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A New Computational Methodology to Find Appropriate Solutions of Fuzzy Equations

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Abstract

In this paper, a new computational methodology to get an appropriate solution of a fuzzy equation of the form $\mu.\eta = \gamma$, where μ, γ are known continuous triangular fuzzy numbers and η is an unknown fuzzy number, are presented. In support of that some propositions with proofs and theorems are presented. A different approach of the definition of 'positive fuzzy number' and 'negative fuzzy number' have been focused. Also, the concept of 'half-positive and half-negative fuzzy number' has been introduced. The solution of the fuzzy equation can be 'positive fuzzy number' or 'negative fuzzy number' or 'half positive or half negative fuzzy number' which is computed by using the methodology focused in the proposed propositions.

Keywords: Fuzzy number, Fuzzy equation, Positive fuzzy number, Negative fuzzy number, half positive and half negative fuzzy number, $\alpha - cut$ of a fuzzy number.

1. Introduction

In most cases in our life, the data obtained for decision making are only approximately known. The concept of fuzzy set theory to meet those problems have been introduced [11]. The fuzziness of a property lies in the lack of well defined boundaries [i.e., ill-defined boundaries] of the set of objects, to which this property applies. Therefore, the membership grade is essential to define the fuzzy set theory.

The notion of fuzzy numbers has been introduced from the idea of real numbers [4] as a fuzzy subset of the real line. There are arithmetic operations, which are similar to those of the set of real numbers, such that +, –, . , /, on fuzzy numbers [6 – 8]. Fuzzy numbers allow us to make the mathematical model of linguistic variable or fuzzy environment, and are also used to describe the data with vagueness and imprecision.

The definition of 'positive fuzzy number' and 'negative fuzzy number' have been introduced [5, 9]. The shortcoming of the definitions [5] has been focused [10] and the concept of 'nonnegative fuzzy numbers' has been introduced [10] as well. None has introduced the notion of 'half-positive and half-negative fuzzy number'. In this paper, a different approach of the definitions of 'positive fuzzy number' and 'negative fuzzy number' have been focused; and a new notion of 'half-positive and half-negative fuzzy number' has been introduced. There are another notion in the fuzzy set theory is the concept of the solution of fuzzy equations [8] of the form $\mu + \eta = \gamma$ and $\mu.\eta = \gamma$, which have been discussed in [1 – 3, 8]. It is easy to solve the fuzzy equation of the form $\mu + \eta = \gamma$, where μ, γ are known fuzzy numbers and η is an unknown fuzzy number [8], but there are some limitations to solve the fuzzy equation of the form $\mu.\eta = \gamma$, where η is an unknown fuzzy number. Our main objective is to introduce a new computational methodology to overcome the limitations to get a solution, if it exists, of the fuzzy equation of the form $\mu.\eta = \gamma$ where μ and γ are known continuous triangular fuzzy numbers. Here it is noted that the core of a known continuous triangular fuzzy number is a singleton set.

2. Preliminaries

In this section, some definitions [1 – 11] have been reviewed which are important to us for representing our main objective in the later sections. Let $I^{\mathbb{R}}$ be the set of all fuzzy numbers and $\mu \in I^{\mathbb{R}}$ means that μ is a fuzzy number whose membership function is $\mu: \mathbb{R} \to [0,1]$.

2.1 Definition **:** The $\alpha - cut$ of a fuzzy set μ is denoted by ${}^{\alpha}\mu = [\underline{\mu}(\alpha), \mu(1), \overline{\mu}(\alpha)]$ and is defined by ${}^{\alpha}\mu = \{x : \mu(x) \geq \alpha\}$, $\forall \alpha \in [0,1]$.

2.2 Definition **:** The strong $\alpha - cut$ of a fuzzy set μ is denoted by ${}^{\alpha+}\mu = \left(\underline{\mu}(\alpha), \mu(1), \overline{\mu}(\alpha)\right)$ and is defined by ${}^{\alpha+}\mu = \{x : \mu(x) > \alpha\}$, $\forall \alpha \in [0,1]$.

2.3 Definition **:** The support of a fuzzy set μ is denoted by ${}^{0+}\mu = \left(\underline{\mu}(0), \mu(1), \overline{\mu}(0)\right)$ and is defined by ${}^{0+}\mu = \{x : \mu(x) > 0\}$.

2.4 Definition **:** A fuzzy set $\mu \in I^{\mathbb{R}}$ is normal if there exist $x \in \mathbb{R}$, s.t $\mu(x) = 1$.

2.5 Definition **:** A fuzzy number is a fuzzy set, whose membership function is denoted by $\mu: \mathbb{R} \to [0,1]$, which satisfies the conditions as under :

(a) μ is normal fuzzy set;
(b) ${}^{\alpha}\mu$ is a closed interval $\forall \alpha \in (0,1]$;
(c) support of μ, i.e., ${}^{0+}\mu$ is a bounded set in the classical sense.

That is, a fuzzy number satisfies the condition of normality and convexity.

2.6 Definition **[5] :** A fuzzy number μ is called positive (negative), denoted by $\mu > 0$ ($\mu < 0$), if its membership function $\mu(x)$ satisfies $\mu(x) = 0, \forall x < 0$ $(x > 0)$.

2.7 Definition **[10] :** A fuzzy number μ is called positive, denoted by $\mu > 0$, if its membership function $\mu(x)$ satisfies $\mu(x) = 0 \ \forall x \leq 0$.

2.8 Definition **[10] :** A fuzzy number μ is called nonnegative, denoted by $\mu \geq 0$, if its membership function $\mu(x)$ satisfies $\mu(x) = 0 \ \forall x < 0$.

3. Existence of a Solution of a Fuzzy Equation

Consider the fuzzy equation $\mu \cdot \eta = \gamma$, where μ, γ are known fuzzy numbers and η is an unknown fuzzy number. If ${}^{\alpha}\mu = [\underline{\mu}(\alpha), \mu(1), \overline{\mu}(\alpha)]$, ${}^{\alpha}\eta = [\underline{\eta}(\alpha), \eta(1), \overline{\eta}(\alpha)]$ and ${}^{\alpha}\gamma = [\underline{\gamma}(\alpha), \gamma(1), \overline{\gamma}(\alpha)]$ are $\alpha - cuts$ of μ, η and γ, respectively, then the fuzzy equation $\mu \cdot \eta = \gamma$ has a solution if and only if the equation

$$[\underline{\mu}(\alpha), \mu(1), \overline{\mu}(\alpha)] \cdot [\underline{\eta}(\alpha), \eta(\alpha = 1), \overline{\eta}(\alpha)] = [\underline{\gamma}(\alpha), \gamma(\alpha = 1), \overline{\gamma}(\alpha)] \quad \text{(A)}$$

has a solution ${}^{\alpha}\eta = [\underline{\eta}(\alpha), \eta(1), \overline{\eta}(\alpha)]$ and ${}^{\alpha}\eta$ satisfies the following conditions [8] :

Condition 1: $\underline{\eta}(\alpha) \leq \eta(1) \leq \overline{\eta}(\alpha) \quad \forall \alpha \in (0,1]$. (B)

Condition 2 : If $\alpha \leq \beta$ then $\underline{\eta}(\alpha) \leq \underline{\eta}(\beta) \leq \overline{\eta}(\beta) \leq \overline{\eta}(\alpha) \quad \forall 0 < \alpha \leq \beta \leq 1$. (C)

4. New Proposed Definitions

Here we have introduced some definitions which will help us to solve the fuzzy equation of the form $\mu \cdot \eta = \gamma$, where μ, γ are known continuous fuzzy numbers and η is an unknown fuzzy number. The definitions are as follows and will be used in the next section.

4.1 Definition **:** A triangular fuzzy number μ is called negative, denoted by $\mu < 0$, if $\forall \alpha \in [0,1]$ there exist ${}^{\alpha}x_1, {}^{\alpha}x_2, {}^{\alpha}x_3 \in \mathbb{R}$ where ${}^{\alpha}x_1 = \underline{\mu}(\alpha)$, ${}^{\alpha}x_2 = \overline{\mu}(\alpha)$, ${}^{\alpha}x_3 = \mu(\alpha = 1)$ such that

$$\underline{\mu}(\alpha) \leq \mu(\alpha = 1) \leq \overline{\mu}(\alpha) \leq 0, \text{ and } {}^{\alpha}\mu = [\underline{\mu}(\alpha), \mu(1), \overline{\mu}(\alpha)].$$

4.2 Example **:** μ is a negative fuzzy number which is defined by

$$\mu(x)=\begin{cases} x+4 & for -4 \le x < -3 \\ -(x+1)/2 & for -3 \le x \le -1 \\ 0 & otherwise \end{cases},$$

where $\underline{\mu}(\alpha) = \alpha - 4 \le \mu(\alpha = 1) = -3 \le \overline{\mu}(\alpha) = -2\alpha - 1 \le 0$, and ${}^{\alpha}\mu = [\alpha - 4, -3, -2\alpha - 1]$ $\forall\, \alpha \in [\,\overline{0}, 1]$.

4.3 Definition **:** A triangular fuzzy number μ is called positive, denoted by $\mu > 0$, if $\forall\, \alpha \in [\,0, 1]$ there exist ${}^{\alpha}x_1, {}^{\alpha}x_2, {}^{\alpha}x_3 \in \mathbb{R}$ where ${}^{\alpha}x_1 = \underline{\mu}(\alpha)$, ${}^{\alpha}x_2 = \overline{\mu}(\alpha)$, ${}^{\alpha}x_3 = \mu(\alpha = 1)$ such that

$$0 \le \underline{\mu}(\alpha) \le \mu(\alpha = 1) \le \overline{\mu}(\alpha), \text{ and } {}^{\alpha}\mu = [\underline{\mu}(\alpha), \mu(1), \overline{\mu}(\alpha)].$$

4.4 Example **:** μ is a positive fuzzy number which is defined by

$$\mu(x)=\begin{cases} x-1 & for\ \ 1 \le x < 2 \\ (5-x)/3 & for\ \ 2 \le x \le 5 \\ 0 & otherwise \end{cases},$$

where $0 \le \underline{\mu}(\alpha) = \alpha + 1 \le \mu(\alpha = 1) = 2 \le \overline{\mu}(\alpha) = 5 - 3\alpha$, and ${}^{\alpha}\mu = [\,\alpha + 1,\ 2,\ 5 - 3\alpha]$ $\forall\, \alpha \in [\,0, 1]$.

4.5 Definition [Half positive and half negative] **:** A triangular fuzzy number μ is called 'half-positive and half-negative', denoted by $\underline{\mu} < 0 < \overline{\mu}$, if $\forall\, \alpha \in [\,0, 1]$ there exist ${}^{\alpha}x_1, {}^{\alpha}x_2, {}^{\alpha}x_3 \in \mathbb{R}$ where ${}^{\alpha}x_1 = \underline{\mu}(\alpha)$, ${}^{\alpha}x_2 = \overline{\mu}(\alpha)$, ${}^{\alpha}x_3 = \mu(\alpha = 1)$ such that

$$\underline{\mu}(\alpha) < 0 = \mu(\alpha = 1) < \overline{\mu}(\alpha), \text{ and } {}^{\alpha}\mu = [\underline{\mu}(\alpha), \mu(1), \overline{\mu}(\alpha)].$$

4.6 Example **:** μ is a half-positive and half-negative fuzzy number which is defined by

$$\mu(x)=\begin{cases} (x+2)/2 & for -2 \le x < 0 \\ (5-x)/5 & for\ \ 0 \le x \le 5 \\ 0 & otherwise \end{cases}$$

where $\underline{\mu}(\alpha) = 2\alpha - 2 < \mu(\alpha = 1) = 0 < \overline{\mu}(\alpha) = 5 - 5\alpha$ and ${}^{\alpha}\mu = [\,2\alpha - 2,\ 0,\ 5 - 5\alpha]$.

Figure 1 represents the fuzzy numbers which are given in examples 4.2, 4.4, and 4.6.

5. Problems, Discussions, and Results

In this section, we have proposed some propositions with their proofs, which will help us to solve the fuzzy equation $\mu \cdot \eta = \gamma$ without any difficulties and within a reasonable time. We have also established related theorems. In support of that some problems and their solutions have also been investigated.

5.1 Proposition **:** If $\mu, \gamma \le 0$ are known fuzzy numbers and η is any unknown fuzzy number, then the solution of the fuzzy equation $\mu \cdot \eta = \gamma$ is a positive fuzzy number.

Proof : Given that $\mu, \gamma \le 0$ and the fuzzy equation $\mu \cdot \eta = \gamma$. Then, ${}^{\alpha}\mu = [\underline{\mu}(\alpha), \mu(\alpha = 1), \overline{\mu}\,(\alpha)]$ and ${}^{\alpha}\gamma = [\underline{\gamma}(\alpha), \gamma(\alpha = 1), \overline{\gamma}\,(\alpha)]$, where $\underline{\mu}(\alpha) \le \mu(\alpha = 1) \le \overline{\mu}\,(\alpha) \le 0$ and $\underline{\gamma}(\alpha) \le \gamma(\alpha = 1) \le \overline{\gamma}\,(\alpha) \le 0$. Now, via $\alpha - cut$ representation, we have, $\mu \cdot \eta = \gamma \implies {}^{\alpha}\mu \cdot {}^{\alpha}\eta = {}^{\alpha}\gamma$.

Then, $\exists\ \underline{\eta}(\alpha) \ge 0$ and $\overline{\eta}\,(\alpha) \ge 0$ such that $\forall\ x \in [\underline{\eta}(\alpha), \eta(\alpha = 1), \overline{\eta}\,(\alpha)]$, $\underline{\mu}(\alpha) \cdot x = \underline{\gamma}(\alpha) \le 0$ and $\overline{\mu}(\alpha) \cdot x = \overline{\gamma}\,(\alpha) \le 0$. That is, $[\underline{\mu}(\alpha), \mu(\alpha = 1), \overline{\mu}\,(\alpha)] \cdot [\underline{\eta}(\alpha), \eta(\alpha = 1), \overline{\eta}\,(\alpha)] = [\underline{\gamma}(\alpha), \gamma(\alpha = 1), \overline{\gamma}\,(\alpha)]$ is true if each $x \in [\underline{\eta}(\alpha), \eta(\alpha = 1), \overline{\eta}\,(\alpha)]$ is positive. Hence, the solution η of the fuzzy equation $\mu \cdot \eta = \gamma$ is a 'positive fuzzy number'.

5.2 Problem **:** Suppose that μ and γ are two triangular negative continuous fuzzy numbers, where

$$\mu(x)=\begin{cases}(x+7)/2 & if\ -7<x\le -5\\ -(x+3)/2 & if\ -5<x<-3\\ 0 & otherwise.\end{cases};\qquad \gamma(x)=\begin{cases}(x+6)/3 & if\ -6<x\le -3\\ -x/3 & if\ -3<x<0\\ 0 & otherwise.\end{cases}.$$

Solve the fuzzy equation $\mu\cdot\eta=\gamma$ for the unknown fuzzy number η.

Solution : Given the fuzzy equation $\mu\cdot\eta=\gamma$, (1)

where μ and γ are known negative fuzzy numbers and η the unknown fuzzy number. Here, ${}^{\alpha}\mu=[\,2\alpha-7,-5,-2\alpha-3\,]$ and ${}^{\alpha}\gamma=[\,3\alpha-6,-3,-3\alpha\,]$. Now, we solve the following equation for the unknown ${}^{\alpha}\eta$,

i.e., ${}^{\alpha}\mu\,.\,{}^{\alpha}\eta={}^{\alpha}\gamma$ (2)

Since $\mu,\ \gamma<0$, we choose three cases for unknown fuzzy number :

$(i)\ \eta>0,(ii)\ \eta<0,(iii)\ \underline{\eta}<0<\overline{\eta}$.

Case (i) : Consider $\eta>0$. Then, ${}^{\alpha}\eta=[\underline{\eta}(\alpha),\ \eta(\alpha=1),\ \overline{\eta}(\alpha)]$, where $\underline{\eta}(\alpha),\ \eta(\alpha=1),\overline{\eta}(\alpha)\ge 0$. Therefore, $\mu\cdot\eta=\gamma\ \Rightarrow\ {}^{\alpha}\mu\,.\,{}^{\alpha}\eta={}^{\alpha}\gamma$

$$\Rightarrow[2\alpha-7,-5,-2\alpha-3]\,.\,[\,\underline{\eta}(\alpha),\ \eta(1),\ \overline{\eta}(\alpha)\,]=[3\alpha-6,-3,-3\alpha].$$

So, ${}^{\alpha}\eta=\left[\underline{\eta}(\alpha),\ \eta(1),\ \overline{\eta}(\alpha)\right]=[3\alpha/(2\alpha+3)\,,3/5,\ (3\alpha-6)/(2\alpha-7)\,]$. Since ${}^{\alpha}\eta$ satisfies (A), (B) and (C) $\forall\ \alpha\in(0{,}1]$, it is a solution of equation (2) and hence, $\underline{\eta}$ is the solution of the fuzzy equation (1) whose membership function is as follows :

$$\eta(x)=\begin{cases}3x/(3-2x) & if\ \ 0<x\le 3/5\\ (7x-6)/(2x-3) & if\ \ 3/5\le x<6/7\\ 0 & otherwise\end{cases}.$$

The graphical representation of μ, γ and η are shown in Figure 2 where the graph of μ is shown by dashed lines.

Case (ii) : Consider $\eta<0$. Then, ${}^{\alpha}\eta=[\,\underline{\eta}(\alpha),\ \eta(\alpha=1),\ \overline{\eta}(\alpha)]$, where $\underline{\eta}(\alpha),\ \eta(\alpha=1),\ \overline{\eta}(\alpha)\le 0$.

So, ${}^{\alpha}\eta=\left[\underline{\eta}(\alpha),\ \eta(1),\ \overline{\eta}(\alpha)\right]=[3\alpha/(7-2\alpha)\,,3/5\ ,(3\alpha-6)/(-2\alpha-3)]$, and it does not satisfy the equation (A) for $\alpha=0.5$. Therefore, $\eta<0$ is not a solution of (1).

Case (iii) : Suppose that $\underline{\eta}<0<\overline{\eta}$. Then, ${}^{\alpha}\eta=[\,\underline{\eta}(\alpha),\ \eta(\alpha=1),\ \overline{\eta}(\alpha)]$,

where $\underline{\eta}(\alpha)\ \le 0,\ \eta(\alpha=1)=0,\ \overline{\eta}(\alpha)\ge 0$. Now, we have

${}^{\alpha}\eta=\left[\underline{\eta}(\alpha),\ \eta(1),\ \overline{\eta}(\alpha)\right]=[3\alpha/(7-2\alpha)\,,3/5\ ,(3\alpha-6)/(2\alpha-7)]$, and it does not satisfy the equation (A) for $\alpha=0.5$. So, for the case $\underline{\eta}<0<\overline{\eta}$, η is not a solution of (1).

5.3 Proposition **:** If $\mu,\gamma\ge 0$ are known fuzzy numbers and η is any unknown fuzzy number, then the solution of the fuzzy equation $\mu\cdot\eta=\gamma$ is a positive fuzzy number.

Proof : Given that $\mu,\gamma\ge 0$ and the fuzzy equation $\mu\cdot\eta=\gamma$. Then, ${}^{\alpha}\mu=[\underline{\mu}(\alpha),\mu(\alpha=1),\overline{\mu}\,(\alpha)]$ and ${}^{\alpha}\gamma=[\underline{\gamma}(\alpha),\gamma(\alpha=1),\overline{\gamma}\,(\alpha)]$, where $0\le\underline{\mu}(\alpha)\le\mu(\alpha=1)\le\overline{\mu}(\alpha)$ and $0\le\underline{\gamma}(\alpha)\le\gamma(\alpha=1)\le\overline{\gamma}(\alpha)$. Now, via $\alpha-cut$ representation, we have $\mu\cdot\eta=\gamma\Rightarrow\ {}^{\alpha}\mu\cdot{}^{\alpha}\eta={}^{\alpha}\gamma$. Then, $\exists\ \underline{\eta}(\alpha)\ge 0$ and $\overline{\eta}\,(\alpha)\ge 0$ such that $\forall\ x\in[\underline{\eta}(\alpha),\eta(\alpha=1),\overline{\eta}\,(\alpha)]$, $\underline{\mu}(\alpha)\cdot x=\underline{\gamma}(\alpha)\ge 0$ and $\overline{\mu}(\alpha)\cdot x=\overline{\gamma}\,(\alpha)\ge 0$. That is, $[\underline{\mu}(\alpha),\mu(\alpha=1),\overline{\mu}\,(\alpha)]\cdot[\underline{\eta}(\alpha),\eta(\alpha=1),\overline{\eta}\,(\alpha)]=[\underline{\gamma}(\alpha),\gamma(\alpha=1)\,,\overline{\gamma}\,(\alpha)]$ is true only if each $x\in[\underline{\eta}(\alpha),\eta(\alpha=1),\overline{\eta}\,(\alpha)]$ is positive. Hence, the solution η of the fuzzy equation $\mu\cdot\eta=\gamma$ is a 'positive fuzzy number'.

5.4 Problem **:** Suppose that $\mu,\gamma>0$ are two triangular fuzzy numbers, where

$$\mu(x) = \begin{cases} (x-1)/2 & if \quad 1 \le x \le 3 \\ (5-x)/2 & if \quad 3 \le x \le 5 \\ 0 & if \quad x < 1 \; and \; x > 5 \end{cases} ; \qquad \gamma(x) = \begin{cases} x/4 & if \quad 0 \le x \le 4 \\ (8-x)/4 & if \quad 4 \le x \le 8 \\ 0 & if \quad x < 0 \; and \; x > 8 \end{cases} .$$

Show that the solution of the fuzzy equation $\mu \cdot \eta = \gamma$ is a positive fuzzy number η.

Solution : Given the fuzzy equation $\mu \cdot \eta = \gamma$. (1)

where μ and γ are known positive fuzzy numbers and η the unknown fuzzy number. Here, ${}^{\alpha}\mu = [2\alpha + 1, 3, 5 - 2\alpha]$ and ${}^{\alpha}\gamma = [4\alpha, 4, 8 - 4\alpha]$. Now, we solve the following equation for the unknown ${}^{\alpha}\eta$,

${}^{\alpha}\mu \,.\, {}^{\alpha}\eta = {}^{\alpha}\gamma.$ (2)

Since $\mu,\ \gamma > 0$, we choose three cases for unknown fuzzy number :

$(i)\ \eta > 0, (ii)\ \eta < 0, (iii)\ \underline{\eta} < 0 < \overline{\eta}$.

Case (i) : Suppose that $\eta > 0$. Then, ${}^{\alpha}\eta = [4\alpha/(2\alpha+1), 4/3, (8-4\alpha)/(5-2\alpha)]$. Since ${}^{\alpha}\eta$ satisfies (A), (B) and (C) $\forall\ \alpha \in (0,1]$, it is a solution of equation (2) and hence, η is the solution of the fuzzy equation (1) whose membership function is as follows :

$$\eta(x) = \begin{cases} x/(4-2x) & if \quad 0 \le x \le 4/3 \\ (8-5x)/(4-2x) & if \quad 4/3 \le x \le 8/5 \\ 0 & if \quad x < 0 \; and \; x > 8/5 \end{cases} .$$

The graphical representation of μ, γ and η are shown in Figure 3 where the graph of μ is shown by dashed lines.

Case (ii) : Suppose that $\eta < 0$. Then, ${}^{\alpha}\eta = [4\alpha/(5-2\alpha), 4/3, (8-4\alpha)/(2\alpha+1)]$. Here, ${}^{\alpha}\eta$ satisfies the conditions (B) and (C). and does not satisfy the equation (A) for $\alpha = 0.5$. So, for the case $\eta < 0$, η is not a solution of (1).

Case (iii) : Suppose that $\underline{\eta} < 0 < \overline{\eta}$. Then, ${}^{\alpha}\eta = [4\alpha/(5-2\alpha), 4/3, (8-4\alpha)/(5-2\alpha)]$. Here, ${}^{\alpha}\eta$ satisfies the conditions (B) and (C), but does not satisfy the equation (A) for $\alpha = 0.5$. So, for the case $\underline{\eta} < 0 < \overline{\eta}$, η is not a solution of (1).

5.5 Proposition **:** If $\mu \le 0$ and $\gamma \ge 0$ are known fuzzy numbers and η is any unknown fuzzy number, then the solution of the fuzzy equation $\mu \cdot \eta = \gamma$ is a negative fuzzy number.

Proof : Given that $\mu \le 0$, $\gamma \ge 0$ and the fuzzy equation $\mu \cdot \eta = \gamma$. Then,

${}^{\alpha}\mu = [\underline{\mu}(\alpha), \mu(\alpha = 1), \overline{\mu}(\alpha)]$ and ${}^{\alpha}\gamma = [\underline{\gamma}(\alpha), \gamma(\alpha = 1), \overline{\gamma}(\alpha)]$,

where $\underline{\mu}(\alpha) \le \mu(\alpha = 1) \le \overline{\mu}(\alpha) \le 0$ and $\underline{\gamma}(\alpha) \ge \gamma(\alpha = 1) \ge \overline{\gamma}(\alpha) \ge 0$. Now, via $\alpha -$ cut representation, we have $\mu \cdot \eta = \gamma \implies {}^{\alpha}\mu \cdot {}^{\alpha}\eta = {}^{\alpha}\gamma$. Then, $\exists\ \underline{\eta}(\alpha) < 0$ and $\overline{\eta}(\alpha) < 0$ such that

$\forall\ x \in [\underline{\eta}(\alpha), \eta(\alpha = 1), \overline{\eta}(\alpha)]$, either (i) $\underline{\mu}(\alpha) \cdot x = \underline{\gamma}(\alpha) \ge 0$and $\overline{\mu}(\alpha) \cdot x = \underline{\gamma}(\alpha) \ge 0$;

or (ii) $\overline{\mu}(\alpha) \cdot x = \overline{\gamma}(\alpha) \ge 0$ and $\underline{\mu}(\alpha) \cdot x = \overline{\gamma}(\alpha) \ge 0$.

That is, $[\underline{\mu}(\alpha), \mu(\alpha = 1), \overline{\mu}(\alpha)] \cdot [\underline{\eta}(\alpha), \eta(\alpha = 1), \overline{\eta}(\alpha)] = [\underline{\gamma}(\alpha), \gamma(\alpha = 1), \overline{\gamma}(\alpha)]$ is verified only if each $x \in [\overline{\eta}(\alpha), \eta(\alpha = 1), \overline{\eta}(\alpha)]$ is negative. Hence, the solution of the fuzzy equation $\mu \cdot \eta = \gamma$ is a 'negative fuzzy number'.

5.6 Problem **:** Suppose that $\mu < 0$ and $\gamma > 0$ are two triangular fuzzy numbers, where

$$\mu(x)=\begin{cases} (x+12)/4 & if -12 < x \le -8 \\ -(x+4)/4 & if -8 < x < -4 \\ 0 & otherwise. \end{cases} \quad ; \quad \gamma(x)=\begin{cases} (x-2)/4 & if\ 3 < x \le 7 \\ (11-x)/4 & if\ 7 < x < 11 \\ 0 & otherwise. \end{cases}$$

Then, show that the solution of the fuzzy equation $\mu \cdot \eta = \gamma$ is a negative fuzzy number.

Solution : Given that $\mu < 0 < \gamma$ and the fuzzy equation $\mu \cdot \eta = \gamma$. (1)

That is, ${}^{\alpha}\mu \cdot {}^{\alpha}\eta = {}^{\alpha}\gamma$ (2)

We have ${}^{\alpha}\mu = [4\alpha - 12, -8, -4\alpha - 4]$ and ${}^{\alpha}\gamma = [4\alpha + 3, 7, 11 - 4\alpha]$. Since $\mu < 0 < \gamma$, so we choose three cases for unknown fuzzy number : $(i)\ \eta > 0, (ii)\ \eta < 0, (iii)\ \underline{\eta} < 0 < \overline{\eta}$.

Case (i) : Suppose that $\eta > 0$. Then, ${}^{\alpha}\eta = [(4\alpha - 11)/(4\alpha + 4), -7/8, (4\alpha + 3)/(4\alpha - 12)]$. Here, ${}^{\alpha}\eta$ satisfies the conditions (B) and (C), but does not satisfy the equation (A) for $\alpha = 0.5$. So, for the case $\eta > 0$, η is not a solution of (1).

Case (ii) : Suppose that $\eta < 0$. Then, ${}^{\alpha}\eta = [(11 - 4\alpha)/(4\alpha - 12), -7/8, -(4\alpha + 3)/(4\alpha + 4)]$. Here, ${}^{\alpha}\eta$ satisfies the conditions (A), (B) and (C) $\forall\ \alpha \in (0, 1]$.

Therefore, ${}^{\alpha}\eta = [(11 - 4\alpha)/(4\alpha - 12), -7/8, -(4\alpha + 3)/(4\alpha + 4)]$ is a solution of (2) and hence $\eta < 0$ is the solution of the fuzzy equation $\mu \cdot \eta = \gamma$. The membership function η is as follows :

$$\eta(x) = \begin{cases} (12x + 11)/4(x + 1) & if \quad -11/12 < x \le -7/8 \\ -(4x + 3)/4(x + 1) & if \quad -7/8 \le x < -3/4 \\ 0 & otherwise \end{cases}.$$

The graphical representation of μ, γ and η are shown in Figure 4 where the graph of μ is shown by dashed lines.

Case (iii) : Suppose that $\underline{\eta} < 0 < \overline{\eta}$. Then, ${}^{\alpha}\eta = [(11 - 4\alpha)/(4\alpha - 12), -7/8, (4\alpha + 3)/(4\alpha - 12)]$. Here, ${}^{\alpha}\eta$ does not satisfy the equation (A) for $\alpha = 0.5$. So, for the case $\underline{\eta} < 0 < \overline{\eta}$, ${}^{\alpha}\eta$ is not a solution of (2).

5.7 Proposition **:** If $\mu \ge 0$ and γ, a half positive and half negative, are known fuzzy numbers and η is any unknown fuzzy number, then the solution of the fuzzy equation $\mu \cdot \eta = \gamma$ is a half positive and half negative fuzzy number.

Proof : Given that $\mu \ge 0$, γ is a half positive and half negative fuzzy number, and the fuzzy equation $\mu \cdot \eta = \gamma$, where η is an unknown fuzzy number. Then, ${}^{\alpha}\mu = [\underline{\mu}(\alpha), \mu(\alpha = 1), \overline{\mu}(\alpha)]$ and

${}^{\alpha}\gamma = [\underline{\gamma}(\alpha), \gamma(\alpha = 1), \overline{\gamma}(\alpha)]$, where $0 \le \underline{\mu}(\alpha) \le \mu(\alpha = 1) \le \overline{\mu}(\alpha)$ and $\underline{\gamma}(\alpha) < \gamma(\alpha = 1) = 0 < \overline{\gamma}(\alpha)$. Now, via $\alpha - cut$ representation, we have $\mu \cdot \eta = \gamma \implies {}^{\alpha}\mu \cdot {}^{\alpha}\eta = {}^{\alpha}\gamma$.

Then, $\exists\ x_1, x_2 \in [\underline{\mu}(\alpha), \mu(\alpha = 1), \overline{\mu}(\alpha)]$ and $\exists\ y_1, y_2 \in [\underline{\eta}(\alpha), \eta(\alpha = 1), \overline{\eta}(\alpha)]$ such that

$$\underline{\gamma}(\alpha) = x_1 \cdot y_1 < 0, \quad \overline{\gamma}(\alpha) = x_2 \cdot y_2 > 0 \text{ and } x \cdot \eta(1) = \gamma(\alpha = 1) = 0.$$

Which implies that $y_1 = \underline{\eta}(\alpha) < 0$ and $y_2 = \overline{\eta}(\alpha) > 0$. Therefore, ${}^{\alpha}\eta = [\underline{\eta}(\alpha), \eta(\alpha = 1), \overline{\eta}(\alpha)]$ is the solution of $[\underline{\mu}(\alpha), \mu(\alpha = 1), \overline{\mu}(\alpha)] \cdot [\underline{\eta}(\alpha), \eta(\alpha = 1), \overline{\eta}(\alpha)] = [\underline{\gamma}(\alpha), \gamma(\alpha = 1), \overline{\gamma}(\alpha)]$, that is, the corresponding fuzzy number η , which is a 'half positive and half negative fuzzy number', is the solution of $\mu \cdot \eta = \gamma$.

5.8 Problem **:** Suppose that $\mu > 0$ and γ, a half positive and half negative, are two triangular fuzzy numbers, where

$$\mu(x)=\begin{cases}(x-1)/2 & if\ \ 1<x\leq 3\\ (5-x)/2 & if\ \ 3<x<5\\ 0 & otherwise.\end{cases}\ ;\qquad \gamma(x)=\begin{cases}(x+4)/4 & if -4<x\leq 0\\ (4-x)/4 & if\ \ 0<x<4\\ 0 & otherwise.\end{cases}.$$

Prove that the solution of the fuzzy equation $\mu\cdot\eta = \gamma$ is a half positive and half negative fuzzy number.

Solution : Given the fuzzy equation $\mu\cdot\eta = \gamma$ (1)

That is, ${}^{\alpha}\mu\cdot{}^{\alpha}\eta = {}^{\alpha}\gamma$ (2)

Case (i) : Suppose that $\eta > 0$. Then, ${}^{\alpha}\eta = [(4\alpha-4)/(2\alpha+1), 0, (4-4\alpha)/(5-2\alpha)]$. Here, ${}^{\alpha}\eta$ satisfies the conditions (B) and (C), but does not satisfy the equation (A) for $\alpha = 0.5$. So, for the case $\eta > 0$, η is not a solution of (1).

Case (ii) : Suppose that $\eta < 0$. Then, ${}^{\alpha}\eta = [(4\alpha-4)/(5-2\alpha), 0, (4-4\alpha)/(2\alpha+1)]$. Here, ${}^{\alpha}\eta$ satisfies the conditions (B) and (C), but does not satisfy the equation (A) for $\alpha = 0.5$. So, for the case $\eta > 0$, η is not a solution of (1) too.

Case (iii) : Suppose that $\underline{\eta} < 0 < \overline{\eta}$. Then, ${}^{\alpha}\eta = [(4\alpha-4)/(5-2\alpha), 0, (4-4\alpha)/(5-2\alpha)]$. Here, ${}^{\alpha}\eta$ satisfies the conditions (A), (B) and (C) $\forall\alpha\in(0,1]$.

Therefore, ${}^{\alpha}\eta = [(4\alpha-4)/(5-2\alpha), 0, (4-4\alpha)/(5-2\alpha)]$ is a solution of (2) and hence η is a solution of the fuzzy equation $\mu\cdot\eta = \gamma$. The membership function η is as follows :

$$\eta(x) = \begin{cases}(5x+4)/2(x+2) & if\ \ -4/5<x\leq 0\\ (5x-4)/2(x-2) & if\ \ 0\leq x<4/5\\ 0 & otherwise\end{cases}.$$

So, for the case $\underline{\eta} < 0 < \overline{\eta}$, η is the solution of the fuzzy equation $\mu\cdot\eta = \gamma$. The graphical representation of μ, γ and η are shown in Figure 5 where the graph of μ is shown by dashed lines.

5.9 Proposition **:** If $\mu\leq 0$ and γ, a half positive and half negative fuzzy number, are known fuzzy number and η is any unknown fuzzy number, then than the solution of the fuzzy equation $\mu\cdot\eta = \gamma$ is a half positive and half negative fuzzy number.

Proof : The proof is similar to Proposition.5.7.

5.10 Problem **:** Let $\mu\leq 0$ and γ, a half positive and half negative be two triangular fuzzy numbers, where

$$\mu(x)=\begin{cases}(x+6)/2 & if\ \ -6<x\leq -4\\ -(x+2)/2 & if\ \ -4<x<-2\\ 0\ , & otherwise.\end{cases}\ ;\qquad \gamma(x)=\begin{cases}(x+2)/2 & if\ \ -2<x\leq 0\\ (2-x)/2 & if\ \ 0<x<2\\ 0 & otherwise.\end{cases}.$$

Then, the solution of the fuzzy equation $\mu\cdot\eta = \gamma$ is a 'half positive and half negative fuzzy number' η, where

$$\eta(x) = \begin{cases}(3x+1)/(x+1) & if\ \ -\frac{1}{3}<x\leq 0\\ (3x-1)/(x-1) & if\ 0\leq x<\frac{1}{3}\\ 0 & otherwise\end{cases}.$$

The graphical representation of μ, γ and η are shown in Figure 6, where the graph of μ is shown by dashed lines.

6. Conclusion

In this paper we have established a new methodology to overcome the discussed shortcomings or limitations of the method [8] of the solutions of a fuzzy equation of the form $\mu \cdot \eta = \gamma$, where μ, γ are known positive or negative continuous fuzzy numbers and η is an unknown fuzzy number. For this reason, different approaches of the definitions of 'positive fuzzy number' and 'negative fuzzy number' have been introduced. A new notion of 'half positive and half negative fuzzy number' has also been innovated. Some propositions with their proofs and some related problems with their solutions have been discussed. The propositions will help to assume the sign of unknown fuzzy number η of the fuzzy equation $\mu \cdot \eta = \gamma$ for which we will be able to get a solution of the fuzzy equation easily. After that, some related theorems are presented. There is none who has discussed these notions yet. Without this notion it is very difficult to solve a fuzzy equation of the form discussed above.

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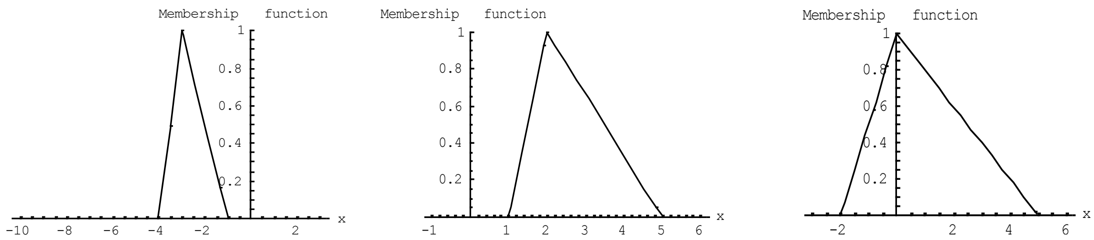

Figure 1 : Graphs of fuzzy numbers which are given in examples 4.2, 4.4, and 4.6.

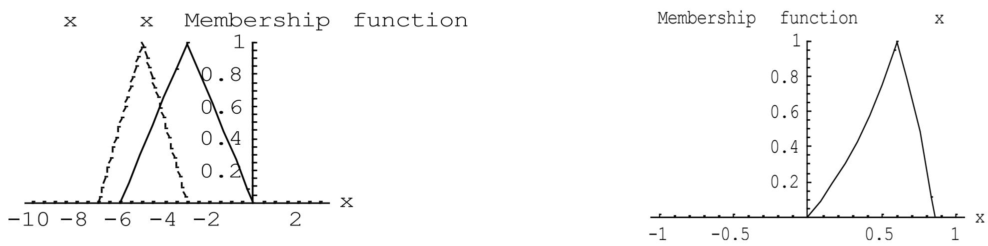

Figure 2 : Graphs of fuzzy numbers μ, γ and the solution fuzzy number η, respectively.

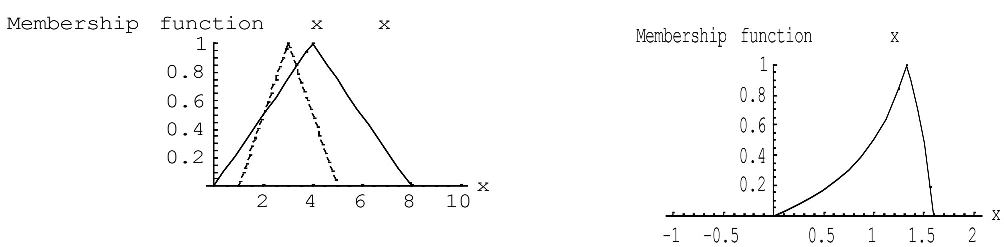

Figure 3 : Graphs of fuzzy numbers μ, γ and the solution fuzzy number η, respectively.

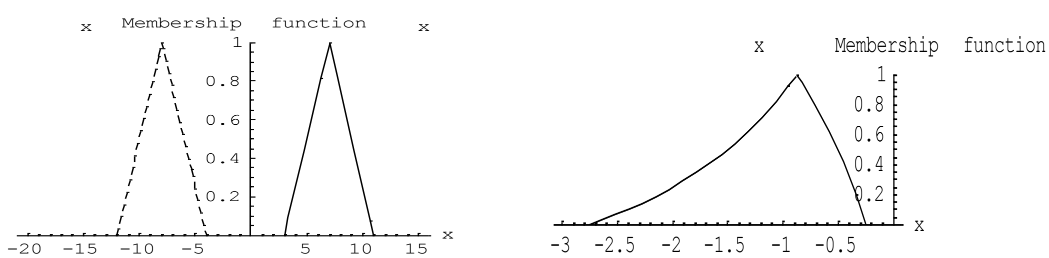

Figure 4 : Graphs of fuzzy numbers μ, γ and the solution fuzzy number η, respectively.

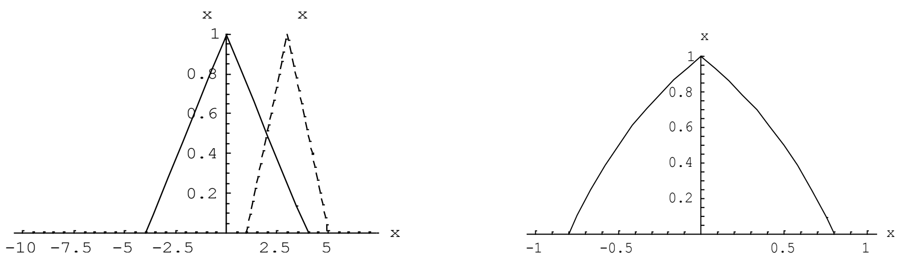

Figure 5 : Graphs of fuzzy numbers μ, γ and the solution fuzzy number η, respectively.

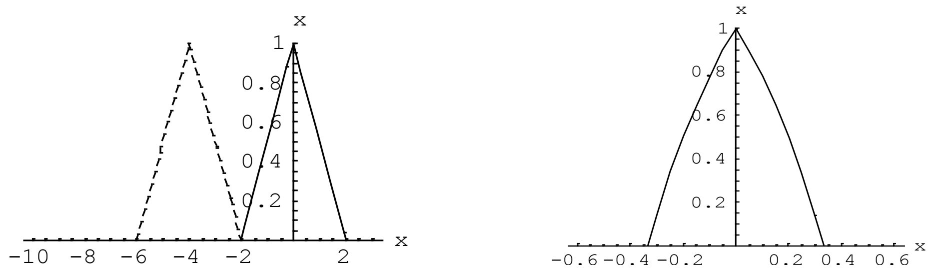

Figure 6 : Graphs of fuzzy numbers μ, γ and the solution fuzzy number η, respectively.

The above tables and figures have been discussed to the relevant sections of this paper.