = y(0) + \frac{1}{2!} y_2(0)x^2 - x^3 + \left(-\frac{3}{4}\right)(y_2(0) - 2)[\frac{1}{2!}x^4 - \frac{5}{6}.\frac{1}{6!}x^6 + \frac{7}{8}\frac{5}{6}.\frac{1}{9!}x^9 -''' \ldots$$

Now taking limit(x→0) and using y(0)=2(2-$y_2(0)$) from (3.5.4) we obtain $(y_2(0) - 2)$=0. Thus the solution is y(x)=$x^2 - x^3$. Also in spite of above simplifications if we use (2.3.2) abinitio we must have $y_n(0)$=0 for every n≥4 rendering $y_2(0) = 2$ again therefore implying that y(x)=$x^2 - x^3$ is the required solution.

4. Conclusion

In this paper, we have applied Taylor series method successfully to find solutions of linear as well as nonlinear subclass of boundary (initial) value problems. We have used a subsidiary method to maneuver exact solution through simple and easy simplifications. Referenced problems from literature have been solved successfully to prove importance and compatibility of proposed process. In a nutshell, the involvement of subsidiary telescoping process improvises the exploring process of solution to the given class of such scientifically modeled problems very successfully.

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