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On Generalized Dislocated Quasi Metrics

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Abstract
The notion of dislocated quasi metric is a generalization of metric that retains, an analogue of the illustrious Banach's Contraction principle and has useful applications in the semantic analysis of logic programming. In this paper we introduce the concept of generalized dislocated quasi metric space.The purpose of this note is to study topological properties of a gdq metric, its connection with generalized dislocated metric space and to derive some fixed point theorems.
Keywords: Generalized dislocated metric, Generalized dislocated quasi metric, Contractive conditions, coincidence point, β-property.

Introduction: Pascal Hitzler [1] presented variants of Banach's Contraction principle for various modified forms of a metric space including dislocated metric space and applied them to semantic analysis of logic programs. In this context Hitzler raised some related questions on the topological aspects of dislocated metrics.

In this paper we establish the existence of a topology induced by a gdq metric, induce a generalized dislocated metric (D metric) from a gdq metric and prove that fixed point theorems for gdq metric spaces can be derived from their analogues for D metric spaces . We then prove a D metric version of Ciric's fixed point theorem from which a good number of fixed point theorems can be deduced.

Definition 1.1:[2]Let binary operation $\lozenge : R^+ \times R^+ \to R^+$ satisfies the following conditions:

(I) $\lozenge$ is Associative and Commutative,

(II) $\lozenge$ is continuous.

Five typical examples are $a \lozenge b = \max\{a, b\}$, $a \lozenge b = a + b$, $a \lozenge b = ab$,

$a \lozenge b = ab + a + b$ and $a \lozenge b = \dfrac{ab}{\max\{a,b,1\}}$ for each $a, b \in R^+$

Definition 1.2: [2] The binary operation $\lozenge$ is said to satisfy β-property if there exists a positive real number β such that $a \lozenge b \le \beta \max\{a, b\}$ for every $a, b \in R^+$.

Definition 1.3: Let X be a nonempty set. A generalized dislocated quasi metric (or gdq metric) on X is a function $gdq : X^2 \to R^+$ that satisfies the following conditions for each $x, y, z \in X$.

(1) $gdq(x, y) \ge 0$,
(2) $gdq(x, y) = 0$ implies $x = y$
(3) $gdq(x, z) \le gdq(x, y) \lozenge gdq(y, z)$

The pair (X , gdq) is called a generalized dislocated quasi metric (or gdq metric) space.
If gdq satisfies $gdq(x, y) = gdq(y, x)$ also, then gdq is called generalized dislocated metric space.
Unless specified otherwise in what follows (X , gdq) stands for a gdq metric space.

Definition 1.4: If ($x_\alpha / \alpha \in \Delta$)is a net in X and $x \in X$ we say that $(x_\alpha)_{\alpha\in\Delta}$ gdq converges to x and write $gdq \lim_{\alpha} x_\alpha = x$ if $\lim_{\alpha} gdq(x_\alpha\ x) = \lim_{\alpha} gdq(x\ x_\alpha)$=0 .i.e For each $\in > 0$ there exists $\alpha_o \in \Delta$ such that for all $\alpha \geq \alpha_0 \Rightarrow gdq(x, x_\alpha) = gdq(x_\alpha, x) < \in$.We also call x, gdq limit of (x_α).We introduce the following

Definition 1.5: Let $A \subseteq X$. $x \in X$ is said to be a gdq limit point of A if there exists a net $(x_\alpha)_{\alpha\in\Delta}$ in X such that $gdq \lim_{\alpha} x_\alpha = x$.

Notation1.6: Let $x \in X$, $A \subseteq X$ and r >0,We write

$D(A)=\{x / x \in X$ and x is a gdq limit point of $A\}$.

$B_r(x) = \{y / y \in X \ and\ \min\{gdq(x,y), gdq(y,x)\} < r\}\ and\ V_r(x) = \{x\} \cup B_r(x).$

Remark 1.7: $gdq \lim_{\alpha\in\Delta} x_\alpha = x$ iff for every $\delta > 0$ there exists $\alpha_0 \in \Delta$ such that $x_\alpha \in B_\delta(x)\ if\ \alpha \geq \alpha_0$

Proposition 1.8: Let (X , gdq) be a gdq metric space such that $\lozenge$ satisfies β -property with $\beta > 0$.

If a net ($x_\alpha / \alpha \in \Delta$) in X gdq converges to x then x is unique.

Proof: Let ($x_\alpha / \alpha \in \Delta$) gdq converges to y and $y \neq x$

Since ($x_\alpha / \alpha \in \Delta$) gdq converges to x and y then for each $\in > 0$ there exists $\alpha_1, \alpha_2 \in \Delta$ such that

for all $\alpha \geq \alpha_1 \Rightarrow gdq(x, x_\alpha) = gdq(x_\alpha, x) < \frac{\in}{\beta}$ and $\alpha \geq \alpha_2 \Rightarrow gdq(y, x_\alpha) = gdq(x_\alpha, y) < \frac{\in}{\beta}$

From triangular inequality we have, $gdq\ (x, y) \leq gdq(x, x_\alpha) \lozenge gdq(x_\alpha, y)$

$$\leq \beta \max\{gdq(x, x_\alpha), gdq(x_\alpha, y)\}$$

$$< \beta \max\{\frac{\in}{\beta}, \frac{\in}{\beta}\} = \in$$

Which is a contradiction.

Hence $gdq(x,\ y)$=0 similarly $gdq(y, x)$=0

$\therefore x = y$

Proposition 1.9: $x \in X$ is a gdq limit point of $A \subset X$ iff for every r>0 , $A \cap B_r(x) \neq \phi$

Proof: Suppose $x \in D(A)$. Then there exists a net ($x_\alpha / \alpha \in \Delta$) in A such that $x = gdq \lim_{\alpha} x_\alpha$.

If $r > 0, \exists\ \alpha_0 \in \Delta$ such that $B_r(x) \cap A \neq \phi\ \ for\ \ \alpha \geq \alpha_0$.

Conversely suppose that for every r >0 $B_r(x) \cap A \neq \phi$.

Then for every positive integer n, there exists $x_n \in B_{\frac{1}{n}}(x) \cap A$ so that

$gdq\ (x_n\ , x) < \frac{1}{n}$, $gdq\ (x\ ,\ x_n\ ) < \frac{1}{n}$ and $x_n \in A$

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Hence $gdq \lim_{n} gdq(x_n\ x) = 0$ so that $x \in D(A)$

Theorem 1.10: Let (X , gdq) be a gdq metric space such that $\lozenge$ satisfies β -property with $\beta \le 1$ and

$A \subseteq X$ and $B \subseteq X$ then

i. $D(A) = \phi$ *if* $A = \phi$

ii. $D(A) \subseteq D(B)$ *if* $A \subseteq B$

iii. $D(D(A)) \subseteq D(A)$

iv. $D(A \cup B) = D(A) \cup D(B)$

Proof: (i) and (ii) are clear. That $D(A) \cup D(B) \subseteq D(A \cup B)$ follows from (ii). To prove the reverse inclusion, let $x \in D(A \cup B)$, $x = gdq \lim_{\alpha\in\Delta} (x_\alpha)$ where $(x_\alpha / \alpha \in \Delta)$ is a net in $A \cup B$.

If $\exists\ \lambda \in \Delta$ such that $x_\alpha \in A$ for $\alpha \in \Delta$ and $\alpha \ge \lambda$ then ($x_\alpha / \alpha \ge \lambda, \alpha \in \Delta$) is a cofinal subnet of ($x_\alpha / \alpha \in \Delta$) and $\lim_{\alpha\ge\lambda} gdq\ (x, x_\alpha) = \lim_{\alpha\in\Delta} gdq\ (x, x_\alpha) = \lim_{\alpha\in\Delta} gdq\ (x_\alpha, x) = \lim_{\alpha\ge\lambda} gdq\ (x_\alpha, x) = 0$ so that $x \in D(A)$. If no such λ exists in Δ then for every $\alpha \in \Delta$, choose $\beta(\alpha) \in \Delta$ such that $\beta(\alpha) \ge \alpha$ and $x_{\beta(\alpha)} \in B$. Then ($x_{\beta(\alpha)} / \alpha \in \Delta$) is a cofinal subnet in B of ($x_\alpha / \alpha \in \Delta$) and $\lim_{\alpha\in\Delta} gdq\,(x_{\beta(\alpha)}, x) = \lim_{\alpha\in\Delta} gdq\ (x_\alpha, x) = \lim_{\alpha\in\Delta} gdq\ (x, x_\alpha) = \lim_{\alpha\in\Delta} gdq\ (x\ ,\ x_{\beta(\alpha)}) = 0$ so that $x \in D(B)$.It now follows that $D(A \cup B) \subseteq D(A) \cup D(B)$ and hence (iii) holds. To prove (iv)

let $x \in D(D(A))$, $x = gdq \lim_{\alpha\in\Delta} x_\alpha$, $x_\alpha \in D(A)$ for $\alpha \in \Delta$, and $\forall\ \alpha \in \Delta$,let ($x_{\alpha(\beta)} / \beta \in \Delta(\alpha)$) be a net in A Such that $x_\alpha = gdq \lim_{\beta\in\Delta(\alpha)} x_{\alpha_\beta}$ .For each positive integer i $\exists\ \alpha_i \in \Delta$ such that

$gdq\ (x_{\alpha_i}, x) = gdq\ (x, x_{\alpha_i}) < \frac{1}{i}$, and $\beta_i \in \Delta(\alpha_i) \ni$ $gdq\ (x_{\alpha_{i_{\beta_i}}}, x_{\alpha_i}) = gdq\,(x_{\alpha_i}, x_{\alpha_{i_{\beta_i}}}) < \frac{1}{i}$.

If we write $\alpha_{i_{\beta_i}} = \gamma_i$ $\forall$ i, then $\{\gamma_1, \gamma_2 ....\}$ is directed set with $\gamma_i < \gamma_j$

$$\begin{aligned} \text{if}\quad i < j,\ gdq\left(x_{\gamma_i}, x\right) &\le gdq(x_{\gamma_i}, x_{\alpha_i}) \lozenge gdq(x_{\alpha_i}, x) \\ &\le \beta \max\{gdq(x_{\gamma_i}, x_{\alpha_i}), gdq(x_{\alpha_i}, x)\} \\ &\le gdq(x_{\gamma_i}, x_{\alpha_i}) + gdq(x_{\alpha_i}, x) \\ &< \frac{2}{i} \end{aligned}$$

Similarly, $gdq\left(x, x_{\gamma_i}\right) < \frac{2}{i}$

Hence $x \in D(A)$.

Corollory 1.11: If we write $\overline{A} = A \cup D(A)$ for $A \subset X$ the operation $A \to \overline{A}$ satisfies Kurotawski's

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Closure axioms[6] so that the set $\Im = \{A / A \subset X \text{ and } \bar{A}^C = A^C\}$ is a topology on X. We call (X, gdq, $\Im$) topological space induced by gdq. We call $A \subset X$ to be closed if $\bar{A} = A$ and open if $A \in \Im$.

Corollary 1.12: $A \subset X$ is open (i.e $A \in \Im$) iff for every $x \in A$ there exists $\delta > 0 \ni V_\delta(x) \subseteq A$

Proposition 1.13: Let (X, gdq) be a gdq metric space such that $\Diamond$ satisfies β-property with $\beta \le 1$. If $x \in X$ and $\delta > 0$ then $V_\delta(x)$ is an open set in (X, gdq, $\Im$).

Proof: Let $y \in B_\delta(x)$ and $0 < r < \min\{\delta - d(x,y), \delta - d(y,x)\}$.

Then $B_r(y) \subset B_\delta(x) \subset A$, since $z \in B_r(y) \Rightarrow \min\{gdq(y,z), gdq(z,y)\} < r$

$$\Rightarrow gdq(y,z) < r$$

$$< \min\{\delta - gdq(x,y), \delta - gdq(y,x)\}$$

Now

$$\begin{aligned} gdq(x,z) &\le gdq(x,y) \Diamond gdq(y,z) \\ &\le \beta \max\{gdq(x,y), gdq(y,z)\} \\ &\le gdq(x,y) + gdq(y,z) \\ &< \delta \end{aligned}$$

Similarly $gdq(z,x) < \delta$ therefore $z \in B_\delta(x)$

Hence $V_\delta(x)$ is open.

Proposition 1.14: Let (X, gdq) be a gdq metric space such that $\Diamond$ satisfies β-property with Then (X, gdq, $\Im$) is a Hausodorff space and first countable.

Proof: Suppose $x \ne y$ we have to find δ such that $A_\delta = (B_\delta(x) \cup \{x\}) \cap (B_\delta(y) \cup \{y\}) = \phi$

Since $x \ne y$. One of $gdq(x,y)$, $gdq(y,x)$ is non zero. We may assume $gdq(y,x) > 0$

Choose $\delta > 0$ such that $2\delta < gdq(y,x)$ .we show that $A_\delta = \phi$.

If $z \in A_\delta$ and $z = y$, $z \ne x$. $z \in B_\delta(x)$.

$$\Rightarrow gdq(z,x) < \delta$$

$$\Rightarrow gdq(y,x) < \delta < \frac{gdq(y,x)}{2}$$ which is a contradiction.

Similarly if $z \ne y \text{ and } z = x$, $z \notin A_\delta$

If $y \ne z \ne x$ then

$$\begin{aligned} gdq(y,x) &\le gdq(y,z) \Diamond gdq(z,x) \\ &\le \beta \max\{gdq(y,z), gdq(z,x)\} \\ &\le gdq(y,z) + gdq(z,x) \\ &< 2\delta < gdq(y,x) \qquad \textit{which is a contradiction.} \end{aligned}$$

Hence $A_\delta = \phi$

Hence (X, gdq, $\Im$) is a Hausodorff space.

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If $x \in X$ then the collection $V_{\frac{1}{n}}(x)$ is base at x .Hence (X , gdq , $\Im$) is first countable.

Remark 1.15: Proposition 1.14 enables us to deal with sequences instead of nets.

Definition 1.16: A sequence $\{x_n\}$ in X is a gdq Cauchy sequence if for every $\in > 0$ there corresponds a positive integer $N_0 \ni gdq(x_n, x_m) < \in \ or \ gdq(x_m, x_n) < \in whenever\ n \geq N_0$ and $m \geq N_0$ $ie \min\{gdq(x_n, x_m) < \in, gdq(x_m, x_n) < \in\}$. And (X , gdq) is said to be gdq complete if every gdq Cauchy sequence in X is gdq convergent.

Result 1.17: Define D (x , y)= gdq (x , y) $\Diamond$ gdq (y , x), where a $\Diamond$ b = a + b, for a , b $\in R^+$.

1. D is a generalized dislocated metric(D metric) on X .
2. For any $\{x_\alpha / \alpha \in \Delta\} in\ X\ and\ x \in X\ gdq \lim(x_\alpha) = x \Leftrightarrow D \lim(x_\alpha) = x$
3. X is a gdq Complete $\Leftrightarrow$ X is $D complete.$

Proof:(i) and (ii) are clear.we prove (iii)

Let X is a gdq Complete

Let $\{x_n\}$ be a Cauchy sequence in (X , D) and $\in > 0$ then there exist a positive integer $n_0 \ni m, n \geq n_0$. $\lim D(x_n, x_m) < \in$

$$\lim[gdq(x_n\ x_m) \Diamond gdq(x_m\ x_n)] < \in$$
$$\lim[gdq(x_n\ x_m) + gdq(x_m\ x_n)] < \in$$
$$\min\{gdq(x_n\ x_m), gdq(x_m\ x_n)\} < \in$$

$\therefore \{x_n\}$ is a gdq Cauchy sequence.

Hence convergent.

$\therefore \lim gdq(x_n, x) = \lim gdq(x, x_n) = 0$

$\therefore D(x_n, x) = 0$

Hencce X is $D complete.$

Conversely suppose that,

Let X is $D complete.$

Let $\{x_n\}$ be a gdq Cauchy sequence in X and $\in > 0$ there exist a positive integer $n_0 \ni$

$$\min\{gdq(x_n\ x_m), gdq(x_m\ x_n)\} < \frac{\in}{2}$$

$gdq(x_n\ x_m) + gdq(x_m\ x_n)\} < \in$

$gdq(x_n\ x_m) \Diamond gdq(x_m\ x_n)\} < \in$

$D(x_n\ x_m) < \in$

Hence $\{x_n\}$ be a Cauchy sequence in (X , D)

$\therefore$ there exist x in X $\ni$ $D \lim x_n = x \Rightarrow \lim D(x_n\ x) = 0$

$\Rightarrow \lim[gdq(x_n\ x) + gdq(x\ x_n)] = 0$

$\Rightarrow \lim gdq(x_n\ x) = \lim gdq(x\ x_n)] = 0$

Hence X is gdq $D complete.$

Remark: As a consequence of 1.17 we can derive a fixed point theorem for gdq metric space if we can prove the same for D metric space and derive the contractive inequality for D from gdq .

The D metric induced by a gdq metric on a set X is very useful in deriving fixed point theorems for self maps on (X , gdq) from their analogues for (X , D).If a self map f on a gdq metric space (X , gdq) satisfies a contractive inequality $gdq(f(x), f(y)) \le \Phi_{gdq}(x, y)$, Where Φ_{gdq} is a linear function of { $gdq(u,v) / \{u,v\} \subseteq \{x, y, f(x), f(y)\}$ } then f satisfies the contractive inequality $D(f(x), f(y)) \le \Phi_D(x, y)$ Where Φ_D is obtained by replacing gdq in Φ_{gdq} by D.

B.E Rhodes[4] collected good number of contractive inequalities considered by various authors and established implications and nonimplications among them. We consider a few of them here.

Let (X , d) be a metric space, $x \in X$, $y \in Y$, f a self map on X ,

And $a, b, c, h, a_1, a_2, a_3, a_4, a_5, \alpha, \beta, \gamma$ nonnegative real numbers (=constants), $a(x, y), b(x, y), c(x, y), p(x, y), q(x, y), r(x, y) s(x, y)$ and $t(x, y)$ be nonnegative real valued continuous function on X x X.

1. (Banach) : $d(f(x), f(y)) \le a\, d(x, y)$, $0 \le a < 1$
2. (Kannan) : $d(f(x), f(y)) \le a\{d(x, f(x)) + d(y, f(y))\}$, $0 \le a < \dfrac{1}{2}$
3. (Bianchini): $d(f(x), f(y)) \le$ h max $\{d(x, f(x)), d(y, f(y))\}$, $0 \le h < 1$
4. $d(f(x), f(y)) \le a\, d(x, f(x)) + b\, d(y, f(y)) + c\, d(x, y)$, $a + b + c < 1$
5. $d(f(x), f(y)) \le a(x,y)\, d(x, f(x)) + b(x,y)\, d(y, f(y)) + c(x,y)\, d(x, y)$,
   $\sup\limits_{x,y \in X} \{a(x,y) + b(x,y) + c(x,y) / x \in X , y \in Y\} < 1$
6. (Chatterjea): $d(f(x), f(y)) \le a[d(x, f(y)) + d(y, f(x))]$, $a < \dfrac{1}{2}$
7. $d(f(x), f(y)) \le$ h max $[d(x, f(y)), d(y, f(x))]$, $0 \le h < 1$
8. $d(f(x), f(y)) \le a\, d(x, f(y)) + b\, d(y, f(x)) + c\, d(x, y)$, $a + b + c$ <1
9. $d(f(x), f(y)) \le a(x,y)\, d(x, f(y)) + b(x,y)\, d(y, f(x)) + c(x,y)\, d(x, y)$,
   $\sup\limits_{x,y \in X} \{a(x,y) + b(x,y) + c(x,y) / x \in X , y \in Y\} < 1$
10. (Hardy and Rogers):

$$d(f(x), f(y)) \le a_1 d(x, y) + + a_2 d(x, f(x)) + a_3 d(y, f(y)) + a_4 d(x, f(y)) + a_5 d(y, f(x)) ,$$

$$\sup_{x,y \in X} \{ \sum_i a_i(x, y)\} < 1 \quad For\ every\ \ x \ne y$$

11. (Zamfirescu):For each $x, y \in X$ at least one of the following is true:

I. $d(f(x), f(y)) \leq \alpha d(x,y)$, $0 \leq \alpha < 1$

II. $d(f(x), f(y)) \leq \beta\, [d(x, f(x)) + d(y, f(y))]$, $0 \leq \beta < \frac{1}{2}$

III. $d(f(x), f(y)) \leq \gamma\, [d(x, f(y)) + d(y, f(x))]$, $0 \leq \gamma < \frac{1}{2}$

12. (Ciric): For each $x, y \in X$

$$d(f(x), f(y)) \leq q(x,y)\, d(x,y) + r(x,y)\, d(x, f(x)) + s(x,y)\, d(y, f(y)) +$$

$$t(x,y)[d(x, f(y)) + d(y, f(x))]$$

$$\sup_{x,y \in X}\{q(x,y) + r(x,y) + s(x,y) + 2t(x,y) \leq \lambda < 1$$

13. (Ciric): For each $x, y \in X$

$d(f(x), f(y)) \leq$ h

max{ $d(x,y)$ $d(x, f(x))$, $d(y, f(y))$, $d(x, f(y))$, $d(y, f(x))$ } ,

$0 \leq h < 1$

B.E Rhoades[4] established the following implications among the above inequalities:

(2) $\Rightarrow$ (3) $\Rightarrow$ (5) $\Rightarrow$ (12) $\Rightarrow$(13)
(2) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (12) $\Rightarrow$(13)
(6) $\Rightarrow$ (7) $\Rightarrow$ (9) $\Rightarrow$(13)
(6) $\Rightarrow$ (8) $\Rightarrow$ (9) $\Rightarrow$(13)
(6) $\Rightarrow$ (10) $\Rightarrow$ (11) $\Rightarrow$ (12) $\Rightarrow$(13).

If d is a D metric instead of a metric, it is possible that $d(x,x) \neq 0.$ As such these implications hold good in a D metric space as well when "$x \neq y$" is replaced by "$D(x,y) \neq 0$". More over all these implications end up with (13). Thus a fixed point theorem for f satisfying the D metric version of Ciric's Contraction principal (13) yields fixed point theorem for f satisfying the D metric version of other inequalities.

Moreover $d(x, f(x))$=0 $\Rightarrow$ $f(x) = x$ when d is a metric. However "$f(x) = x$"does not necessarily imply $d(x, f(x))$=0 where d is a D metric. We in fact prove the existence of x such that $D(x, f(x))$=0 which we call a coincidence point of f .We now prove the following analogue of Ciric's Contraction principle.

2 Main Results

Theorem 2.1 : Let (X , D) be a complete D metric space such that ◊ satisfies β-property with $\beta \leq 1$, f a self map on X and $0 \leq h < 1$.If for all x, y with $D(x,y) \neq 0.$

$D(f(x), f(y)) \leq h$ max{ $D(x,y), D(x, f(x)), D(x, f(y)), D(y, f(x)), D(y, f(y))$} ----(*)

Then f has a unique coincidence point.

Proof: Assume that f satisfies (*).

For $x \in X$ and any positive integer n write $0(x,m)$={ x , $f(x)$…… $f^m(x)$} and $\delta[0(x,m)]$=sup { $D(u,v) / \{u,v\} \subset 0(x,m)$ } .

We first prove the following

Lemma[7]: For every positive integer 'm' there exists a positive integer $k \le m$ such that $\delta[0(x,m)] \le D(x, f^k(x))$

Proof: To prove this it is enough if we prove that $\delta[0(x,m)] \le \gamma_m$

Where γ_m=max { $D(x,x)$…….. $D(x, f^m(x))$ } ________________(1)

We prove this by using Induction,

Assume that (1) is true for 'm' i.e $\delta[0(x,m)] \le \gamma_m$

Now we have to prove for $m+1$ i.e $\delta[0(x,m+1)] \le \gamma_{m+1}$ __________(2)

We have $\delta[0(f(x),m)] \le$ max { $D(x,x)$,…….. $D(x, f^m(x)), D(x, f^{m+1}(x))$ }

Also $D(f^i(x), f^{m+1}(x)) \le$ max { $D(x,x)$,…….. $D(x, f^m(x)), D(f^{i-1}(x), f^{m+1}(x))$ } ______(3)

$\forall \ 1 \le i \le m$

Hence $\delta[0(x,m+1)]$=Sup { $D(f^i(x), f^j(x)) / 0 \le i \le j \le m+1$} ,

=Sup { $D(x,x)$,…….. $D(x, f^m(x))$, $D(x, f^{m+1}(x))$ } $\cup$

Sup { $D(f^i(x), f^j(x)) / 0 \le i \le j \le m+1$} ,

$\le$ max{ $D(x,x)$…….. $D(x, f^m(x)), D(x, f^{m+1}(x)), \delta(0(f(x),m))$ }

$\le \gamma_{m+1}$ from (1) and (3)

Hence $\delta[0(x,m+1)] \le \gamma_{m+1}$. This proves the lemma.

Proof of the Theorem:

If $1 \le i \le m, 1 \le j \le m$

$D(f^i(x), f^j(y)) = D(f(f^{i-1}(x)), f(f^{j-1}(y)))$

$\le h$ max { $D(f^{i-1}(x), f^{j-1}(y))$, $D(f^{i-1}(x), f^i(x))$, $D(f^{i-1}(x), f^j(y))$,

$D(f^{j-1}(x), f^i(x))$, $D(f^{j-1}(x), f^j(x))$}

$$\le h\ \delta[0(x,m)]\text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(1)$$

Also $\delta[0(x,m)] \le \max\ \{D(f^{j-1}(x), f^{j}(x)), D(x,x), D(x, f(x)), \ldots D(x, f^{m}(x))\}$
_____(2)

If m , n are positive integers such that $m > n$ then by (1)

$D(f^{m}(x), f^{n}(x)) = D(f^{m-n+1}(f^{n-1}(x)), f(f^{n-1}(x)))$

$$\le h\ \delta(0(f^{n-1}(x), m-n-1))$$

$$\le h\ D(f^{n-1}(x), f^{k_1+n-1}(x)) \text{ for some } k_1\ ;0 \le k_1 \le m-n-1$$

(by above lemma)

$$\le h^{2}\ \delta(0(f^{n-2}(x), m-n+2)$$

$$\le h^{n}\ \delta[0(x,m)]$$

By the lemma $\exists\ k \ni 0 \le k \le m$ and $\delta[0(x,m)] \le D(x, f^{k}(x))$

Assume $k \ge 1$, $D(x, f^{k}(x)) \le D(x, f(x)) \Diamond D(f(x), f^{k}(x))$

$$\le \beta \max\{D(x, f(x)), D(f(x), f^{k}(x))\}$$

$$\le D(x, f(x)) + D(f(x), f^{k}(x))$$

$$\le D(x, f(x)) + h\ \delta[0(x,m)]$$

$$\le D(x, f(x)) + h\ D(x, f^{k}(x))$$

$$\Rightarrow D(x, f^{k}(x)) \le \frac{1}{1-h}\ D(x, f(x))$$

If $k = 0$, $\delta[0(x,m)] \le D(x, x) \le D(x, f(x)) \Diamond D(f(x), x)$

$$\le \beta \max\{D(x, f(x)), D(f(x), x)\}$$

$$\le D(x, f(x)) + D(f(x), x)$$

$$\le D(x, f(x)) + h\ D(x, x)$$

$$\Rightarrow D(x,x) \le \frac{1}{1-h} D(x, f(x))$$

Hence $D(f^{m}(x), f^{n}(x)) \le h^{n}\ \delta[0(x,m)]$

$$\le \frac{h^{n}}{1-h} D(x, f(x))$$

This is true for every $m > n$ Since $0 \le h < 1$. $\lim h^{n} = 0$.Hence $\{f^{m}(x)\}$ is a Cauchy sequence in (X,D).

Since X is complete, $\exists\ z \in X$ so that $\lim f^{n}(x) = z$

We prove that $D(z, f(z)) = 0$

$$\begin{aligned}0 \le D(z, f(z)) &\le D(z, f^{n+1}(z)) \Diamond D(f^{n+1}(z), f(z)) \\ &\le \beta \max\{D(z, f^{n+1}(z)), D(f^{n+1}(z), f(z))\} \\ &\le D(z, f^{n+1}(z)) + D(f^{n+1}(z), f(z))\end{aligned}$$

By continuity of f, lim $D(f^{n+1}(x), f(z)) = 0$

Hence $D(z, f(z)) = 0$, hence z is a coincidence point of f.

Suppose z_1, z_2 are coincidence point of f then $D(z_1, z_1) = D(z_1, f(z_1)) = 0$, *similarly* $D(z_2, z_2) = 0$

If $D(z_1, z_2) \neq 0$. Then by (*),

$D(z_1, z_2) = D(f(z_1), f(z_2))$

$$\le h \max\{D(z_1, z_2), D(z_1, f(z_1), D(z_1, f(z_2)), D(z_2, f(z_1)), D(z_2, f(z_2))\}$$

$\le h\, D(z_1, z_2)$ a contradiction

Hence $D(z_1, z_2) = 0$.Hence $z_1 = z_2$.This completes the proof.

We now prove a fixed point theorem for a self map on a D metric space satisfying the analogue of (12).

Theorem2.2: Let (X, D) be a complete D metric space such that $\Diamond$ satisfies β-property with $\beta \le 1$ and $f: X \to X$ be a continuous mapping such that there exist real numbers $\alpha, \beta_0, \gamma \ni, 0 \le \alpha < \frac{1}{2}, 0 \le \beta_0 < \frac{1}{2}, \gamma < \min\{\frac{1}{4}, \frac{1}{2} - \alpha, \frac{1}{2} - \beta_0\}$ satisfying at least one of the following for each $x, y \in X$

i. $D(f(x), f(y)) \le \alpha D(x, y)$
ii. $D(f(x), f(y)) \le \beta_0 \{D(x, f(x)) \Diamond D(y, f(y))\}$
iii. $D(f(x), f(y)) \le \gamma \{D(x, f(y)) \Diamond D(y, f(x))\}$

Then f has a unique coincidence point.

Proof: Putting $y = x$ in the above and $\delta = \max\{2\alpha, 2\beta_0, 2\gamma\}$ we get $D(f(x), f(x)) \le \delta D(x, f(x))$

Again putting $y = f(x)$ in the above (i) (ii) (iii) yield

$$D(f(x), f^2(x)) \le 2\alpha D(x, f(x))$$

$$D(f(x), f^2(x)) \leq \frac{\beta_0}{1-\beta_0} D(x, f(x))$$

$$D(f(x), f^2(x)) \leq \frac{\gamma+\delta}{1-\gamma} D(x, f(x))$$

If $h = \max\{2\alpha , \frac{\beta_0}{1-\beta_0} , \frac{\gamma+\delta}{1-\gamma}\} then\, 0 \leq h < 1 and$

$$D(f(x), f^2(x)) \leq h \; D(x, f(x))$$

If m, n are positive integers such that $m > n$ then we can show that

$$D(f^n(x), f^m(x)) \leq \frac{h^n}{1-h} D(x, f(x))$$

since $0 \leq h < 1$; $\lim h^n = 0$

Hence { $f^n(x)$ } is a Cauchy sequence in (X, D).

Since X is complete, $\exists\, z \; in \; X \ni \lim\limits_{n} f^n(x) = z$

Since f is continuous, lim $f^{n+1}(x) = f(z) \; in \; (X, D)$.

$$\sin ce\; 0 \leq D(z, f(z)) \leq D(z, f^{n+1}(x)) \Diamond D(f^{n+1}(x), f(z))$$

$$\leq \beta \max\{D(z, f^{n+1}(x)), D(f^{n+1}(x), f(z))\}$$

$$\leq D(z, f^{n+1}(x)) + D(f^{n+1}(x), f(z))$$

It follows that $D(z, f(z))$=0 , Hence z is a Coincidence point of f .

Uniqueness : If z_1 , z_2 are coincidence points of f then by hypothesis,

Either $D(z_1, z_2) \leq \alpha\; D(z_1, z_2)$ or 0 or $2\gamma\; D(z_1, z_2)$

Since $0 < \alpha < \frac{1}{2}$ and $0 < \alpha < \frac{1}{4}$ we must have $D(z_1, z_2)$=0

Hence $z_1 = z_2$. This completes the proof.

The D metric version for the contractive inequality (10) in the modified form (**) given below yields the following

Theorem2.3 : Let (X, D) be a complete D metric space such that $\Diamond$ satisfies β-property with $\beta \leq 1$ and $f : X \to X$ be a continuous mapping. Assume that there exist non-negative constants a_i satisfying $a_1 + a_2 + a_3 + 2a_4 + 2a_5 < 1$ such that for each $x, y \in X$ with $x \neq y$

$D(f(x), f(y)) \leq a_1\, D(x, y) \Diamond\, a_2\, D(x, f(x)) \Diamond\, a_3\, D(y, f(y)) \Diamond\, a_4\, D(x, f(y)) \Diamond\, a_5$
$D(y, f(x))$

-----(**). Then f has a unique coincidence point.

Proof:
Consider

$$
\begin{aligned}
D(f(x), f^2(x)) &\le a_1 D(x, f(x)) \Diamond a_2 D(x, f(x)) \Diamond a_3 D(f(x), f^2(x)) \Diamond a_4 D(x, f^2(x)) \Diamond a_5 D(f(x), f(x)) \\
&\le \beta \max\{a_1 D(x, f(x)), a_2 D(x, f(x)), a_3 D(f(x), f^2(x)), a_4 D(x, f^2(x)), a_5 D(f(x), f(x))\} \\
&\le a_1 D(x, f(x)) + a_2 D(x, f(x)) + a_3 D(f(x), f^2(x)) + a_4 D(x, f^2(x)) + a_5 D(f(x), f(x)) \\
&= (a_1 + a_2) D(x, f(x)) + a_3 D(f(x), f^2(x)) + a_4 D(x, f^2(x)) + 2a_5 D(x, f(x))
\end{aligned}
$$

$$(\because D(f(x), f(x)) \le D(x, f(x)) + D(f(x), x))$$

$\Rightarrow$

$$D(f(x), f^2(x)) \le \left[\frac{a_1 + a_2 + 2a_5}{1 - a_3}\right] D(x, f(x)) + \left[\frac{a_4}{1 - a_3}\right] D(x, f(x)) + \left[\frac{a_4}{1 - a_3}\right]$$

$D(f(x), f^2(x))$

$$\Rightarrow D(f(x), f^2(x)) \le \left[\frac{a_1 + a_2 + 2a_5 + a_4}{1 - a_3 - a_4}\right] D(x, f(x))$$

$$\Rightarrow D(f(x), f^2(x)) \le \beta\ D(x, f(x)) \quad \text{where } \beta = \left[\frac{a_1 + a_2 + 2a_5 + a_4}{1 - a_3 - a_4}\right],\ 0 < \beta < 1$$

If $m > n$ then

$$
\begin{aligned}
D(f^n(x), f^m(x)) &\le D(f^n(x), f^{n+1}(x)) \Diamond D(f^{n+1}(x), f^{n+2}(x)) \Diamond ----\Diamond D(f^{m-1}(x), f^m(x)) \\
&\le \beta \max\{D(f^n(x), f^{n+1}(x)), D(f^{n+1}(x), f^{n+2}(x)), ---, D(f^{m-1}(x), f^m(x))\} \\
&\le D(f^n(x), f^{n+1}(x)) + D(f^{n+1}(x), f^{n+2}(x)) + ----+ D(f^{m-1}(x), f^m(x)) \\
&\le (\beta^n + \beta^{n+1} + ... + \beta^{m-1})\ D(x, f(x)) \\
&= \beta^n (1 + \beta + \beta^2 .... + \beta^{m-n-1})\ D(x, f(x)) \\
&< \frac{\beta^n}{1 - \beta}\ D(x, f(x))
\end{aligned}
$$

Hence $\{ f^n(x) \}$ is Cauchy sequence in (X, D), hence convergent.

Let $\xi = \lim_n (f^n(x))$ then $f(\xi) = \lim_n (f^{n+1}(x))$ (since f is continuous)

So $D(\xi, f(\xi)) = \lim_n D(f^n(x), f^{n+1}(x))$

$$\leq \lim_{n} \frac{\beta^n}{1-\beta} D(x, f(x))$$

Since $0 < \beta < 1$, $D(\xi, f(\xi))=0$ .Hence $f(\xi)=\xi$.

Hence ξ is a coincidence point for f .

Uniqueness: If $D(\xi, f(\xi)) = D(\eta, f(\eta)) = 0$

$\Rightarrow f(\xi)=\xi$ and $f(\eta)=\eta$

Consider

$$\begin{aligned} D(\xi,\eta) = D(f(\xi), f(\eta)) &\leq a_1 D(\xi,\eta) \Diamond a_2 D(\xi, f(\xi)) \Diamond a_3 D(\eta, f(\eta)) \Diamond a_4 D(\xi, f(\eta)) \Diamond a_5 D(\eta, f(\xi)) \\ &\leq \beta \max\{a_1 D(\xi,\eta), a_2 D(\xi, f(\xi)), a_3 D(\eta, f(\eta)), a_4 D(\xi, f(\eta)), a_5 D(\eta, f(\xi))\} \\ &\leq a_1 D(\xi,\eta) + a_2 D(\xi, f(\xi)) + a_3 D(\eta, f(\eta)) + a_4 D(\xi, f(\eta)) + a_5 D(\eta, f(\xi)) \\ &\leq \gamma D(\xi,\eta) \qquad \text{where } \gamma = a_1 + a_4 + a_5 < 1 \end{aligned}$$

$\Rightarrow D(\xi,\eta) = 0$ $\qquad$ *Hence* $\xi = \eta$

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References:

[1] Pascal Hitzler: Generalised metrices and topology in logic programming semantics, Ph. D Thesis,2001.
[2] S.Sedghi: fixed point theorems for four mappings in d*-metric spaces, Thai journal of mathematics, vol 7(2009) November 1:9-19
[3] S.G. Mathews: Metric domains for completeness, Technical report 76 , Department of computer science , University of Warwick, U.K, Ph.D Thesis 1985.
[4] B.E.Rhoades : A comparison of various definitions of contractive mappings,Trans of the Amer.Math.Society vol 226(1977) 257-290.
[5] V.M.Sehgal : On fixed and periodic points for a class of mappings , journal of the London mathematical society (2), 5, (1972) 571-576.
[6] J. L. Kelley. General topology. D. Van Nostrand Company, Inc., 1960.
[7] Lj. B. Ciric : A Generalisation of Banach's Contraction Principle, American Mathematical Society, volume 45, Number 2, August 1974.