Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.4, No.6, 2014 www.iiste.org

Existence And Uniqueness Solution For Second-Partial Differential Equation By Using Conditions And Continuous Functions

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Abstract

We study the second- partial differential equation with some conditions and depending arbitrary continuous functions of the bar at the point x at the time t .

1.Introduction

In[1],they are obtained an asymptotic expansion, containing regular boundary corner functions in the small parameter ε, for the solution of a second partial differential equation , and constructed the asymptotic expansion $u_n(x,t,\varepsilon)$ for the modified problem and proved it is the unique solution .In[2], they are constructed and justify the asymptotic solutions as two series in the powers of small parameters consisting four parts and derivatives degenerate into the systems of partial differential equation of first order, also proved that the solution is uniform in the domain ,and they are a unique solution and the asymptotic approximation is within $0(\varepsilon^{n+1})$. In[3], they are studied the development wave equation with some conditions and proved the existence and uniqueness solution by using the reflection method. In[4],they are considered the problem of periodic solution of ordinary differential equations of arbitrary $4-^{th}$order with a rapidly oscillating coefficient proportional to the frequency of oscillations, and proved the existence and local uniqueness of solution close to the corresponding asymptotic solution of the original and averaged problems with natural additional conditions of smoothness.In[5], they are construct asymptotic solution of a partial differential equation with small parameter , also proved the solution is unique and uniform in the domain Ω and, further, the asymptotic approximation is within $0(\varepsilon^2)$ .studied a modification of an initial-boundary-value problem in the critical case for the heat-conduction equation in a thin domain, and they justified asymptotic expansions of the solutions of the problems with respect to a small parameter ε>0,In[6].[7], they are constructed and justify the asymptotic solutions as two series in the powers of –small parameters consisting four parts and derivatives degenerate into the systems of partial differential equation of first order .they are proved that the solution is uniform in the domain, and they are a unique solution and the asymptotic approximation is within $0(\varepsilon^{n+1})$ .In[8], for a system of two partial differential equations of second order, they obtained and justify two asymptotic solutions in the form of two series with respect to the small parameter ε. They are proved the solutions is unique and uniform in the domain Ω, and, further, each the asymptotic approximations are within $0(\varepsilon^{n+1})$ .they presented an efficient integral equation approach to solve the forced heat equation,$u_t(x) - \Delta u(x) = F(x,u,t)$,in a two-dimensional, multiply connected domain, with Dirichlet boundary conditions, Instead of using an integral equation formulation based on the heat kernel, we discretize in time, first, In[9].In[10], they proved that the temperature distribution in the limit one-dimensional rod with time-averaged sources of heat is the uniform asymptotic approximation of the temperature distribution in the initial problem in an arbitrary sub domain of the plane rod and in an arbitrary time interval, which are located at a positive distance from the ends of the rod and the initial time instance, respectively of course, the temperature in the one-dimensional rod, which is a function of the longitudinal coordinate x and the time t, is identified with the function of (x, y ,t), which is independent of the transversal coordinate y of the plane rod. In[11], the presented a numerical method that solved heat equations using he's variation iteration method , it showed that the solutions obtained from the developed method converged rapidly to the exact solutions within three iterations .it is also found that he's variation iteration method gives very trivial solutions for the nonlinear differential equations with zero initial condition. In[12]the equation describing the conduction of heat in solid had, over the past two centuries, proved to be a powerful tool for analyzing the dynamic motion of heat as well as for solving an enormous array of diffusion –type problems in physical sciences, biological sciences, earth sciences, and social sciences. In[13],this articled provides a practical overview of numerical solution to the heat equation using the finite difference method. The forward time, centered space , the backward time, centered space ,and Crank-Nicolson schemes are developed, and applied to a simple problem involving the one-dimensional heat equation. In[14], they investigated a free boundary problem for the heat equation derived from combustion theory and study the development of the boundary, Γ this problem described the propagation solutions to this problem is reviewed and major results are summarized. .our principal aim in the present paper is concerned the second-partial differential equation and proving the existence and uniqueness solution by using some conditions.

2. Formulation of The Problem

We consider the second-partial differential equation for the following problem

$$c_1(t)u_t - c_2(x)u_{xx} = 0$$

$$u(x,0) = f(x) \qquad 0 \le x \le h \qquad (2.1)$$

$$u(0,t) = T_1 \ , \ \ u(l,t) \qquad t > 0$$

$$\frac{\partial u(x,t)}{\partial x}|_{x=0} = \frac{\partial u(x,t)}{\partial x}|_{x=h} = 0 \qquad t > 0$$

Where $u(x,t)$the function of the bar at the point x at the time t $c_1(t), c_2(x)$ be a continuous function depends on the variable t and x f(x) is a given function .In order to determine the temperature in the bar at any time t. However it turns out that suffices to consider the case $T_1 = T_2$=0 only.We can also assume that the ends of the bar are insulated so that no heat can pass through them which impliesv$c_1(t)u_{t-}c_2(x)u_{xx} = 0 \, , 0 < x < h \ \ , t > 0$ (2.2)

Where u(x ,t) satisfies the initial condition

$$u(x,0) = f(x), \qquad 0 < x < h, \qquad (2.3)$$

And the boundary conditions

$$u(0,t) = u(h,t) = 0 \qquad (2.4)$$

In the same way, we can obtain

$$\frac{\partial u(x,t)}{\partial x}|_{x=0} = \frac{\partial u(x,t)}{\partial x}|_{x=h} = 0 \qquad t > o \qquad (2.5)$$

The problem (2.2),(2,3),(2,5) is known as the Neumann problem while (2.2),(2,3),(2.4) as the Dirichlet problem for the diffusion equation .At first we discuss a property of the diffusion equation, known as the maximum-minimum principle

Let: $F = \{(x,t):\ o \le x \le h\ \ , 0 \le t \le T\}$ be a closed rectangle and $L = \{(x,t) \in F :\ t = 0 \text{ or } = 0 \text{ or } x = h\}$.

3.Procedure of Solving The Problem

We study the following theorem .

Theorem 3-1 : Let u(x ,t) be a continuous function in F which satisfies equation (2.2) in F/ L

$$\max_F u(x,t) = \max_L u(x,t) \tag{3.6}$$

$$\min_F u(x,t) = \min_L u(x,t) \tag{3.7}$$

$$\max_F c_2(x) = \max_L c_2(x) \tag{3.8}$$

$$\min_F c_2(x) = \min_L c_2(x) \tag{3.9}$$

Proof : We use the method of contradiction . assume that the minimum value of u(x ,t)attained at an interior point (x_0, t_0) . Let $G = \min_l u$, thus there exist a finite $\xi > 0$ such that $u(x_0, t_0) = G + \xi$. At the minimum point (x_0, t_0) . We have $u_x(x_0, t_0) = 0$, $u_{xx}(x_0, t_0) \le 0$, $u_t \ge 0$. We need to rule out the possibility of equality consider $v(x,t) = -u(x,t) + \varphi(t - t_o)$. For positive constant $\varphi > 0$ at the point (x_0, t_0),we have $v(x,t) = G + \xi$ since both t ,$t_0 \le T$, $\varphi(t - t_0) \le \varphi T$. Now we choose φ such that $\varphi T \le \frac{\xi}{2}$ since $\min_l u = G$,we have $\min_l v \le G + \frac{\xi}{2^-}$, since u is continuous , so is v. thus , v must have a minimum value at some point(x_1, t_1) in the interior $(0 < t_1 \le T, o < xh)$

$v(x_1, t_1) \ge v(x_0, t_0) = G + \xi$,therefore. $v_{xx} \le 0$, $v_t \ge 0$. Since

$-u_{xx}(x_1, t_1) = v_{xx}(x_1, t_1), -u_t(x_1, t_1) = v_t(x_1, t_1) + \varphi$

$-u_{xx}(x_1, t_1) \le 0, \quad -u_t(x_1, t_1) \ge \varphi > 0$

Which is contradictory to $c_1(t)u_t = c_2(x)u_{xx}$ therefore $\min_F u(x,t) = \min_L u(x,t)$. The same way above we get :$\min_F c_2(x) = \min_L c_2(x)$ considering the function $v(x,t) = u(x,t)$ we have $\max_F u(x,t) = \max_L u(x,t)$ and considering the function $s(x,t) = -c_2(x)$ we get $\max_F c_2(x) = \max_L c_2(x)$.

4.Non –Homogenous for The Second-Partial Differential Equations

The solution for non-homogenous second-order partial differential equations by using maximum-minimum principle it follows the uniqueness

$c_1(t)u_t - c_2(x)u_{xx} = f(x,t) \qquad 0 < x < h,\ \ o < t \le T$

$u(x,0) = f(x) \qquad 0 \le x \le h \qquad (4.1)$

$u(0,t) = g_1(t), u(h,t) = g_2(t) \qquad 0 \le t \le T$

Suppose

$f(x,t) \in C(F) \qquad f(x) \in C[0.1]$

$g_{1(t)} \in C[0,T] \qquad g_2(t) \in [0,T] \qquad (4.2)$

$f(0) = g_1(0) \qquad f(h) = g_2(0)$

By a solution we mean a function $u \in C(F)$ which is differentiable inside F and satisfies the equation along with the initial and the boundary condition of (4.1) .

Theorem 4-1 : The problem in (4.1) and(4.2) has no more than one solution .

Proof : Suppose u(x ,t)and w(x ,t)are two solution of (4.2) , let $p(x,t) = u(x,t) - w(x,t)$ then

$c_{1(t)}p_t - c_{2(x)}p_{xx} = 0 \qquad o < x < h, o < t \le T$

$p(x,0) = 0 \qquad o \le x \le h$

$p(0,t) = p(h,t) = 0 \qquad 0 \le t \le T$

By theorem (3.1) it follows $\max_F p(x,t) = \min_F p(x,t) = 0$

$\max_F c_2(x) = \min_F c_2(x) = 0$. Therefore $p(x,t) \equiv 0$, so that $u(x,t) \equiv w(x,t)$ For every $(x,t) \in F$. Consider the problem (4.2), with f= $g_1 = g_2 = 0$ that is $c_1(t)u_t - c_2(x)u_{xx} = 0$, $0 < x < h, 0 < t \le T$ $u(x,0) = \varphi(x) \qquad 0 \le x \le h \qquad (4.3)$

$u(0,t) = u(h,t) = 0 \qquad 0 \le t \le T$

As a corollary of theorem(3.1) the continuous dependence of solution of (4.3) with respect to initial data follows .

Corollary 4-1 : Let $u_i(x,t)$be a solution of (4.3) with initial data $f_i(x) \quad i = 1,2$ then

$\max_{0<x\le h}|u_1(x,t) - u_2(x,t)| \le \max_{0<x\le h}|f_1(x) - f_2(x)| \qquad (4.4)$ for every $t \in [0,T]$.

Proof : Consider the function $v(x,t) = u_1(x,t) - u_2(x,t)$,which satisfies

$c_1(t)v_t - c_2(x)v_{xx} = 0 \qquad 0 < x < h, 0 < t \le T$

$v(x,0) = f_1(x) - f_2(x) \qquad 0 \le x \le h$

$v(0,t) = v(h,t) = 0 \qquad 0 \le x \le T$

By theorem(3.1) it follows that $u_1(x,t) - u_2(x,t) \le \max\left\{\max_{0 \le x \le h}\left(f_1(x) - f_2(x)\right), 0\right\} \le \max_{0\le x\le h}|f_1(x) - f_2(x)|$.

And $u_1(x,t) - u_2(x,t) \geq \min\{\min(f_1(x) - f_2(x)), 0\}$

$$\geq -\max\left\{\max_{0 \leq x \leq h}(f_1(x) - f_2(x)), 0\right\}$$

$$\geq -\max_{0\leq x\leq h}|f_1(x) - f_2(x)|$$

Which imply (4.4) . The uniqueness and stability to (4.3) can be derived another approach ,known as the energy method. Let u be a solution of the problem (4.3). the quantity $H(t) = \int_0^h u^2(x,t)dx$. Is referred to as the thermal energy at the instant t .we shall show that H(t) is a decreasing function .

Theorem 4-2 : Let u(x ,t)be a solution of (4.3) then $H(t_1) \geq H(t_2)$ if $,0 \leq t_1 \leq t_2 \leq T$.

Let $u_i(x,t)$be a solution of(4.3)correspondingto the initial data $f_i(x), i = 1,2$ then $\int_0^h (u_1(x,t) - u_2(x,t))^2 dx \leq \int_0^h (f_1(x) - f_2(x))^2 dx$.

Proof : Multiplying the equation by u ,using

$$u[c_1(t)u_t] = \frac{c_1(t)}{2}\frac{\partial}{\partial t}(u^2)$$

$$u[c_2(x)u_{xx}] = c_2(x)\left[\frac{\partial}{\partial x}uu_x - \frac{\partial}{\partial x}u^2\right]$$

$$0 = \int_0^h [c_1(t)u_t - c_2(x)uu_x]u\, dx$$

$$= \int_0^h \left[\frac{c_1(t)}{2}\frac{\partial}{\partial t}u^2 - c_2(x)\left[\frac{\partial}{\partial x}uu_x - \frac{\partial}{\partial x}u^2\right]\right]dx$$

$$= \frac{c_1(t)}{2}\frac{\partial}{\partial t}u_x^2|_0^h - c_2(h)u(h,t)u_x|_{x=h} + C(0)u(o,t)u_x|_{x=0} + \int_0^h (c_2(x)\frac{\partial}{\partial x}u^2)dx$$

$$= c_2(x)\frac{\partial}{\partial x}u_x^2|_0^h - \int_0^h \frac{\partial}{\partial x}u_x^2\, c_{2x}(x)dx$$

$$= c_2(x)\frac{\partial}{\partial x}u_x^2|_0^h - \frac{\partial}{\partial x}u_x^2 c_2(x)|_0^h + \int_0^h c_2(x)\frac{\partial}{\partial x}u^2 dx$$

$$\frac{dH}{dt}(t) = -2\int_0^h c_2(x)u_x^2\, dx$$

$$\frac{dH}{dt}(t) \leq 0$$

Thus H(t)is a non-increasing function of time t . $H(t_1) \geq H(t_2)$ for all $t_2 \geq t_1 \geq 0$. The function $v(x,t) = u_1(x,t) - u_2(x,t)$satisfies(4.3)with $\varphi(x) = \varphi_1(x) - \varphi_2(x)$therefore for t≥0 by (a)

$$\int_0^h (u_1(x,t) - u_2(x,t))^2 \leq \int_0^h (u_1(x,0) - u_2(x,0))^2 dx = \int_0^h (\varphi_1(x) - \varphi_2(x))^2 dx$$

now to show that $\frac{d^2H(t)}{dt^2} = 4\int_0^h u_x^2\, dx$. We can multiply by u_t and integrate with respect to and get $\int_0^h c_1(t)\frac{\partial^2 u(x,t)}{\partial t}dx = \int_0^h c_2(x)\frac{\partial u(x,t)}{\partial t}\frac{\partial^2 u(x,t)}{\partial x^2}dx$

$$= \int_0^h c_2(x)\left[\frac{\partial}{\partial x}\left(\frac{\partial u(x,t)}{\partial x}\frac{\partial u(x,t)}{\partial t}\right) - \frac{\partial u(x,t)}{\partial x}\frac{\partial^2 u(x,t)}{\partial x\,\partial t}\right]dx$$

$$\int_0^h c_1(t)\frac{\partial^2 u(x,t)}{\partial t}dx = \int_0^h c_2(x)\frac{\partial}{\partial x}\left(\frac{\partial u(x,t)}{\partial x}\frac{\partial u(x,t)}{\partial t}\right)dx -$$

$$\int_0^h c_2(x)\frac{\partial u(x,t)}{\partial x}\frac{\partial^2 u(x,t)}{\partial x\,\partial t})dx$$

$$\int_0^h c_1(t)\frac{\partial^2 u(x,t)}{\partial t}dx = c_2(h)\frac{\partial u(x,t)}{\partial x}|_0^h\frac{\partial u(x,t)}{\partial t}|_0^h -$$

$$c_2(0)\frac{\partial u(x,t)}{\partial x}|_{x=0}\frac{\partial u(x,t)}{\partial t}|_{x=0} \quad -\int_0^h c_2(x)\frac{\partial u(x,t)}{\partial x}\frac{\partial^2 u(x,t)}{\partial x\,\partial t}dx$$

We get by the chain rule $\frac{\partial}{\partial t}\frac{\partial u^2(x,t)}{\partial x} = 2\frac{\partial u(x,t)}{\partial x}\frac{\partial^2 u(x,t)}{\partial x\,\partial t}$

$$\int_0^h c_1(t)\frac{\partial u^2(x,t)}{\partial t}dx = c_2(h)\frac{\partial u(h,t)}{\partial x}\frac{\partial u(h,t)}{\partial t} -$$

$$c_2(0)\frac{\partial u(0,t)}{\partial x}\frac{\partial u(0,t)}{\partial t} - \frac{1}{2}\int_0^h c_2(x)\frac{\partial}{\partial t}\frac{\partial u^2(x,t)}{\partial x}dx$$

$$\frac{1}{2}\frac{d}{dt}\int_0^h c_2(x)\frac{\partial u^2(x,t)}{\partial x}dx = -\int_0^h\frac{\partial u^2(x,t)}{\partial t}dx +$$

$$c_2(h)\frac{\partial u(h,t)}{\partial x}\frac{\partial u(h,t)}{\partial t}c_2(0)\frac{\partial u(0,t)}{\partial x}\frac{\partial u(0,t)}{\partial t}$$

According to the boundary condition (2), u(0,t)=u(h ,t)=0 for all t>0 since u(0,t)and u(h ,t)are constant with respect to time ,we conclude that $u_t(0,t) = u_t(h,t) = 0$ for $t > 0$,thus , we get that $\frac{d}{dt}\int_0^h c_2(x)\frac{\partial u^2(x,t)}{\partial x}dx = -2\int_0^h\frac{\partial u^2(x,t)}{\partial t}dx$

$$\frac{d}{dt}\int_0^h -2c_2(x)\frac{\partial u^2(x,t)}{\partial x}dx = 4\int_0^h\frac{\partial u^2(x,t)}{\partial t}dx$$

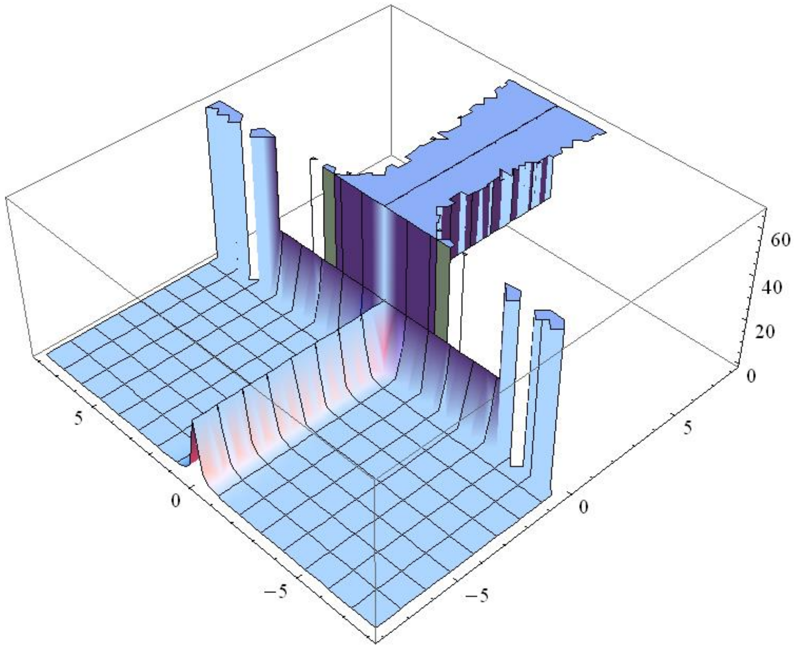
$$\frac{d^2H(t)}{dt^2} = 4\int_0^h\frac{\partial u^2(x,t)}{\partial t}dx\,.$$

Problem 1: solve the equation $C_1(t)u_t - C_2(x)u_{xx} = 0$ where $C_1(t) = t^2+3$, $C_2(x) = 2x^2$, $f(x) = x^3 + x$, 0< x <4

Solution : $B_n = \frac{2}{l}\int_0^h f(x)\sin\frac{n\pi x}{h}\,dx$

$$B_n = \frac{2}{4}\int_0^4 (x^3+x)\sin\frac{n\pi x}{4}\,dx = \frac{44}{3}\sin\frac{n\pi x}{4} = \frac{44(-1)^n}{12}$$

$$u(x,t) = \sum_{n=1}^{8}\frac{44(-1)^n}{12}e^{\frac{2x^2}{t^2+3}n^2\pi^2 t}\,\mathrm{Sin}[\frac{nx\pi}{4}]$$



Plot3D[%1, {t, −8,8}, {x, −8,8}], {nxπ, −2,2}] .

Problem 2 : solve the equation $c_1(t)u_t - c_2(x)u_{xx} = 0$ where

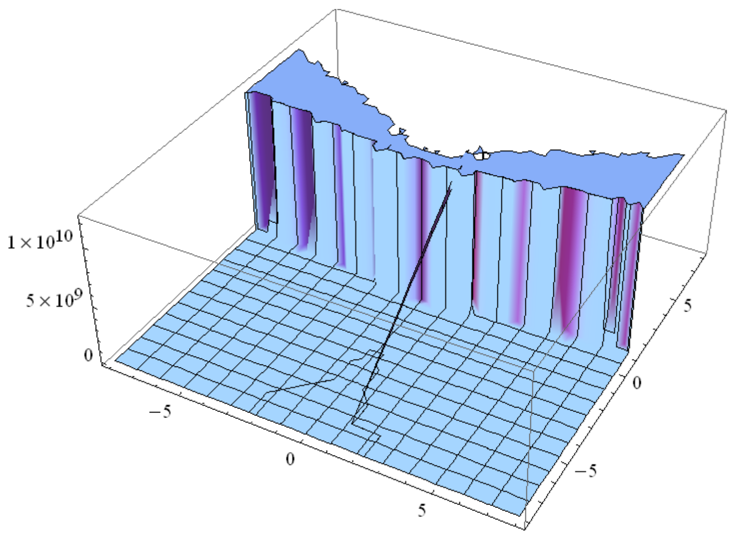
$c_1(t) = 2t^3 + \tan(t)$, $c_2(x) = 3x^3$, $f(x) = e^{\frac{x^3}{2x}}$

Solution : $B_n = \frac{2}{h}\int_0^h f(x)\sin\frac{nx\pi}{h}\,dx$

$$B_n = \frac{2}{2}\int_0^2 e^{\frac{x^3}{2x}}\sin\left(\frac{nx\pi}{2}\right)dx$$

$$= \sqrt{\frac{\pi}{2}}\,Erfi\left(\sqrt{2}\right)\sin\left(\frac{nx\pi}{2}\right)$$

$$u(x,t) = \sum_{n=1}^{5}\sqrt{\frac{\pi}{2}}\,\mathrm{Erfi}\left(\sqrt{2}\right)\sin\left(\frac{nx\pi}{2}\right)e^{\frac{3x^3n^2\pi^2 t}{2t^3+\tan(t)}}$$



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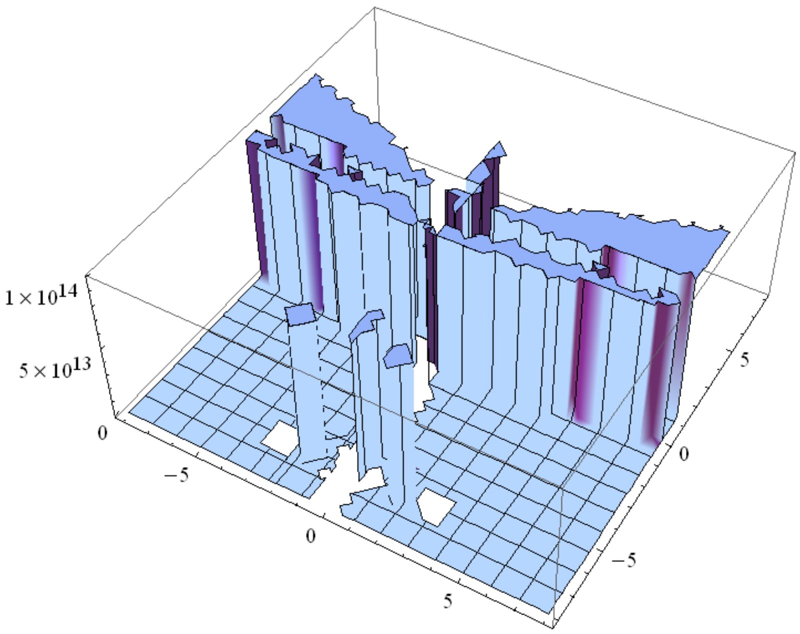
Problem 3: solve the heat equation $c_1(t)u_t - c_2(x)u_{xx} = 0$ where $f(x) = \frac{(3x^2-2)}{2x}$, $c_1(t) = 2t^3 + \cos t$, $c_2(x) = 3x^3 - \cot\frac{x}{2}$

Solution : $B_n = \frac{2}{12}\int_0^{12} \frac{(3x^2-2)}{2x} \sin(\frac{nx\pi}{12})dx$

$=143\sin(\frac{nx\pi}{12})$

$$u(x,t) = \sum_{n=1}^{7} 143\sin(\frac{nx\pi}{12}) e^{\frac{(3x^3-\cot\frac{x}{2})}{2t^3+\cos t} n^2\pi^2 t} \sin(\frac{nx\pi}{12})$$



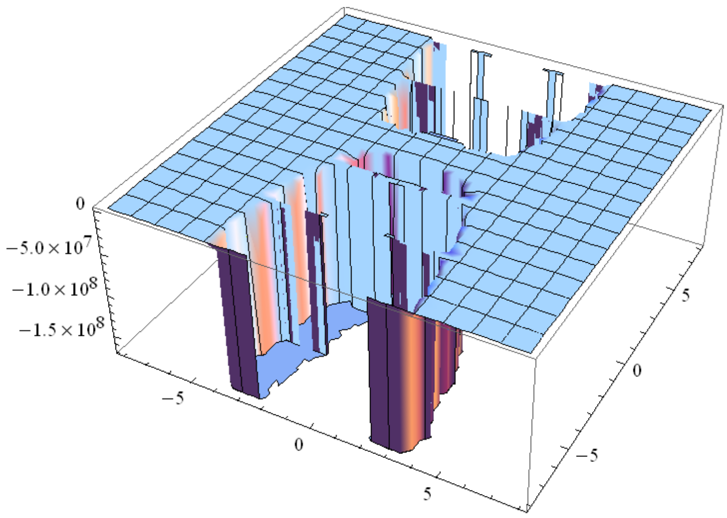
Manipulate[Plot3D[%1, {t, −8,8}, {x, −8,8}], {nxπ, −2,2}, {nπx, −2,2}] .

Problem 4 : solve the heat equation $c_1(t)u_t - c_2(x)u_{xx} = 0$ where $f(x) = x + 2x^3 - x^5$, $c_1(t) = \sinh 3t$, $c_2(x) = 2x^2 - e^{\tan x}$.

Solution: $B_n = \frac{2}{7}\int_0^{7} (x + 2x^3 - x^5) \sin(\frac{nx\pi}{7})dx$

$=\frac{-15757}{3}\sin(\frac{nx\pi}{7})$

$$u(x,t) = \sum_{n=1}^{9} \frac{-15757}{3}\sin(\frac{nx\pi}{7})\, e^{\frac{2x^2-e^{\tan x} n^2\pi^2 t}{\sinh 3t}} \sin(\frac{nx\pi}{7})$$



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