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# Adomian Decomposition Method for Solving Higher Order Boundary Value Problems

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**Abstract**

In this paper, we present efficient numerical algorithms for the approximate solution of linear and non-linear higher order boundary value problems. Algorithms are, based on Adomian decomposition. Also, the Laplace Transformation with Adomian decomposition technique is proposed to solve the problems when Adomian series diverges. Three examples are given to illustrate the performance of each technique.

**Keyword:** Higher order Singular boundary value problems, Adomian decomposition techniques, Laplace transformations.

## 1. Introduction

The Adomian decomposition method, proposed initially with the aims to solve frontier physical problems, has been applied to a wide class of deterministic and stochastic problems, linear and nonlinear problems, in physics, biology and chemical reactions etc (Adomian 1992). Inspired and motivated by the ongoing research in this area, the method assumes a series solution for the unknown quantity. It has been shown ( Abbaooui & Cherruault 1999, Hosseini & Nasabzadeh 2006, that the series converges fast, and with a few terms this series approximate the exact solution with a fairly reasonable error, normally very less. Each term of this series is a generalized polynomial called the Adomian polynomial. Besides, it also has certain advantages over standard numerical methods as it is free from rounding-off errors and computationally inexpensive since it does not involve discretization. In this paper, we adopt the algorithm to the solution of boundary value problems arising in the modelling of real life problems. The fourth order two point boundary value problems have received a lot of attention in the literature due to their many applications in elasticity, was investigated by Kosmatov (2004). The objective of this paper is to implement a symbolic code (discussed in Kumar at el. 2010 and Kumar at el. 2011) for fully reflecting a simple and reliable technique for third order singular boundary value problems and fourth order beam bending problem using Mathematica 6.0.

The balance of this paper is as follows. In the next Section, we briefly introduce the Adomian decomposition method. In Section 3, we will explain Adomian decomposition method for higher order singular boundary value problems. In Section 4, Laplace transformation with Adomian decomposition method for oscillatory solutions for which Adomian method diverges is described. In Section 5, we present the numerical experiment which shows effectiveness of the proposed method. In the last Section, concluding remark with summary of the paper is given.

## 2. Adomian decomposition method:

In this section, we describe the Adomian decomposition method as it applies to a general nonlinear equation of the form

$$y - Ny = f \tag{2.1}$$

where $N$ is a nonlinear operator from a Hilbert space $H$ into $H$, $f$ is a given function in $H$. By the decomposition method $y$ is a series solution given by

$$y = \sum_{n=0}^{\infty} y_n \tag{2.2}$$

Nonlinear operator $N$ is decomposed by

$$Ny = \sum_{n=0}^{\infty} A_n \tag{2.3}$$

From (2.1), (2.2) and (2.3) we have

$$\sum_{n=0}^{\infty} y_n - \sum_{n=0}^{\infty} A_n = f \tag{2.4}$$

$A_n$ are Adomian polynomials that can be constructed for various classes of nonlinearity according to specific algorithms set by Adomian 1992.

$$A_0 = f(y_0)$$
$$A_1 = y_1 f'(y_0)$$
$$A_2 = y_2 f'(y_0) + \frac{y_1^2}{2!} f''(y_0),$$
$$\vdots$$
$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ f\left(\lambda^i y_i\right) \right]_{\lambda=0}, \quad n = 0,1,2...... \tag{2.5}$$

If the series (2.3) is convergent, then (2.4) hold as

$$y_0 = f$$
$$y_1 = A_0(y_0)$$
$$y_2 = A_1(y_0, y_1)$$
$$\vdots$$
$$y_n = A_{n-1}(y_0, y_1, ......., y_{n-1})$$

Thus, we can recursively determines every term of the series $\sum_{n=0}^{\infty} y_n$ .

## 3. Adomian decomposition method for higher order singular boundary value problems

For the higher order singular boundary value problems of type

$$y^{n+1} + \frac{m}{x} y^n + q(x)F(y) = g(x), \tag{3.1}$$

$$y(0) = a_0, y'(0) = a_1, \ldots, y^{n-1}(0) = a_{n-1}, y'(b) = c \tag{3.2}$$

where $F$ is a nonlinear operator of order less than $n$. Therefore equation (3.1) and (3.2) can be rewritten in operator form as

$$Ly = g(x) - q(x)F(y) \tag{3.3}$$

where $n$ fold operator $L$ is defined by Hasan et. al., 2009 as

$$L_1 = x^{-1}\frac{d^n}{dx^n}x^{1+n-m}\frac{d}{dx}x^{m-n}(.) \tag{3.4}$$

$$L_2 = x^{-1}\frac{d^n}{dx^n}x^{n-m}\frac{d}{dx}x^{m-n+1}\frac{d}{dx}(.) \tag{3.5}$$

Hasan et al., 2009 defined the inverse operator for (3.4) and (3.5) respectively as:

$$L_1^{-1} = x^{n-m}\int_b^x x^{m-n-1}\int_0^x\int_0^x\cdots\int_0^x x(.)\,dx\ldots dx \tag{3.6}$$

$$L_2^{-1} = \int_0^x x^{n-m-1}\int_b^x x^{m-n}\int_0^x\cdots\int_0^x x(.)\,dx\ldots dx \tag{3.7}$$

By applying $L_1^{-1}$ and $L_2^{-1}$ to the equation (3.1) we have

$$y^{(1)}(x) = \sum_{n=0}^{\infty} y_n = \phi_1(x) + L_1^{-1}g(x) - L_1^{-1}[q(x)\sum_{n=0}^{\infty} A_n] \tag{3.8}$$

$$y^{(2)}(x) = \sum_{n=0}^{\infty} y_n = \phi_2(x) + L_2^{-1}g(x) - L_2^{-1}[q(x)\sum_{n=0}^{\infty} A_n] \tag{3.9}$$

where $Ny = \sum_{n=0}^{\infty} A_n$, specific formula for finding Adomian polynomial is explained in (2.5) for nonlinear term $F(y)$. $y_1(x)$ and $y_2(x)$ are approximate solution using inverse operator (3.6) and (3.7) respectively. The components $y_n$ can be obtained by applying modified Adomian decomposition method from the recurrence relation:

$$\begin{cases} y_0 = \phi(x) \\ y_1 = L^{-1}g(x) + L^{-1}A_1 \\ y_{n+1} = -L^{-1}A_n,\ n \geq 1 \end{cases} \tag{3.10}$$

The $n$ term approximation can be obtained by,

$$\psi_n = \sum_{i=0}^{n} y_i \tag{3.11}$$

The Adomian decomposition method (ADM) outlined above (detailed in Hassan 2009) is easy to implement and does not need discretization. However, it has some drawbacks. Its efficiency and accuracy rely on the convergence and the rate of convergence of the series solution. We found, that the ADM gives a series solution which may have a slow rate of convergence over wider regions. Furthermore, if the solution of the problem is oscillatory, then the ADM series solution may be divergent. To overcome these drawbacks, ADM needs to be modified in order to work for problems where the solutions are of oscillatory in nature. Due to this difficulty the idea of Laplace transform is introduced with Adomian decomposition method to solve such problems.

## 4. Laplace Transformation with Adomian decomposition method for higher order boundary value problems

In this section, the Laplace Transform with Adomian decomposition Method discussed (Hajji et al., 2008) for solving nonlinear higher order boundary value problems in the interval $[0,b]$ of the form

$$y^{(n)} + q(x)F(y) = g(x), \qquad 0 < x < b \tag{4.1}$$

subject to the condition

$$y(0) = a_0\,, y'(0) = a_1, \ldots, y^{n-1}(0) = a_{n-1}, y'(b) = c \tag{4.2}$$

Appling Laplace Transform integral operator (denoted by $L$ ) both side of equation (4.1)

$$L[y^{(n)}] + L[q(x)F(y)] = L[g(x)], \qquad 0 < x < b \tag{4.3}$$

It gives

$$s^n L[y] - s^{n-1}a_0 - s^{n-2}a_1 - \cdots\cdots - sa_{n-2} - a_{n-1} + L[q(x)F(y)] = L[g(x)], \tag{4.4}$$

Substituting boundary conditions from (4.2) we have

$$s^n L[y] - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - sy^{n-2}(0) - y^{n-1}(0) + L[q(x)F(y)] = L[g(x)], \tag{4.5}$$

where the constants which are unknown are determined by imposing the boundary conditions at $x = b$.

$$L[y] = \frac{1}{s^n}(s^{n-1}a_0 + s^{n-2}a_1 + \cdots + s\,a_{n-2} + a_{n-1} - L[q(x)F(y)] + L[g(x)]) \tag{4.6}$$

Putting $y(x) = \sum_{n=0}^{\infty} y_n(x)$ and $F(y) = \sum_{n=0}^{\infty} A_n$ in (4.6) and comparing the terms we will get

$$L[y_0] = \frac{1}{s^n}(s^{n-1}a_0 + s^{n-2}a_1 + \cdots + s\,a_{n-2} + a_{n-1} + L[g(x)]) \tag{4.7}$$

$$L[y_1] = -\frac{1}{s^n}(L[q(x)A_0]) \tag{4.8}$$

$$L[y_2] = -\frac{1}{s^n}(L[q(x)A_1]) \tag{4.9}$$

$$L[y_3] = -\frac{1}{s^n}(L[q(x)A_2]) \tag{4.10}$$

$$\vdots$$

Applying the inverse Laplace transform to equation (4.7), gives the zero$^{th}$ component of Adomian solution. After substituting the value of $A_0, A_1, A_2, \cdots, A_n$ in (4.8-4.10) respectively and applying the inverse Laplace transform, we obtain the solution components $y_0, y_1, y_2, \cdots, y_n$ successively. The $n$ term approximation is given by

$$\psi_n(x,c) = \sum_{i=1}^{n-1} y_i(x,c), \tag{4.11}$$

To determine the constant $c$ , we require that $\psi_n(x,c)$ satisfies the boundary conditions at $x = b$. Solving the algebraic equation, we obtain the required constant, which complete the numerical solution of our nonlinear higher-order boundary value problems.

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## 5. Computer Simulation

In this section, we apply the proposed algorithm on two third order non-linear singular boundary value problems and one fourth order oscillatory boundary value problem.

**Example 1 :**( Third order singular problem)

$$y''' - \frac{2}{x} y'' = y + y^2 + 7x^2 e^x + 6x e^x - 6e^x - x^6 e^{2x}, \tag{5.1}$$

$$y(0) = y'(0) = 0, \; y'(1) = e, \tag{5.2}$$

Operator form of the equation is

$$Ly = q(x) + F(y, x), \tag{5.3}$$

where

$$q(x) = 7x^2 e^x + 6xe^x - 6e^x - x^6 e^{2x} \text{ and } F(y,x) = y^2$$

So the inverse operators for the equation (5.1), (5.2) from (3.6), (3.7) are

$$L_1^{-1} = x^4 \int_1^x x^{-5} \int_0^x \int_0^x x(.)\, dx\, dx\, dx \tag{5.4}$$

$$L_2^{-1} = \int_0^x x^{-4} \int_1^x x^{-5} \int_0^x x(.)\, dxdxdx \tag{5.5}$$

Applying $L^{-1}$ on the both side of equation (5.2) we get

$$y_{(i)}(x) = 2.71828\, x^4 + L_i^{-1} q(x) + L_i^{-1} \sum_{k=1}^{\infty} A_n, \tag{5.7}$$

Nonlinear term $y^2$, is calculated by the Adomian polynomial as,

$$A_0 = y_0^2,$$
$$A_1 = 2y_0 y_1,$$
$$A_2 = 2y_0 y_2 + y_1^2,$$
$$\vdots$$

After computing the values of series components $y_1$, $y_2$, $y_3$, we get the third order approximation of series solution

$$\phi_3 = \sum_{i=1}^{3} y_i, \tag{5.8}$$

The exact solution of the (5.1), subject to the (5.2) is $y = x^3 e^x$, the comparison of exact solution and approximate solution is given in table 1. Figure 1 shows the graphical representation of the approximate solutions which is very close to the exact solution.

**Table 1**

| x | y(ADM ) | y(Exact) | Error |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0.1 | 0.00110664 | 0.00110666 | 1.88E-8 |
| 0.2 | 0.00986585 | 0.00986616 | 3.01E-7 |
| 0.3 | 0.03751990 | 0.03752150 | 1.52E-6 |
| 0.4 | 0.10147900 | 0.10148400 | 4.83E-6 |
| 0.5 | 0.22886500 | 0.22887700 | 1.19E-5 |
| 0.6 | 0.46120300 | 0.46122900 | 2.56E-5 |
| 0.7 | 0.86027700 | 0.86032800 | 5.07E-5 |
| 0.8 | 1.51512000 | 1.51522000 | 9.82E-5 |
| 0.9 | 2.55015000 | 2.55035000 | 1.92E-4 |
| 1.0 | 4.13454000 | 4.13492000 | 1.92E-4 |

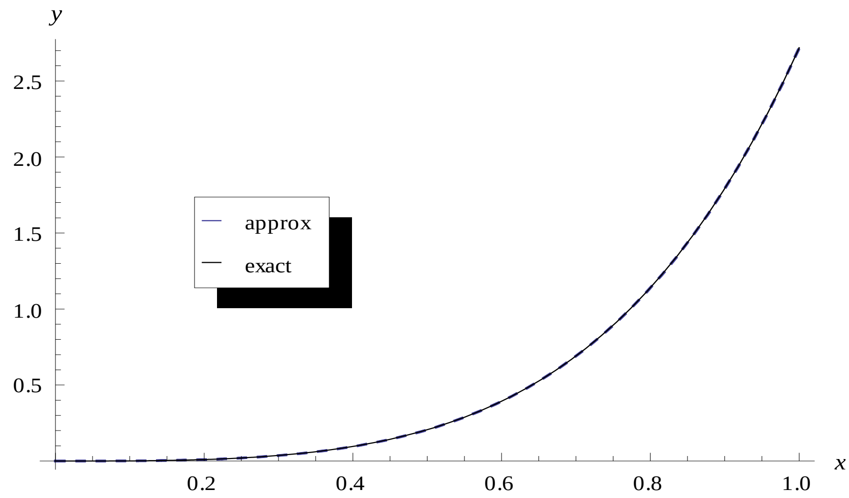


**Figure 1:** Exact and Approximate solution of example 1

**Example 2** :( Third order singular problem)

$$y''' - \frac{2}{x}y'' - y^3 = g(x), \quad (5.9)$$

$$y(0) = y'(0) = 0, \ y'(1) = 10.873, \quad (5.10)$$

Operator form of the equation is

$$Ly = g(x) + F(y, x), \quad (5.11)$$

where $g(x) = 7x^2e^x + 6xe^x - 6e^x - x^9e^{3x} + x^3 e^x$ and $F(y,x) = y^3$

So the inverse operator for the equation (5.10) and (5.9) is

$$L^{-1}(\cdot) = x^4 \int_1^x x^{-5} \int_0^x \int_0^x x(\cdot)\, dx\, dx\, dx \quad (5.12)$$

Applying $L^{-1}$ on the both side of equation (5.11) we get

$$y(x) = L^{-1}g(x) + L^{-1}\sum_{k=1}^{\infty} y + L^{-1}\sum_{k=1}^{\infty} A_n + y(1)x^4, \tag{5.13}$$

Nonlinear term $y^2$ is calculated by the Adomian polynomial as,

$$A_0 = y_0^3$$
$$A_1 = 3y_0^2 y_1,$$
$$A_2 = 3y_0^3 y_2 + 3y_1^2 y_0,$$
$$\vdots$$

After computing the values of series components $y_1$, $y_2$, $y_3$, third approximation of series solution is

$$\phi_3 = \sum_{i=1}^{3} y_i, \tag{5.14}$$

The exact solution of the (5.9) with (5.10) is $y = x^3 e^x$, the comparison of exact solution and approximate solution is given in table 2. Figure 2 shows the graphical representation of the approximate solutions which is very close to the exact solution.

**Table 2**

| x | y(ADM) | y(Exact) | Error |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0.1 | 0.00110664 | 0.00110666 | 1.88E-8 |
| 0.2 | 0.00986585 | 0.00986616 | 3.01E-7 |
| 0.3 | 0.03751990 | 0.03752150 | 1.52E-6 |
| 0.4 | 0.10147900 | 0.10148400 | 4.83E-6 |
| 0.5 | 0.22886500 | 0.22887700 | 1.19E-5 |
| 0.6 | 0.46120300 | 0.46122900 | 2.56E-5 |
| 0.7 | 0.86027700 | 0.86032800 | 5.07E-5 |
| 0.8 | 1.51512000 | 1.51522000 | 9.82E-5 |
| 0.9 | 2.55015000 | 2.55035000 | 1.92E-4 |
| 1.0 | 4.13454000 | 4.13492000 | 1.92E-4 |

 

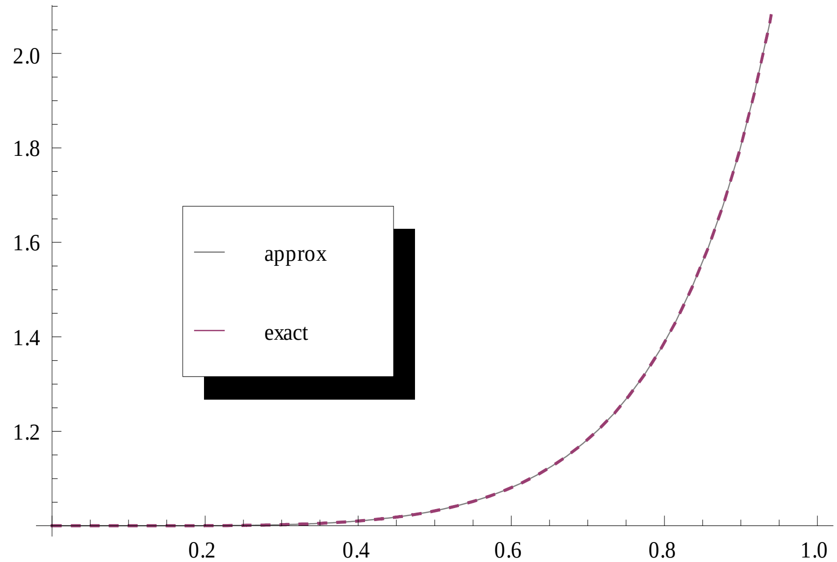


**Figure 2:** Exact and Approximate solution of example 2

**Example 3: (Elastic beam equation rigidly fixed at both ends)**

Fourth-order nonlinear differential equations have many applications such as balancing condition of an elastic beam whose two ends are simply supported, may be described by nonlinear fourth-order ordinary differential equations. Beams can be used in many different settings, as long as their capabilities are understood. Many different types of beams are available; each with their own identity which depends on the types of material, length, width, depth, and external forces being placed on the beam. One major concern that needs to be considered, when deciding what type of beam to use in a certain structure, is deflection.

Deflection is the displacement of any point along the beam from its original position, measured in the y direction. It is generally shown in a graph of the deflection curve, representing the deflection versus incremental load values. The specific values of deflection can be found through differential equations. The moment-curvature equation is a second-order, ordinary differential equation; whereas, the load-deflection equation is a fourth-order, ordinary differential equation. Both are very useful equations and can be easily programmed for solution by computer. Beams are used everywhere (stadiums, airports, bridges, etc.), but only when their capabilities are known can they be used safely.

We consider the problem of bending of a long uniformly rectangular plate supported over the entire surface by an elastic foundation and rigidly supported along the edges. The vertical deflection $y$ at every point satisfies the system

$$D\,y^{(4)}(x) = f(x,y) - g\,y, \qquad 0 < x < b, \tag{5.15}$$

$$y(0) = y'(0) = y(1) = y'(1) = 0 \tag{5.16}$$

where $D$, is the flexural rigidity of the plate, $f$ is the intensity of the load acting on the plate, and $g$ is the reaction of the foundation. The more details we refer Agarwal et al., (1982).

**Another mathematical model:**

Suppose $y$, represents an elastic beam of length *b*, which is clamped at its left side $x = 0$, and resting on a kind of elastic bearing at its right side $x = b$, along its length, a load $f$ is added to cause deformations. Then the differential equation that models this phenomenon is given by

$$y^{(4)}(x) = f(x, y), \qquad 0 < x < b, \tag{5.17}$$

$$y(0) = y'(0) = 0, y''(b) = 0, \ y'''(b) = g(y(b)) \tag{5.18}$$

where $f \in C([0,1] \times R)$ *and* $g \in C(R)$ .Owing to its importance in physics, the existence of solutions to this nonsingular problem has been studied by many authors (Pang et al., (2006), Wazwaz, (2002), Cabada, (1994)), However, in practice only its positive solutions are significant. Corresponding problems modeling vibrating beams on elastic bearings were considered in Ma, 2001. The detail of the mechanical interpretation of the above two models belongs to a general class of boundary value problems of the form

$$y^4(x) + g(x)F(x, y) = f(x), \qquad 0 < x < 1, \tag{5.19}$$

$$y(0) = \alpha_0, y'(0) = \alpha_1, y(b) = \beta_0, \ y'(b) = \beta_1 \tag{5.20}$$

We consider the problem of bending a rectangular clamped beam of length *b*. $\pi$ resting on an elastic foundation. The vertical deflection $y(x)$ of the beam satisfies the system

$$y^{(4)} + 64y = Sin\ 2x, \qquad 0 < x < \pi \tag{5.21}$$

$$y(0) = 0, \ y'(0) = 0, \ y(\pi) = 0, \ y''(\pi) = 0 \tag{5.22}$$

**Adomian decomposition method:**

$$y_0 = a_0$$

$$y_1 = \sum_{i=0}^{3} a_i \frac{1}{i!} x^i - 64L^{-1}[y_0] + L^{-1}[Sin\, 2x]$$

$$y_2 = -64 L^{-1}[y_1]$$

$$y_3 = -64 L^{-1}[y_2]$$

$$\ldots$$

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{5.23}$$

$$y(x) = \sum_{k=0}^{\infty} y_k(x) = \sum_{k=0}^{\infty} \left[ \sum_{i=0}^{3} a_i x^i \frac{\left(-64x^4\right)^k}{(i+4k)!} + \frac{(-64)^k}{2^{4k+4}} Sin\left(2x\right) \right] \tag{5.24}$$

The series solution (5.24) shows the divergent geometric series. Thus we can observe that the ADM approach diverges, therefore it is not applicable to that type of problem in which solution is oscillatory. To overcome this difficulty we will use Laplace Transformation (discussed in Hajji et al., (2008))

**Laplace Adomian decomposition technique:**

Now applying Laplace transform integral operator to the both sides of the equation (5.19), we get

$$L[y^{(4)}] = -L[64y] + L[Sin\ 2x]$$

Using the formula of the Laplace transform of the derivatives, we obtain

 

$$L[y_0]=\frac{\alpha}{s^3}+\frac{\beta}{s^4}+\frac{2}{s^4(s^2+4)} \tag{5.25}$$

$$L[y_n]=-\frac{(64)^n}{s^4}(L[y_{n-1}]),\ n\geq 1 \tag{5.26}$$

Following the ADM procedure outlined in the previous section, we find that the general form of the series solution by Laplace transformation is given by

$$L[y]=\frac{\alpha}{s^3}+\frac{\beta}{s^4}+\frac{2}{s^4(s^2+4)}-\frac{(64)^n}{s^4}(L[y_{n-1}])$$

$$y(x)=\frac{\alpha}{64+s^4}+\frac{\beta s}{64+s^4}+\frac{2}{(s^4+64)(s^2+4)} \tag{5.27}$$

For this example, it can be easily verified that the exact solution can be recovered by taking the inverse Laplace transform. By the above analysis, we see that the coupling of the Laplace transform with the Adomian decomposition made it possible to obtain a convergent series expansion in the Laplace domain. A n-term approximate solution is obtained by calculating n solution components, as described above $\phi_n(x,\alpha,\beta)$. The constants $\alpha$ and $\beta$ are determined by imposing the boundary conditions at $x=\pi$ . Using Mathematica 6.0, $\alpha$ and $\beta$ were found to be $\alpha=0.099672$ and $\beta=-0.3$. Then we obtain the approximate solution is given by

$$\begin{aligned}\phi_7(x) =\ & 1638.38x + 0.0498136\,x^2 - 1092.3\,x^3 + 218.467\,x^5 - 0.00885575\,x^6 - 20.8025\,x^7 \\ & + 1.15556\,x^9 + 0.000112454\,x^{10} - 0.042051\,x^{11} + 0.00107876\,x^{13} - 2.99578\times10^{-7}\,x^{14} \\ & - 0.0000204876\,x^{15} + 3.00699\times10^{-7}\,x^{17} + 2.6107\times10^{-10}\,x^{18} - 3.55832\times10^{-9}\,x^{19} + \\ & 3.41515\times10^{-11}\,x^{21} - 9.51724\times10^{-14}\,x^{22} - 2.57512\times10^{-13}\,x^{23} + 1.66137\times10^{-15}\,x^{25} + \\ & 1.69761\times10^{-17}\,x^{26} - 1.13598\times10^{-17}\,x^{27} + 6.21772\times10^{-20}\,x^{29} - 1.65188\times10^{-21}\,x^{30} \\ & - 1.06971\times10^{-22}\,x^{31} - 819.19\,\sin(2x).\end{aligned}$$

The above series is the convergent series in Laplace domain, shows that the Adomian decomposition method coupled with Laplace transformation gives convergent series solution for oscillatory problems.

## 6. Conclusion

In this paper, we explained the Adomian decomposition method for solving higher order singular boundary value problems and Laplace Adomian decomposition method for solving fourth-order boundary value problems for which the Adomian decomposition method diverges. It can be a potential tool to solve the oscillatory nonlinear higher order boundary value problems. Although there are some other methods which can be used to solve such systems or more complex ones, but Adomian decomposition method shows its advantages is that calculations are relatively easy to follow and understand, besides, it can be fulfilled by mathematical software like Mathematica 6.0 though the solution is of the form of an infinite series in many cases, it can be written in a closed form in some cases, otherwise, it can also be satisfactorily represented by proper truncations for it shares a relatively rapid convergence.

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