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Conditional Heteroscedasticity: GARCH model with application to interest rate in Ghana (2003:01 – 2013:12).

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Abstract
The development of time series model for analysis has seen a major patronage in recent times. This can mainly be attributed to the precision that is associated with these models and hence its dependence in the field of finance, statistics and economics. The theory of Generalized Autoregressive Conditional Heteroscedasticity (GARCH) was explored and monthly interest rate of Ghana from 2003:01 to 2013:12 was applied. The results shows that the best GARCH model to adequately capture the volatility in interest rest is the GARCH (1, 2). The estimated model was used to forecast interest rate for a year in Ghana and the result shows that interest rate is predicted not to hit above 30% by the end of 2014.
Keywords: Autocorrelation**,** Conditional, GARCH, heteroscedasticity, and volatility.

1.0 Introduction

One major time series methodology that has benefited from the current development is the Box-Jenkins approach. This may due to the fact that the Box-Jenkins ARIMA model is simple to construct and applied to many time series processes but it has a shortfall of losing some observations through ordinary and seasonal differencing. Again, one major assumption in modelling with time series data is the invariant or constant variance. However, in real life, variance may change with time. Many time series models assume time - invariant or constant variance by the underlining process. This changing variance with respect to time is called heteroscedasticity, and it shall be more appropriate to accommodate the possible variation in variance in forecasting any time series process.

Campbell et al., (1997) maintained that it is logically inconsistent and statistically inefficient to model and use volatility measures that are based on the assumption of constant variance over some period when the resulting series moves or progress through time. In general, economic and financial data large and small errors occur in clusters, which implies large returns are followed by larger returns and small returns or observation are again followed by yet smaller observation. Actually, according to Akuffo and Ampaw (2013) high inflation are usually followed by further period of high inflation and small inflation period, are followed by much smaller inflation. That is the changing variance in time series process have real implication on forecast power where we assumed constant variance.

A family of models that can take care of the dynamics of conditional heteroscedasticity is the autoregressive conditional heteroscedastic (ARCH) models and its extension as the generalized auto regression conditional heteroscedastic (GARCH) models. Engle (1982) developed and introduced the ARCH models and was later generalized by Bollerslev (1986) as GARCH and have been applied in many processes. In the ARCH model, the dynamics of the conditional heteroscedastic is accounted for by relating the error variance to the previous errors. However, in the case of the GARCH, previous conditional variances are nested in the model.

In practice, the forecast the accuracy of confidence intervals can be greatly affected by the presence of non-constant variance, heterosdasticity, and as such should be adequately taken care of. Two most frequently used tests for heroscedasticity are the Engle's Lagrange multiplier and the portmanteau test statistic. The ARCH-GARCH modelling make use of conditional error variance as a function of the past realization of the data.

Stochastic volatility models, autoregressive conditional heteroscedastic (ARCH) and Generalized ARCH (GARCH) models can be used to capture and model the volatility behaviour of time series data with the phenomena of heteroscedasticity. The ARCH-GARCH models have a demerit of having very little theory available hence they are difficult to construct. However, they have more precision power on prediction than ARIMA and SARIMA models (Chinomona, 2010).

The ARCH-GARCH models have been applied in many areas and proven to be statistically efficient model. Ling and Li (1997) considered fractionally integrated autoregressive moving-average time series models with conditional heteroscedasticity, which combined the popular generalized autoregressive conditional heteroscedastic (GARCH) and the fractional (ARMA) models. Drost and Klaassen (1997) said that it is well-known that financial data sets exhibit conditional hereroskedasticity. GARCH-type models are often used to model this phenomenon. They constructed adaptive and hence efficient estimators in a general GARCH in mean-type context including integrated GARCH models.

2.0 MATERIALS AND METHODS

Notwithstanding the highly acknowledged strengths of ARCH model, its formulation can lead to highly parametric model when there is a large lag *q* under consideration. It is therefore necessary to seek and extension to the ARCH model which can adequately accommodate the possibility of large lag q and this leads us to the development of the GARCH model.

2.2 The GARCH model

Developed by Bollerslev (1986), the generalized *ARCH* model is the extension of the *ARCH* mode just as the autoregressive (AR) model has its extension as the autoregressive moving average *(ARMA)*. Other extensions to the *ARCH* model includes exponential *GARCH (EGARCH)* and the integrated *GARCH (IGARCH)*, which is not the focus of this paper. The main issue with the ARCH as mentioned earlier is the fact that when there is a large lag, the *ARCH* model has the tendency of modelling with many parameters which we seek to avoid in time series modelling theory. It should be noted that a good stochastic or time series model should be parsimony; have fewer number of parameters as possible, which is one of the cardinal principles of a good time series model.

2.2.1 The GARCH (1, 1) model

The *GARCH (1, 1)* depends on both the conditional variance and the previous conditional variance. Let $\{y_t\}$ which is *i. i. d.* Again, let us consider the series at time t $W_t = \{y_1, y_2, y_3, ..., y_{t-1}\}$ as in the ARCH model, then the $\{y_t\}$ is a *GARCH (1, 1)* if a

$$y_t = \sigma_t \varepsilon_t \text{ and } \sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \quad \ldots \quad \textbf{(1)}$$

To obtain positive variance, it is sufficient to have

$\alpha_0 > 0, and\ \alpha_1, \beta_1 \geq 0$ and also $\alpha_1 + \beta_1 < 1$. This is to allow the next period forecast of variance as a blend of our last period forecast and the last period squared return. The *GARCH (1, 1)* can be seen as an *ARMA (1, 1)* model on squared residuals by making some substitution for $\sigma_t^2 = y_t^2 - w_t^2$

Hence

$$\sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

$$y_t^2 - w_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 \left(y_{t-1}^2 - w_{t-1}\right) \quad \ldots \quad \textbf{(2)}$$

$$y_t^2 = \alpha_0 + (\alpha_1 + \beta_1) y_{t-1}^2 + w_t - \beta_1 w_{t-1}$$

Equation [2] is an example of *ARMA (1, 1)* on the squared residuals. Again, the unconditional variance of [2] is

$$Var(y_t) = E[y_t^2] - (E[y_t])^2$$

$$
\begin{aligned}
Var(y_t) &= E\left[y_t^2\right] \\
&= E\left[\sigma_t^2 \varepsilon_t^2\right] \\
&= E\left[\sigma_t^2\right] \\
&= E\left[\alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 \sigma_{t-1}^2\right] \quad \ldots \quad \textbf{(3)} \\
&= \alpha_0 + \alpha_1 E\left[y_{t-1}^2\right] + \beta_1 \sigma_{t-1}^2 \\
&= \alpha_0 + (\alpha_1 + \beta) E\left[y_{t-1}^2\right]
\end{aligned}
$$

Now since y_t is a stationary process then,

$$
Var(y_t) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1} \quad \ldots \quad \textbf{(4)}
$$

Again, the *GARCH (1, 1)* can be written as $ARCH(\infty)$ in a similar form as follows:

$$
\begin{aligned}
\sigma_t^2 &= \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \\
&= \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 \left(\alpha_0 + \alpha_1 y_{t-2}^2 + \beta_1 \sigma_{t-2}^2\right) \\
&= \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 \alpha_0 + \alpha_1 \beta_1 y_{t-2}^2 + \beta_1^2 \left(\alpha_0 + \alpha_1 y_{t-3}^2 + \beta_1 \sigma_{t-3}^2\right) \\
&= \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 \alpha_0 + \alpha_1 \beta_1 y_{t-2}^2 + \beta_1^2 \alpha_0 + \alpha_1 \beta_1^2 y_{t-3}^2 + \beta_1^3 \left(\alpha_0 + \alpha_1 y_{t-4}^2 + \beta_1 \sigma_{t-4}^2\right) \\
&\quad \vdots \quad \ldots \quad \textbf{(5)} \\
&= \frac{\alpha_0}{1 - \beta_1} + \alpha_1 \sum_{i=0}^{\infty} y_{t-1-i}^2 \beta_1^i
\end{aligned}
$$

From the above equation, it can be deduced that the conditional variance at time t is the weighted sum of past squared residuals and the weights decreases as we retrogress in time. From equation [5] the *GARCH (1, 1)* can be written as

$$
\begin{aligned}
\sigma_t^2 &= \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \\
&= (1 - \alpha_0 - \beta_1) E\left[\sigma^2\right] + \sigma_1 y_{t-1}^2 + \beta_1 \sigma_{t-1}^2
\end{aligned} \quad \ldots \quad \textbf{(6)}
$$

(Roger, 2009) Here one can see that the next period's conditional variance is a weighted combination of the unconditional variance $E\left[\sigma^2\right]$, the last period's squared residuals y_{t-1}^2, and the last period conditional variance σ_{t-1}^2, with the weights $(1 - \alpha_1 - \beta_1), \alpha_1, \beta_1$

It is necessary to look at the forecast with the model starting with *GARCH (1, 1)*

For σ_t^2 to derive the forecast for the next period $\hat{\sigma}_{t+1}^2$

$$
\begin{aligned}
\sigma_t^2 &= \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \\
\hat{\sigma}_{t+}^2 &= \alpha_0 + \alpha_1 E\left[y_t^2 \middle| I_{t-1}\right] + \beta_1 \sigma_t^2 \\
&= \alpha_0 + \alpha_1 \sigma_t^2 + \beta_1 \sigma_t^2 \quad \ldots \quad \textbf{(7)} \\
&= \alpha_0 + (\alpha_1 + \beta_1) \sigma_t^2 \\
&= \sigma^2 + (\alpha_1 + \beta_1)\left(\sigma_t^2 - \sigma^2\right)
\end{aligned}
$$

Again,

$$\begin{aligned}\hat{\sigma}_{t+1}^2 &= \alpha_0 + \alpha_1 E\left[y_{t+1}^2 \middle| I_{t-1}\right] + \beta_1 E\left[\sigma_{t+1}^2 \middle| I_{t-1}\right] \\ &= \alpha_0 + (\alpha_1 + \beta_1)\hat{\sigma}_{t+1}^2 \\ &= \sigma^2 + (\alpha_1 + \beta_1)(\sigma_{t+1}^2 - \sigma^2) \\ &= \sigma^2 + (\alpha_1 + \beta_1)^2(\sigma_t^2 - \sigma^2)\end{aligned} \quad \ldots \quad (8)$$

In a similar manner,

$$\begin{aligned}\hat{\sigma}_{t+l}^2 &= \alpha_0 + (\alpha_1 + \beta_1)\hat{\sigma}_{t+l-1}^2 \\ &= \sigma^2 + (\alpha_1 + \beta_1)(\hat{\sigma}_{t+l-1}^2 - \sigma^2) \\ &= \sigma^2 + (\alpha_1 + \beta_1)^l(\sigma_t^2 - \sigma^2) \\ &= \frac{\alpha_0}{1-\alpha_1-\beta} + (\alpha_1 + \beta_1)^l(\sigma_t^2 - \sigma^2)\end{aligned} \quad \ldots \quad (9)$$

From equation [38], it can be noted that $\hat{\sigma}_{t+l}^2 \to \sigma^2$ as $l \to \infty$ so as the forecast horizon goes to infinity, the variance forecast approaches the unconditional variance of y_t. From the l-step ahead variance forecast, we can see that $(\alpha_1 + \beta_1)$ determines how quickly the variance forecast converges to the unconditional variance (Roger, 2009). If the variance spikes up during a crisis, the number of periods, *K*, until it is halfway between the first forecast and the unconditional variance $K = \dfrac{\ln(0.5)}{\ln(\alpha_1 + \beta_1)}$.

2.2.2 Estimation of the GARCH (1, 1) model

Estimation of the parameters of the *GARCH (1, 1)* model is done in the similar manner as in the case of *ARCH (1, 1)*. But since the conditional variance of the *GARCH (1, 1)* model depends also on the past conditional variance, an initial value of the past conditional variance σ^2 is needed. The unconditional variance of y_t can be taken for this variance that is σ_1^2 can be taken to be $\dfrac{\alpha_0}{1-\alpha_1-\beta_1}$ the maximum likelihood estimates are obtained by maximizing the conditional log-likelihood by

$$\ln f\left(y_2, y_3, \ldots, y_t, \sigma_2, \ldots \sigma_t \middle| y_1, \sigma_1^2, \theta\right) = -\frac{1}{2}\sum_{t=2}^{t}\ln\left(2\pi\sigma^2\right) - \frac{1}{2}\sum_{t=2}^{t}\frac{y_t^2}{\sigma_t^2} \quad \ldots \quad (10)$$

Where $\theta = \begin{pmatrix}\alpha_0 \\ \alpha_1 \\ \beta_1\end{pmatrix}$ .the gradient, the Hessian and the optimization procedure are the same as the *ARCH (1)* modelling except that σ_t^2 has a different formulation.

2.2.3 The GARCH (p, q) model

Generalizing the *GARCH (1, 1)* with p as the autoregressive lag and q as the moving average lag give rise to the *GARCH (p, q)* if

$$y_t = \sigma_t \varepsilon_t$$

And

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p}\alpha_i y_{t-i}^2 + \sum_{j=1}^{q}\beta_j \sigma_{t-j}^2 \quad \ldots \quad (11)$$

and

$$\sigma_t^2 = \alpha_0 + \alpha(\mathrm{B})y_t^2 + .\beta(\mathrm{B})\sigma_t^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \textbf{(12)}$$

Where ε_t is a Gaussian white noise as while $\alpha(\mathrm{B})$ and $\beta(\mathrm{B})$ are polynomials in the backshift operator given by

$$\alpha(\mathrm{B}) = \alpha_1 \mathrm{B} + \ldots + \alpha_q \mathrm{B}^{\mathrm{q}} \text{ and } \beta(\mathrm{B}) = \beta_1 \mathrm{B} + \ldots + \beta_q \mathrm{B}^{\mathrm{q}}$$

In order to have the conditional variance remaining positive, we impose the restrictions $\alpha_i \geq 0 \, and \, \beta_j \geq 0 \;\; for\, i = 0, 1, 2, 3, \ldots, q$ and $j = 1, 2, 3, \ldots, p$

It is important to note that *GARCH (0, 1)* model is the same as *ARCH (q)* model and that *p = q = 0* we have a *GARCH (0, 0)* model, which is a white noise (Chinomona, 2009).

Taking a second order stationary process of a *GARCH (p, q)* we have

$$\begin{aligned} Var(y_t) &= E(y_t^2) \\ &= E(\sigma_t^2 \varepsilon_t^2) \\ &= E(\sigma_t^2 E(\varepsilon_t^2 | y_{t-1})) \\ &= E(\alpha_0 + \alpha(\mathrm{B})\mathrm{y}_\mathrm{t}^2 + \beta(\mathrm{B})\sigma_\mathrm{t}^2) \\ &= \alpha_0 + \alpha(\mathrm{B})\mathrm{E}(\mathrm{y}_\mathrm{t}^2) + \beta(\mathrm{B})\mathrm{E}(\sigma_\mathrm{t}^2) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \textbf{(13)} \\ &= \frac{\alpha_0}{1 - \sum_{i=1}^{q} \alpha_i - \sum_{j=1}^{p} \beta_j} \end{aligned}$$

The autocovariance of a *GARCH (1, 1)* model $k \geq 1$ where k is the lag is given by

$$E(y_t y_{t-k}) = 0$$

Since y_t is a martingale difference Gourieroux, et. al (1997). This results shows that the GARCH (*p*, *q*) does not show autocorrelation in the underlining process. It can be shown however that the squared shows autocorrelation. For example considering the difference as $w_t = y_t^2 - \sigma_t^2$, then

$$\begin{aligned} y_t^2 &= \sigma_t^2 + w_t \\ &= \alpha_0 + \sum_{i=1}^{q} \alpha_i y_{t-i}^2 + \sum_{j=1}^{p} \beta_j (y_{t-i}^2 - w_{i-j}) + w_t \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \textbf{(14)} \\ &= \alpha_0 + \sum_{i=1}^{q} \alpha_i y_{t-i}^2 + \sum_{j=1}^{p} \beta_j y_{t-i}^2 - \sum_{j=1}^{p} \beta_j w_{i-j} + w_t \end{aligned}$$

Assuming a maximum order for the process at a discrete order, then we have

$$y_t^2 = \alpha_0 + \sum_{i=1}^{m} \alpha_i y_{t-i}^2 + \sum_{j=1}^{p} \beta_j y_{t-i}^2 - \sum_{j=1}^{p} \beta_j w_{i-j} + w_t \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \textbf{(15)}$$

This can be seen as an ARMA (m, p) for $\alpha_i = 0$ for $i > m$ and $\beta_j = 0$ for $i > p$. To find *GARCH (p, q)* process, we consider solving for α_0 and assume the variance of y_t be σ_ε^2 which yields

$$\alpha_0 = \sigma_\varepsilon^2 \left(1 - \sum_{i=1}^{q} \alpha_i - \sum_{j=1}^{p} \beta_j \right) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \textbf{(16)}$$

By putting the value of α_0 in equation [45] into equation [44] give

$$y_t^2 = \sigma_\varepsilon^2\left[1-\sum_{i=1}^{m}(\alpha_i+\beta_j)\right]+\sum_{i=1}^{m}(\alpha_i+\beta_j)y_{t-i}^2-\sum_{j=1}^{p}\beta_j w_{t-j}+w_t$$

$$=\sigma_\varepsilon^2+\sum_{i=1}^{m}(\alpha_i+\beta_j)(y_{t-i}^2-\sigma_\varepsilon^2)-\sum_{j=1}^{p}\beta_j w_{t-j}+w_t \quad \ldots \quad \ldots \quad \textbf{(17)}$$

Hence

$$y_t^2-\sigma_\varepsilon^2=\sum_{i=1}^{m}(\alpha_i+\beta_j)(y_{t-i}^2-\sigma_\varepsilon^2)-\sum_{j=1}^{p}\beta_j w_{t-j}+w_t \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \textbf{(18)}$$

Taking a multiplier on both sides by $y_{t-k}^2-\sigma_\varepsilon^2$ gives us

$$(y_{t-k}^2-\sigma_\varepsilon^2)(y_t^2-\sigma_\varepsilon^2)=\sum_{i=1}^{m}(\alpha_i+\beta_i)(y_{t-k}^2-\varepsilon_\varepsilon^2)(y_{t-k}^2-\sigma_\varepsilon^2)-\sum\beta_j v_{t-j}(y_{t-k}^2-\sigma_\varepsilon^2)+w_t(y_{t-k}^2-\sigma_\varepsilon^2)$$

Hence

$$E\left[(y_{t-k}^2-\sigma_\varepsilon^2)(y_t^2-\sigma_\varepsilon^2)\right]=E\left[\sum_{i=1}^{m}(\alpha_i+\beta_i)(y_{t-k}^2-\varepsilon_\varepsilon^2)(y_{t-k}^2-\sigma_\varepsilon^2)\right]-E\left[\sum_{j=1}^{p}\beta_j v_{t-j}(y_{t-k}^2-\sigma_\varepsilon^2)+w_t(y_{t-k}^2-\sigma_\varepsilon^2)\right]$$

However,

$$E\left[w_t(y_{t-k}^2-\sigma_\varepsilon^2)\right]=E\left[(y_{t-k}^2-\sigma_\varepsilon^2)E(w_{t-j}|y_{t-k})\right]=0$$

Since w_t is a martingale difference and also

$$E\left[\beta_j w_{t-j}(y_{t-k}^2-\sigma_\varepsilon^2)\right]=E\left[(y_{t-k}^2-\sigma_\varepsilon^2)E(w_{t-j}|y_{t-k})\right]=0$$

For $k<j$. We can say that the autocovariance of the squared returns for the GARCH (*p*, *q*) model is given by

$$Cov(y_t^2,y_{t-k}^2)=E\left[\sum_{i=1}^{m}(\alpha_i+\beta_i)(y_{t-k}^2-\sigma_\varepsilon^2)(y_{t-k}^2-\sigma_\varepsilon^2)\right]$$

$$=\sum_{i=1}^{m}(\alpha_i+\beta_i)Cov(y_t^2,y_{t-k+i}^2). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \textbf{(19)}$$

The Yule-Walker equations for an AR process can derived analogously by dividing [47] by $\operatorname{var}(y_t^2)$, which will results in the autocorrelation function at lag k as

$$\rho_k=\sum_{i=1}^{m}(\alpha_i+\beta_i)\rho_{k-1} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \textbf{(20)}$$

for $k\ge p+1$.

Thus the autocorrelation function (ACF) and the partial ACF (PACF) of the squared process in a GARCH (*p*, *q*) process has the same pattern as those of an ARMA (*m*, *p*) process. Similar to what we have in ARMA modelling, the ACF and the PACF are very important in identifying the orders of p and q of the GARCH process. Again, the ACF are also important for checking model adequacy, in which case, the ACF's residual should be a white noise process should the model be accepted. Hence the first *p* autocorrelations depend on the parameters $\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_p,\beta_1,\beta_1,\beta_1,\ldots,\beta_q$ and $\rho_p,\ldots,\rho_{p+1+m}$ putting it into equation [48] determines uniquely the autocorrelations at higher lags, (Bollerslev, 1986). Hence setting ϕ_{mm} to represent the *m th* partial autocorrelation for y_t^2, then

$$\rho_k=\sum_{i=1}^{m}\phi_{mi}\rho_{k-i},\ k=1,\ldots,m \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \textbf{(21)}$$

According to equation [48] ϕ_{mm} cuts off after lag q for an ARCH (*q*) process such that $\phi_{mm}\ne 0,\ for\ k\le q$ and $\phi_{mm}=0$ for $k=q.$ this is identical to the PACF for an AR (*q*) process and decays exponentially (Bollerslev, 1986). After identifying the orders p and q, we now can estimate the parameters of the GRACH (*p*, *q*) model for forecasting.

2.2.4 Estimating of GARCH (p, q) model

The maximum likelihood estimate can also be used to estimate the parameters of the GARCH (p, q) model. Similar to GARCH (1, 1) model estimation, initial values of both squared returns and past conditional variances are needed in estimating the parameters of the model. As suggested by Bollerslev, (1986) and Tsay, (2002), the unconditional variance given in equation [42] or the past sample variance of the returns for the past variance may be used as initial values. Therefore assuming $y_1, y_2, \ldots, y_q$ and $\sigma_1^2, \ldots, \sigma_p^2$ are known, the conditional maximum like hood estimates can be obtained by maximizing the conditional log-like hood given by

$$l = \ln f\left(y_{q+1}, \ldots, y_t, \sigma_{p+1}^2, \ldots, \sigma_p^2 \middle| \theta, y_1, \ldots, y_q, \sigma_p^2\right)$$

$$= -\frac{1}{2}\sum_{t=m+1}^{T} \ln\left(2\pi\sigma_t^2\right) - \frac{1}{2}\sum_{t=m+1}^{T} \frac{y_t^2}{\sigma_t^2} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \textbf{(22)}$$

With $\theta = \left(\alpha_0, \ldots, \alpha_q, \beta_1 \ldots, \beta_p\right)$ and $m = \max(p, q)$.

2.3 Model Checking

According to Talke (2003) goodness of fit of the ARCH – GARCH model are based on residual and more specifically on the standardized residuals. Normally the residuals are assumed to be independently and identically distributed following either a normal or a standardized *t* – distribution. (Tsay, 2002) and (Gourieroux, 1997). Histograms, normal probability plots and time plot of residuals can be used. If the model fits the data well the histogram of residuals should be approximately symmetric. The normal probability plot should be a straight line while the time plot should exhibit random variation *(http://en.wikipedia.org/wiki/Q - Q - plot).* The ACF and the PACF of the standardized residuals are used for checking the adequacy of the conditional variance model. Again, the Lagrange multiplier and the Ljung Box Q – test are used to check the validity of ARCH effects in the data. Haven established that the model fits the data very well, the fitted model is used to compute forecast as forecasting is the main aim of time series modelling.

2.3.2 Forecasting with GARCH (p, q)

The conditional variance of $\{y_t\}$ in GARCH can be obtained by taking the conditional expectation of the squared mean rates of the process under consideration. Assuming a forecasting origin of T, then 1-step ahead volatility forecast is given by

$$y_t^2(1) = E\left[y_{t+1}^2 \middle| y_t\right]$$

$$= \alpha_0 + \sum_{i=1}^{m}(\alpha_i + \beta_i)E\left(y_{t+1-i}^2 \middle| y_t\right) - \sum_{i=1}^{p}\beta_i\left(w_{t+1-i} \middle| y_t\right) \quad \ldots \quad \ldots \quad \ldots \quad \textbf{(23)}$$

where $y_t^2, \ldots, y_{t+1-m}^2, \sigma_t^2, \ldots, \sigma_{t+1-p}^2$ are assumed known at time t and the true parameter values $y_t^2, \ldots, y_{t+1-m}^2, \sigma_t^2, \ldots, \sigma_{t+1-p}^2$ are assumed known at time t and the true parameters values α_i and β_i for $i = 1, \ldots, m$ are replaced by their estimates. Again, the l – step ahead forecast of the conditional variance in a $GARCH(p, q)$ model can be stated as

$$\sigma_t^2(l) = E\left(y_{t+l}^2 \middle| y_t\right)$$

$$= \alpha_0 + \sum_{i=1}^{m}(\alpha_i + \beta_i)E\left(y_{t+l-i}^2 \middle| y_t\right) - \sum_{i=1}^{p}\beta_i\left(w_{t+l-i} \middle| y_t\right)$$

Where $E\left(y_{t+l-i}^2 \middle| y_t\right)$ for $i < l$ can be given recursively as

$$E\left(y_{t+l-i}^2 \middle| y_t\right) = y_{t+l-i}^2 \quad for\ i \geq l$$

$$E\left(w_{t+l-i} \middle| y_t\right) = 0 \quad for\ i < l \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\textbf{(24)}$$

$$E\left(w_{t+l-i} \middle| y_t\right) = 0 \quad for\ i \geq l.$$

We now proceed to the techniques that are used for selecting the best fitting models in light of several competing models based on the likelihood ratios.

2.5 Model selection criteria and Forecasting Performance

The most common model selection procedures or criteria for deciding on competing *ARCH – GARCH* models are the Akaike Information Criteria (AIC) or the SBC given respectively as:

$$AIC = -2(\log likelihood) + 2(number\ of\ parameters)$$

and

$$AIC = -2(\log likelihood) + 2(number\ of\ parameters)[\log(number\ of\ observations)]$$

Out of several competing models, a desired model is one that minimizes the AIC or the SBC. Another selection consideration is the associated proportion of variability in a data set that is accounted for by the statistical model, R^2. It must be noted that the major limitation of the R^2 is that fact that a model that can pick out the trend reasonably well have an R^2 almost as a unit. Hence, in this study the selection of the best model is done with the AIC and SBC alongside the stated R^2.

One of the criteria for selecting a best time series model can also be the best forecasting model among competing models. In *ARCH – GARCH* models, among the several measures for assessing the predictive accuracy is the mean square error (MSE). The MSE is defined as the average of the squared difference between the actual variance and the volatility forecast denoted by σ_t^2. However, if one have the observed true variance, then the squared time series observation y_t^2 is used. The MSE is given by

$$MSE = \frac{1}{T}\sum\left(y_t^2 - \hat{\sigma}_t^2\right)^2 \quad \text{... ... ... } \textbf{(25)}$$

Where $\hat{\sigma}_t^2$ for $t = 1,..,T$ is the estimated conditional variance obtained from fitting ARCH – GARCH model.

One limitation of the MSE is that although the squared time series observation, y_t^2 is a consistent estimator of σ_t^2, it is noisy and hence unstable (Tsay, 2002).

Lopez (1999) suggested an alternative measures as the mean absolute error (MAE) and the MSE of the log of the squared error (MSEL). That is

$$MAE = \frac{1}{T}\sum_{t=1}^{T}\left|y_t^2 - \hat{\sigma}_t^2\right| \quad \text{... ... ...} \textbf{(26)}$$

and

$$MSEL = \frac{1}{T}\sum_{t=1}^{T}\left(\ln\left(y_t^2\right) - \ln\left(\hat{\sigma}_t^2\right)\right)^2 \quad \text{... ... ...} \textbf{(27)}$$

The advantage of the MSE of the log of the square error is that it penalizes inaccurate variance forecasts more heavily when the squared innovations y_t^2 is low.

3. RESULTS AND DISCUSSION

3.1 Preliminary analysis

3.1.1 Descriptive statistics

The trust of ARCH – GARCH is the ability of modeling with a series or a process with varying volatility. In particular, attention is paid to the behaviour of the mean and standard deviation hence variance of the data. Table 1 shows the yearly average of interest rate in Ghana from 2003 to 2013.

Table 1: Yearly statistic on Ghana's interest rates (2003 – 2013)

Year	Minimum	Maximum	Mean	Standard Deviation
2003	16.30	30.00	26.93	3.97
2004	10.50	22.40	12.74	3.12
2005	11.60	16.70	15.08	1.37
2006	9.50	14.60	10.96	1.34
2007	10.10	12.75	10.72	0.75
2008	12.81	18.41	16.46	2.11
2009	10.06	20.74	17.62	3.79
2010	8.58	14.78	10.79	2.17
2011	8.39	9.16	8.73	0.30
2012	8.60	9.50	9.13	0.33
2013	8.80	13.79	11.61	1.55
Overall period	8.39	30.00	13.71	5.47

According to Table 1 there is evidence of varying mean and standard deviation. Comparing the overall mean of 13.71 with the minimum mean over the period of 8.39, it can be seen that the mean interest rate over the period under study has not be stable. Again, the wide deviation of the overall standard deviation and the minimum standard deviation over the period hence the wide variance give credence to the application of a model that has the ability to capture these variations.

3.1.2 Time plots

Figure 1 shows the graphical representation of the varying mean and standard deviation hence the non-constant variance.

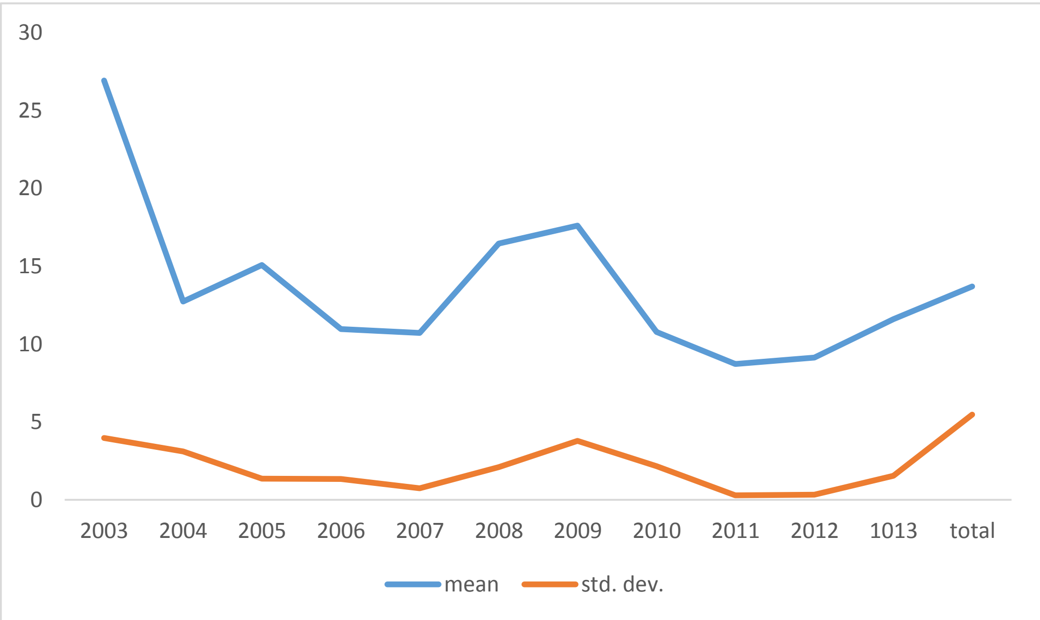


Fig. 1: Yearly mean and standard deviation of interest rates

From Figure 1, the non-constant nature of the mean of interest rate in Ghana for the period under review implies that that interest rate in Ghana for the last decade has not be stable and its prediction can be misleading. In such cases if the right model is not chosen, it can lead to spurious results.

The time plot of the yearly mean and standard deviation hence, variance, as shown in Figure 1, together with the plot of monthly interest rates, according to Figure 2, show that there is changing mean and variance in interest rates over the last decade in Ghana. That is there are evidence of heteroscedasticity in the process.

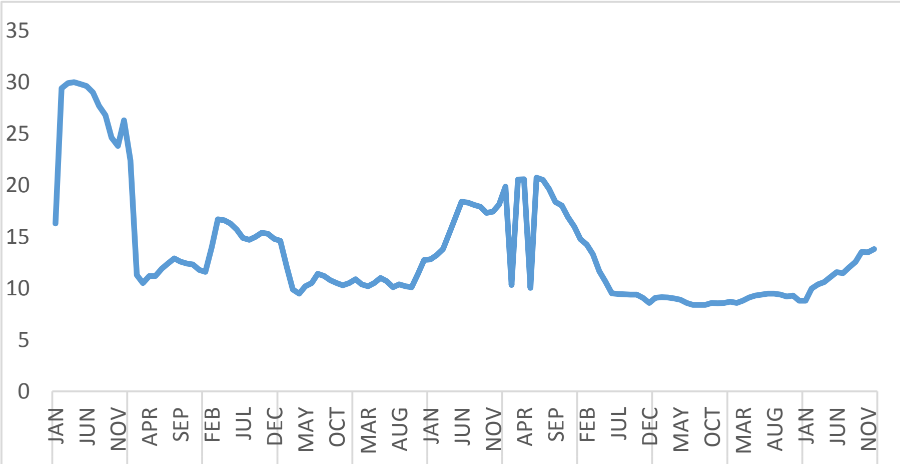


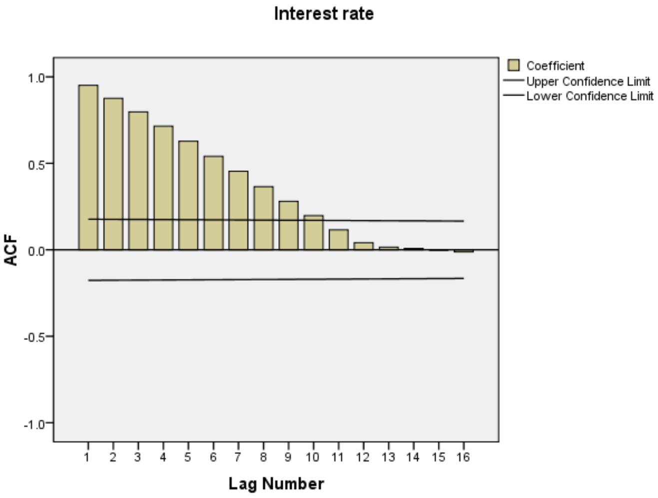
Fig. 2: The time plot of interest rate from January 2003 to December 2013.

Since GARCH modelling accommodates heteroscedasticity, there is no need to transform the data as required in ARIMA processes as a pre-requisite for stationarity. Therefore the presence of heteroscedasticity will further validate the need to use GARCH modelling for the process. Heteroscedasticity is therefore tested formally and the results is as shown in Table 1 showing the Q – test and the Lagrange Multiplier (LM) with the corresponding p – values.

Table 2: Q and LM Tests for disturbances

Lag	Q - value	P > Q	LM	P > LM
1	135.1932	< *0.0001*	126.7402	< *0.0001*
2	213.5320	< *0.0001*	122.5927	< *0.0001*
3	331.0599	< *0.0001*	142.7845	< *0.0001*
4	365.4297	< *0.0001*	142.918	< *0.0001*
5	406.9465	< *0.0001*	143.0367	< *0.0001*
6	414.9814	< *0.0001*	143.0765	< *0.0001*
7	407.2322	< *0.0001*	143.6259	< *0.0001*
8	417.7228	< *0.0001*	143.7114	< *0.0001*
9	417.8562	< *0.0001*	143.7194	< *0.0001*
10	417.9516	< *0.0001*	143.72.12	< *0.0001*
11	418.1016	< *0.0001*	143.7317	< *0.0001*
12	418.5256	< *0.0001*	143.8524	< *0.0001*

From Table 1 it can be seen that all the p – values Q and LM are very small, less than 0.001 at the various lags. Therefore, the errors in the regression model exhibit conditional heteroscedasticity for the interest rate process in Ghana from 2003:01 - 2013:12. It is imperative in time series modelling that before constructing any ARCH – GARCH model, any autocorrelations in the series have to be removed. By regressing the process y_t on the squared of its past observations, $y_{t-1}^2,\ y_{t-2}^2,..$. The resulting autocorrelation (ACF) and partial autocorrelations (PACF) is as shown in Figure 1.



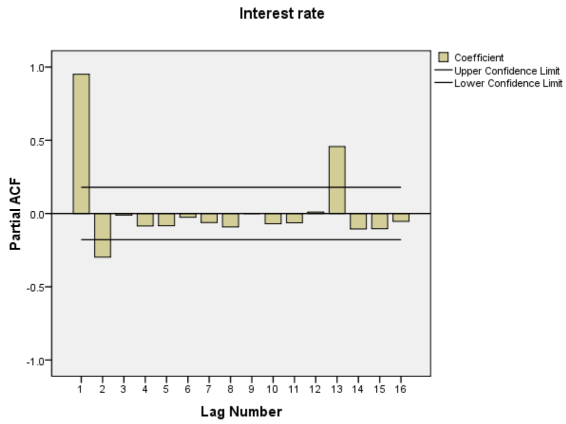


Fig. 3: ACF and PACF

From Figure 3, the spikes in the ACF suggest that there are three AR components to be estimated. This however, depends in the significance of the parameters to be estimated. Hence, an AR (3) model was fitted in order to remove autocorrelations present in the original process.

3.2 Selection of the best fitting GARCH model

Haven removed all autocorrelations, we proceed to select the best fitting ARCH – GARCH model for Ghana's interest rate based on the AIC, SBS, R^2, and the MSE. Again the significance of each estimated parameter was used in the selection of the best fitting model. Several procedure were run tested for models with both intercept and one without intercept. It was revealed that although the model without intercept recorded higher R^2 values, the associated higher values of the other selection criteria estimate makes the model without intercept unfit for our aim. Hence the results of the model with intercepts is as presented in Table 2.

Table 3: Suggested GARCH models with intercepts

Model	AIC	SBC	MSE	R^2
GARCH (0, 1)	135.1932	147.874	0.487	0.8623
GARCH (0, 2)	213.5320	221.003	0.572	0.8921
GARCH (1, 0)	331.0599	338.245	0.397	0.946
GARCH (1, 1)	365.4297	372.980	0.384	0.9532
GARCH (1, 2)	406.9465	412.428	0.497	0.9567
GARCH (1, 3)	414.9814	419.394	0.402	0.9496
GARCH (2, 0)	407.2322	416.035	0.539	0.9473
GARCH (2, 1)	417.7228	421.439	0.414	0.9500
GARCH (2, 2)	417.8562	428.643	0.507	0.9601
GARCH (2, 3)	418.935	430.945	0.476	0.9533

From Table 3, it can be seen that the best fitting model of the generalized autoregressive conditional heteroscedasticity is the GARCH (1, 2). This is because the GARCH (1, 2) model with intercept comes with the minimum measures of AIC, SBC as well as the MSE. Again, the GARCH (1, 2) was seen to have the highest R^2 among the competing models. Therefore, the GARCH (1, 2) model with intercept was chosen for the forecasting of interest rate in Ghana for the period 2003:01 to 2013:12. That is interest rate in Ghana from 2003:01 – 2013:12 can be modelled with a GARCH as

$$y_t = \sigma_t \varepsilon_t$$

Where

$$\sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \beta_1 \sigma_{t-1}^2 + \beta_2 \sigma_{t-2}^2 \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots\textbf{(28)}$$

3.3 Estimation of the parameters of GARCH (1, 2)

The maximum likelihood estimation was use to estimate the parameters of the identified model. The results of the estimation is as presented in Table 4.

Table 4 shows the parameter estimates of the ARCH model.

Table 4: Parameter estimate for GARCH (1, 2)

Variable	df	estimate	approx. error	t - value	$P > \|t\|$
intercept	1	0.4810	0.3910	1.23	0.2187
AR (1)	1	-1.1241	0.1729	-6.50	<0.0001
AR (2)	1	0.6985	0.1167	5.98	<0.0001
α_0	1	0.294	0.095	3.95	<0.0001
α_1	1	0.063	0.102	5.39	<0.0001
β_1	1	0.362	0.075	7.23	<0.0001
β_2	1	0.381	0.125	4.06	<0.0001

It can be seen from Table 4 that all the parameter estimates are statistically significant with the standard error very small but for the constant term for the AR. Hence we have:

$$\sigma_t^2 = 0.294 + 0.063 y_{t-1}^2 + 0.362\sigma_{t-1}^2 + 0.381\sigma_{t-2}^2 . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \textbf{(29)}$$

3.4 Model checking

We consider the residuals from the fitted model to analyze how our chosen model, GARCH (1, 2) fit the process, interest rate in Ghana from January 2003 to December 2013.

If the model fits the process very well then the residuals are expected to be random, independent and identically distributed and the ACF and PACF are to be in control. Figure 4 shows the residual of the ACF and PACF of the ARCH - GARCH (1, 2).

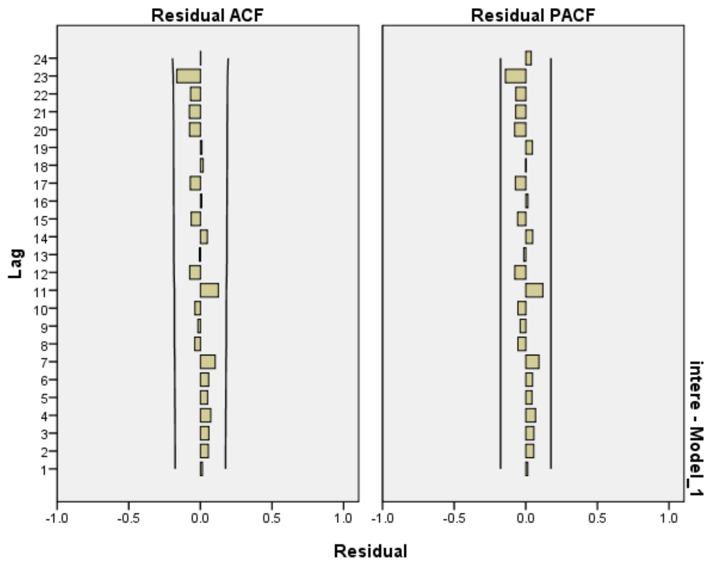


Fig. 4: ACF and PACF of residuals

From the time plot of the residuals in Figure 4, it can be seen that residuals are randomly distributed as expected. The probability time plot of the residuals is as presented in Figure 5. The probability time plot of the model shows that the residual form almost a straight line suggesting that that the residuals follow an approximately a normal distribution.

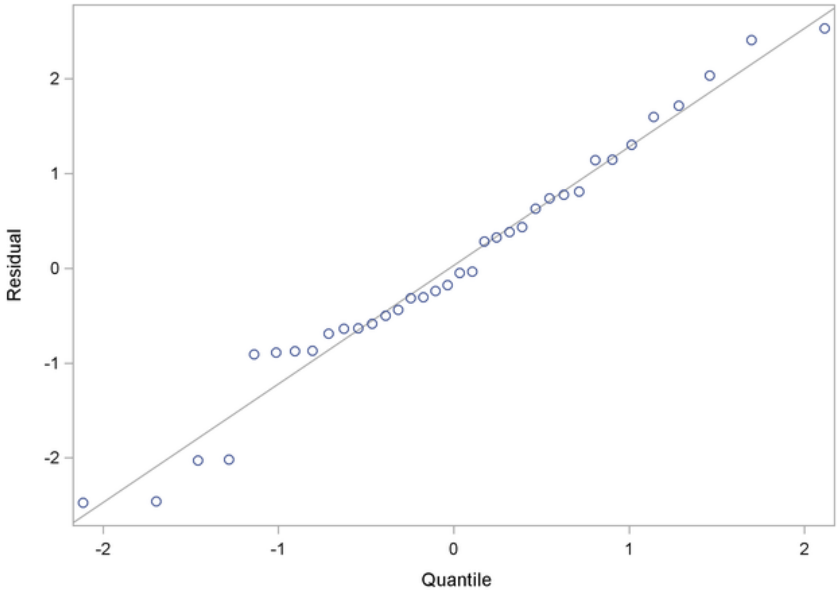


Fig. 5: Plot of residuals from GARCH (1, 2)

The histogram of the residual is as presented in Figure 6.
The bell-shaped distribution of histogram of the residual presented in Figure 4 gives an indication that the residuals of the fitted model follows a normal distribution.

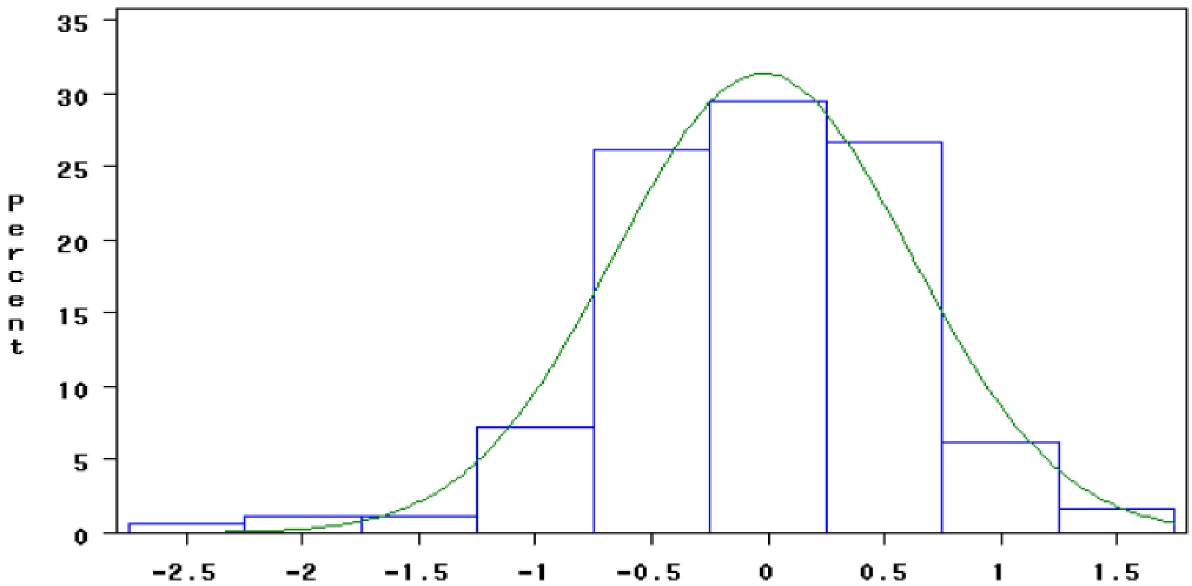


Fig. 6: Histogram of residuals from ARCH – GARCH (1, 2)

Haven satisfied the diagnostics check we proceed in using the model for forecasting.

3.5 Forecasting interest rate with GARCH (1, 2) model

Forecasting is the principal objective of any time series analysis (Akuffo and Ampaw, 2013). Hence haven fitted the model and haven pass the necessary diagnostic tests, we expect this model to give a very good forecast. From Table 4 it can be seen that the GARCH (1, 2) provides fascinating forecast for the year 2014 with the GARCH (1, 2) model. This is use to forecast interest rate for the year 2014. Table 5 gives a one year forecast of interest rates from 2014:01 – 2014:12 in Ghana.

Table 5: Forecast of interest rate with the GARCH (1, 2) for 2014.

Date	Forecast (%)	Actual (%)	Standard Error	95% Lower CI	95% Upper CI
January 2014	17.65	16.31	1.63	10.78	24.96
February 2014	18.83	17.43	2.19	10.22	25.38
March 2014	19.69	18.07	3.07	9.76	26.05
April 2014	19.93	20.72	3.17	9.07	26.94
May 2014	22.22	N/A	2.11	8.65	27.44
June 2014	24.75	N/A	3.00	7.32	28.83
July 2014	25.09	N/A	4.12	5.03	29.49
August 2014	25.95	N/A	4.13	4.72	30.55
September 2014	26.36	N/A	5.047	3.12	31.54
October 2014	27.09	N/A	5.89	3.01	32.89
November 2014	28.65	N/A	6.33	1.03	33.85
December 2014	29.97	N/A	6.24	-0.34	34.86

The nature of the narrowness of the 95% confidence, with larger interval in future time show gives an indication of the high predictive power of the GARCH (1, 2) model.

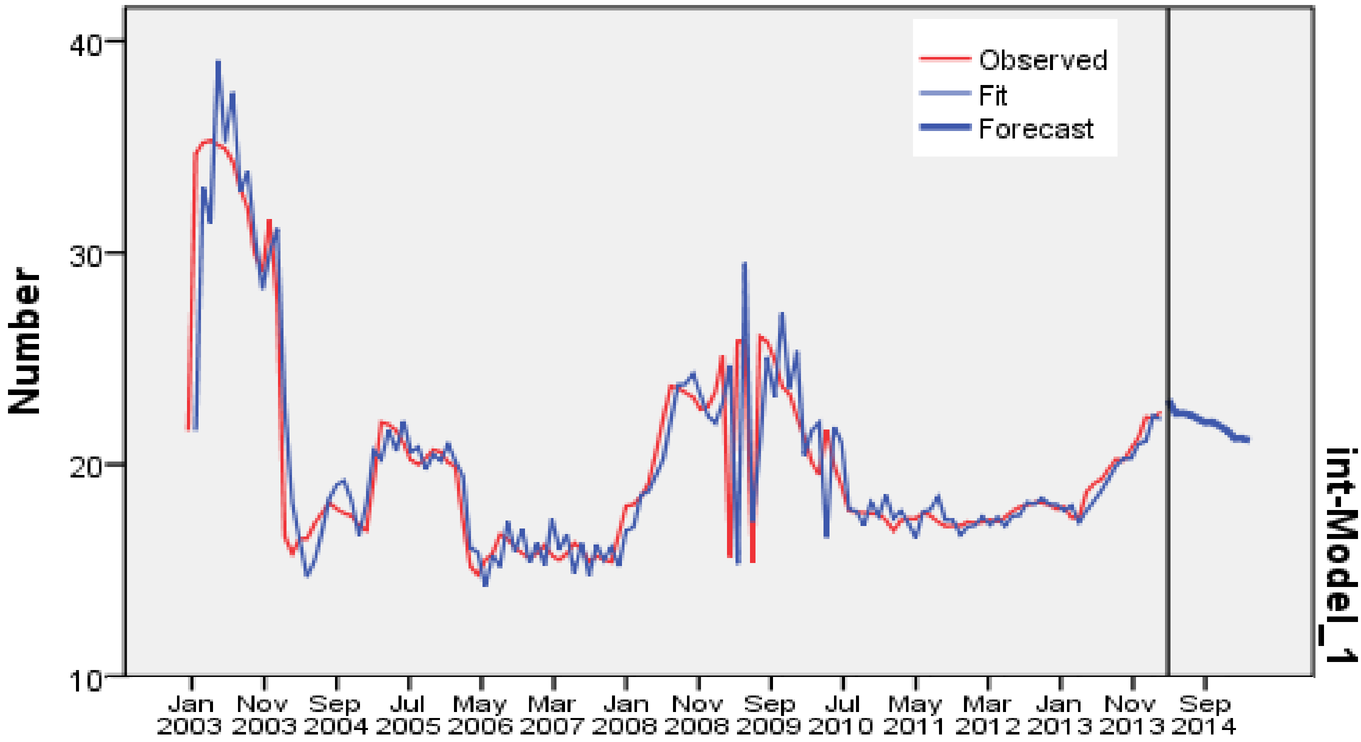


Fig. 7: Monthly forecast of interest rate using the GARCH model for 2014

4. CONCLUSION

The theory of GARCH modeling has been fully explored and applied to Ghana's monthly interest rate data from 2003 to 2013. Unlike ARIMA models, whereby the process need to be transformed to achieve stationarity, if they process was found to be no-stationary at levels, GARCH model has proven superior since the non – stationarity was taken care of. This is due to the fact that the transformation of data makes the respective model rely on rigid assumptions resulting in GARCH mode being superior. From the forecast produced, it can be seen that the GARCH model fits the data well. The closeness of the confidence interval estimated, and the low standard errors registered provide ample evidence of model fitness. With the various argument and discussion on the economy, the study has shown that all things being equal, interest rates in Ghana will be less than 30% by the close of year, 2014. Further studies can be considered as an extensions and improvements to the GARCH models. These may include integrated GARCH, (IGARCH), and the exponential GARCH (EGARCH). The methodology may also be extended to include data which is not necessarily Gaussian to cater for other time series in the form of counts.

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