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Stochastic Modeling of Mortality Risks in Nigeria Using Lee-Carter and Renshaw-Haberman Model

Bamidele, M.A* & Adejumo A.O
Department of Statistics, University of Ilorin, Ilorin, Nigeria
*E-mail of the corresponding author: aodejumo@gmail.com; bamidelemoyo@gmail.com

Abstract

The reductions in mortality rates experienced during the last decades and the resulting increases in life expectancy show that longevity risk, arising from unexpected changes in mortality, cannot be ignored. The study therefore explained the mortality improvements for males' aged 40-65 using Nigeria available data using two stochastic mortality models- Lee Carter Model (M1) and Renshaw-Haberman model (M2). The fitting methodology was applied to the model using the Poisson model; the calibration was done using Life metrics R-code software.

The Lee-Carter class of models allows for greater flexibility in the age effects. On the BIC ranking criterion, the model M2 for the data dominates. However, if we take into account the robustness of the parameter estimates, then model M1 is preferred for the dataset. This model fits the dataset well, and the stability of the parameter estimates over time enables one to place some degree of trust in its projections of mortality rates. The model also shows, for the dataset, that there have been approximately linear improvements over time in mortality rates at all ages.

Keywords: Mortality, Stochastic, Modelling

1. Introduction

The study of mortality relates to the survival and death of individuals within a particular population. The future development of human mortality, together with its wider implications, has attracted increasing interest in recent decades. The historical rise in life expectancy, experienced in the latter half of the twentieth century, shows little sign of slowing. Epidemiological factors seem to have contributed substantially to the increase in life expectancy through prevention of diseases as an important cause of mortality at younger ages.

Historically, the study of human mortality can be seen to fall within the domain of demography and actuarial science, but is increasingly embracing biology, sociology, medicine and finance, thus becoming really interdisciplinary subject. Broadly speaking, we can identify two main approaches to analysing mortality: statistical and biological.

However, several studies have found that official mortality projections in low mortality countries have underestimated the decline in mortality and the gain in life expectancy when compared to the realized outcomes (Keilman, 1997; Boongaarts and Bulatao, 2000; and Lee and Miller, 2001).

1.1 Problem Statement

Given that the future mortality is actually on known, there is a likelihood that the future death rates will turn out to be different from what is being projected, and so for better assessment of mortality and longevity risks would be one that consists of both a mean estimate and a measure of uncertainty. Since the early 1990s a number of

stochastic models have been developed to analyze these mortality improvements, but most of the proposed models exhibit some limitation. This assessment can therefore be performed using stochastic model to describe both demographic and financial risks.

1.2 Study Aim and Objectives

The aim of the study is to explain mortality improvements using stochastic modelling. The specify objectives of this study therefore is to apply Lee Carter Model (M1) and Renshaw-Haberman (M2) to explain male mortality improvement and compare the performance of the model

2. Literature Review

Mortality modelling and forecasting has made considerable progress in the last few years. On the one hand, various stochastic mortality models have been proposed that allow demographers and actuaries to quantify the uncertainty associated with long-run mortality forecasts. Life insurance and annuities are products designed to manage financial uncertainty related to how long an individual will survive. Hence, the lifetime random variable X and its associated mortality model are the basic building blocks in actuarial mathematics.

2.1 Mortality Rates and Survival Probability

In a dynamic context, the changes in mortality are analysed as a function of both age x and time t. In addition, we can consider birth cohorts, that is, people born in a given year. Cohorts are indexed by $c = t - x$, and the development of cohort-specific mortality can be traced through time. Calendar year t is defined as the time period running from t to $t+1$.

Let $m(t,x)$ be the crude or actual death rate for age x in calendar year t,

$$m(t.x)\,\frac{D(t,x)}{E(t,x)} = \frac{\text{number of death during calender } t \text{ aged } x}{\text{average population during calender year } t \text{ aged } x}$$

Where:

T(t, x) denotes the length of the remaining lifetime of an individual aged exactly x at time t. The average population, or the exposure, is usually based on the estimate of the population aged x last birthday in the middle of a calendar year, or on the average of population estimates at the beginning and end of a year (Cairns et al., 2008a).

Using a population estimate at the middle of a year as an approximation to the exposure to mortality risk over that year, we implicitly assume that the population changes uniformly over the year. Therefore, an individual with a (random) remaining lifetime $T(t, x)$ will thus die at age $x + T(t, x)$ in year $t + T(t, x)$. We can construct a model of survival based on $T(t, x)$ by assuming it follows some probability distribution. From the above, surviving to time t is equivalent to attaining an age of $x + t$. The survival function can alternatively be thought as the proportion of a given reference population cohort (aged x at time 0) that are expected to be alive at some future time t. We can also consider the probability, $F(t, x)$, that and individual will not survive to time t, that is, dies before reaching age $x + t$. Clearly,

$$S(t,x) = 1 - F(t,x),$$

where $$S, F : (\mathbb{R} \cup \{0\}) \times (\mathbb{R} \cup \{0\}) \to [0,1]$$

as survival probabilities. We assume the following will also hold $\lim_{n\to\infty} s(t,x) = 1 - \lim_{n\to\infty} F(t,x)$, for each fixed *x*. This simply means that the survival will cease eventually (you cannot live forever).

In some situations it is convenient to define some upper age limit for the population considered. This limit value, which is a characteristic of the population, is usually denoted by $\omega < \infty$ (Atkinson and Dickson, 2000). For humans, for example, this limit could be taken to be on the range of 120 to 130 years, given current observations of the highest attained ages.

Definition 2.1 *The* mortality rate $q(t, x) = \mathbf{P}[T(t, x) \leq 1]$ *is the probability that an individual aged exactly x at time t will die before reaching age x* +1.

In other words, mortality rate is the probability of dying between *t* and *t* + 1.
We can also consider the complement of mortality rate, namely the (one year) survival probabilities.

Definition 2.2 $p(t, x) = \mathbf{P}[T(t, x) > 1]$ *is the probability that an individual aged exactly x at time t will survive to age x* +1.

Therefore, $p(t, x) = 1 - q(t, x)$. We introduce one more mortality related measure, the force of mortality $\mu(t,x)$ (or the hazard rate). This is the instantaneous death rate for individuals aged *x* at time *t*. This means that for small intervals of time, Δt , the probability of death between *t* and $t + \Delta t$ is approximately $\mu(t, x)\,\Delta t$.

Definition 2.5 *The* force of mortality, $\mu(t, x)$, *is defined as*

$$\mu(t,x) = \lim_{n\to\infty} \frac{\mathrm{P}(x < T(t-x,0) \leq x + \Delta t | T(t-x,0) > 0}{\Delta t}$$

That its,

$$\mu(t, x+t) = \lim_{\Delta t \to 0} \frac{F(t+\Delta t, x)F(t,x)}{1 - F(t,x)}$$

$$= \frac{1}{1 - F(t,)} \frac{\partial F(t,x)}{\partial t}$$

$$= \frac{1}{S(t,x)} \frac{\partial}{\partial t} S(t,x)$$

Using equation above, we can express the survival function in terms of the force of mortality as follows:

$$S(t,x) = exp\left(-\int_0^t \mu(s,x+s)ds\right)$$

where we have used the assumption that $S(0, x) = 1 \ \forall \ x$. The relationship between mortality rate and force of mortality can be derived analogously, and is expressed as

$$q(t,x) = 1 - \exp\left[-\int_t^{t+1} \mu(s,x-t+s)ds\right]$$

2.2 Stochastic Mortality Models

We shall therefore review some of the stochastic mortality models

2.2.1 The Lee-Carter Model

Lee and Carter (1992), in their classic paper, proposed the following model for the dynamics of the force of mortality (or the central death rates, $m(t, x)$, depending on the exact specification) to describe the age-pattern of mortality:

$$\log \mu(t,x) = \alpha_x + \beta_x k(t) + e_{t,x}$$

where α_x represents an average log mortality rate over time at age x, whereas β_x represents the improvement rate at age x. k (t) describes the variations in the level of mortality over time, i.e., the random period effect.

The parameters can be estimated from observed mortality data under suitable identifiability constraints, and forecasts can be obtained by specifying a time series model for k (t). Lee and Carter modelled the parameter k (t) as a random walk with drift:

$$k(t+1) = k(t) + \mu + Z(t)$$

where μ is a constant drift parameter and $Z(t)$ a stochastic innovation (noise), with $\{Z(t)_{t\geq 0}\}$ independent and identically distributed.

The time series approach to modelling the pattern of mortality adopted by Lee and Carter has been very influential in the field of mortality forecasts. The notable conclusion from the work of Lee and Carter (1992) was that the time series process could be adequately described by a simple model, such as one-dimensional random walk (Stallard, 2006). However, the model's description of mortality as a function of a single time index is problematic in practice, since it implies that changes in the mortality curve in all ages are perfectly correlated. Apart from this, the model has other drawbacks, as pointed out e.g. by Cairns, Blake and Dowd (2008). In any case, the method by Lee and Carter can be seen as an important step in the introduction and wider acceptance of formal statistical methods to modelling mortality in a dynamic context. The strengths of the model are its simplicity and robustness in the context of linear trends in age-specific death rates, and it has been widely

applied in practice (see, e.g., Lee and Tuljapurkar, 1994; Li, Lee and Tuljapurkar, 2004).

2.2.2 The Renshaw-Haberman Model

Renshaw and Haberman (2006) extended the Lee-Carter model by adding a second age-specific factor to the model:

$$log\mu(t,x) = \alpha_x + \beta_x^1 k_1(t) + \beta_x^2 k_{2(t)} + \beta_x^3 \gamma_{t-x}$$

where k_1, k_2 are the period-related factors, assumed to follow an appropriate stochastic process (e.g., bivariate random walk with drift).

3. Research Methods

3.1 Calibration of the Mortality Models

We specify two models that are applied to Nigeria mortality data and used to model the development of mortality. The first one selected for the study is the Lee-Carter model (1993), while the second one is Renshaw-Haberman model (2006). We denote these models by M1 and M2, respectively.

3.2 Description of Data

We use data for Nigeria males, obtained from National Population Commission. We use data covering years from 1994 to 2003 and age group from 40 to 65 in estimating the parameters for our models.

The dataset was fit to two models described in Cairns et al (2007), namely:

- M1- Lee-Carter model (1992)

$$\log \mu(t,x) = \alpha_x + \beta_x k(t) + e_{t,x}$$

where α_x represents an average log mortality rate over time at age *x*, whereas β_x represents the improvement rate at age *x*. k (*t*) describes the variations in the level of mortality over time, i.e., the random period effect.

- M2- Renshaw-Haberman model (2006), extended the Lee-Carter model by adding a second age-specific factor to the model:

$$log\mu(t,x) = \alpha_x + \beta_x^1 k_1(t) + \beta_x^2 k_{2(t)} + \beta_x^3 \gamma_{t-x}$$

where k_1, k_2 are the period-related factors, assumed to follow an appropriate stochastic process.

3.3 Calibration

Brouhns et al (2002) described a fitting methodology for the Lee-Carter model based on a Poisson model. The main advantage of this is that it accounts for heteroskedasticity of the mortality data for different ages. This method has been used more commonly after that, also for other models. Therefore, the number of deaths is

modelled using the Poisson model, implying:

$$\mathbf{D_{x,t} \sim Poisson\ (E_{x,t}, M_{x,t})}$$

where *Dx,t* is the number of deaths, *Ex,t* is the exposure and *mx,t* in the proposed model. The parameter set φ is fitted with maximum likelihood estimation, where the log-likelihood function of the Poisson model is given by:

$$L(\emptyset; D, E) = \sum_{x,t} \{D_{x,t}\, In[E_{x,t} m_{x,t}(\emptyset)] - E_{x,t} m_{x,t}(\emptyset) - In(D_{x,t}!)\}$$

Based on the implementation of Poisson regression with constraints, we therefore used the R-code of the (free) software package "Lifemetrics" as a basis for fitting (calibration) and stimulation.

3.4 Model Comparison

To evaluate whether the model fits the dataset well, the model was fitted dataset (The data would consists of numbers of deaths *Dx,t* and the corresponding exposures *Ex,t* by year*)* and compared the fitting results. The model fit was compared using the Bayesian Information Criterion measure (BIC). The BIC measure provides a trade-off between fit quality and parsimony of the model and it is defined as:

$$BIC = L(\hat{\phi}) - \frac{1}{2} K In(P)$$

where L(φˆ) is the log-likelihood of the estimated parameter φˆ, P is the number of observations and K is the number of parameters being estimated.

4. Data Analysis and Result of Model Calibration

4.1 Parameter Estimates

In Figures 1–2, we have plotted the maximum-likelihood estimates for the various parameters in the models, using Nigeria males' data, aged 40-65. In this section we will focus on the parameter.

The fitted parameters α_x, β_x and κ_t for Nigeria males are given in figure 1. The figure shows that the pattern of the important parameter α_x and κ_t are well behaved. The patterns of the other parameter all reveal some autoregressive behaviour. Since the factor α_x and κ_t drives a significant part of the uncertainty in mortality rates, its relatively regular behaviour (for this particular dataset) will also show in the projected uncertainty (in other words, the confidence intervals will be relatively narrow). The Model 2 incorporates a cohort parameter; we can see a distinctive cohort effect. The figure shows that the pattern of the important parameter α_x is well behaved. The patterns of the other parameter all reveal some autoregressive behaviour.

4.2 Model Selection Criteria

In this section, we conduct formal model comparisons based on Nigeria data. For each model, we estimate (as appropriate) the $\alpha_x^{(i)}$, $\beta_x^{(i)}$, $\kappa_t^{(i)}$ and $\gamma_c^{(i)}$ for each factor, *i*, age, *x*, year, *t*, and cohort, *c= t- x* by maximizing the log-likelihood function. Estimates of the $\alpha_x^{(i)}$, $\beta_x^{(i)}$, $\kappa_t^{(i)}$ and $\gamma_c^{(i)}$ are plotted in figure 1-2. Values for the maximum likelihood, effective number of parameters (or degrees of freedom in estimation), and the Bayes Information Criterion (BIC) for each model are given in Table below. If one simply compares the maximum likelihoods attained by each model, then it is natural for models with more parameters to fit the data ''better.'' Such

improvements are almost guaranteed if models are nested: if one model is a special case of another, then the model with more parameters will typically have a higher maximum likelihood, even if the true model is the model with fewer parameters.

To avoid this problem, we need to penalize models that are over-parameterized. Specifically, for each parameter that we add to a model, we need to see a ‘‘significant’’ improvement in the maximum likelihood rather than just an increase of any size. A number of such penalties have been proposed. Here we focus on the Bayes Information Criterion (BIC; see, e.g., Hayashi 2000; Cairns 2000). A key point about the use of the BIC is that it provides us with a mechanism for striking a balance between quality of fit (which can be improved by adding in more parameters) and parsimony. The table 1 shows that Renshaw-Haberman model (M2) fits better.

4.3 Model Robustness

An important property of a model is the robustness of its parameter estimates relative to changes in the period of data used to fit a given model. The plots (figure 1-2) reveal that, out of the two models, M1 seems to be the most robust relative to changes in the period of data used: that is, the parameter estimates hardly change even when we use a much shorter data period. M2, on the other hand, seems to produce results that lack robustness, because the parameter estimates jump to a qualitatively quite different solution when we use less data.

5. Conclusion

We have attempted to explain mortality improvements for males aged 40-65 using Nigeria available historical data using two of the stochastic mortality models. The models have different strengths. The Lee-Carter class of models allows for greater flexibility in the age effects. On the BIC ranking criterion, then model M2 for the data dominates. However, if we take into account the robustness of the parameter estimates, then model M1 is preferred for the dataset.

This model fits the dataset well, and the stability of the parameter estimates over time enables one to place some degree of trust in its projections of mortality rates. The lack of robustness in the other models means that we cannot wholly rely on projections produced by them. The model also shows, for the dataset, that there have been approximately linear improvements over time in mortality rates at all ages, but that the improvements have been greater at lower ages than at higher ages.

In further study, we would look at forecasting Nigeria data using Lee-Carter model using uneven data interval approach (Li, Lee and Tuljapurkar, 2004).

Reference

Atkinson, M.E., and Dickson D.C.M., (2000), “An Introduction to Actuarial Studies”. Edward Elgard Publishing Limited, Cheltenham, UK

Boongaarts, J., and Bulatao, R.A. (Eds.), (2000), “Beyond six billion”, Panel on Population Projections, National Research Council, National Academy Press, Washington D.C

Brouhns, N., Denuit, M., and Vermunt, J.K., (2002), “A Poisson log-bilinear regression approach to the construction of projected life tables”, *Insurance: Mathematics and Economics*, 31(3): 373-393

Cairns, A.J.G., (2000), "A discussion of parameter and model uncertainty in insurance", *Insurance: Mathematics and Economics*, 27: 313–330

Cairns, A.J.G., Blake, D., and Dowd, K., (2006a), "Pricing death: Frameworks for the valuation and securitization of mortality risk", *ASTIN Bulletin*, 36: 79-120

Cairns, A.J.G., Blake D., and Dowd, K., (2006b), "A Two-Factor Model for Stochastic Mortality with Parameter Uncertainty", *The Journal of Risk and Insurance*, 73(4): 687-718

Cairns, A.J.G., Blake, D., Dowd, K., Coughlan, G.D., Epstein, D., Ong, A., and Balevich, I., (2007), "A quantitative comparison of stochastic mortality models using data from England & Wales and the United States", Working Paper, Heriot-Watt University, and Pensions Institute Discussion Paper PI- 0701

Cairns, A.J.G., Blake D., Dowd, K., (2008), "Modelling and Management of Mortality Risk: A Review", Pensions Institute Discussion Paper PI-0814

Hayashi, F. (2000), *Econometrics*, Princeton: Princeton University Press.

Renshaw, A.E., and Haberman, S., (2006), "A cohort-based extension to the Lee-Carter model for mortality reduction factors", *Insurance: Mathematics and Economics*, 38: 556–570

Renshaw, A.E., and Haberman, S., (2003c), "Lee-Carter mortality forecasting with age-specific enhancement", *Insurance: Mathematics and Economics*, 33(2), 255–272

Kabbaj, F., and Coughlan, G., (2007), "Managing longevity risk through capital markets", *de Actuaris*, September 2007: 26-29

Keilman, N., (1997), "Ex-post errors in official population forecasts in industrialized countries", *Journal of Official Statistics* (Statistics Sweden) 13(3): 245–277

Lee, R.D., (2000), "The Lee-Carter model for forecasting mortality, with various extensions and applications", *North American Actuarial Journal*, 4: 80–93

Lee, R. D. and Tuljapurkar, S., (1994), "Stochastic population forecasts for the United States: Beyond high, medium, and low", *Journal of the American Statistical Association*, 89: 1175–1189

Lee, R.D., and Miller, T., (2001), "Estimating the performance of the Lee- Carter method for forecasting mortality", *Demography*, 38: 537–549

Li, N., Lee, R. D., and Tuljapurkar, S., (2004), "Using the Lee-Carter method to forecast mortality for populations with limited data", *International Statistical Review*, 72(1): 19–36

Stallard, E., (2006), "Demographic Issues in Longevity Risk Analysis", *The Journal of Risk and Insurance*, 73(4): 575–609

Appendix 1: Parameter Estimation for Model M1

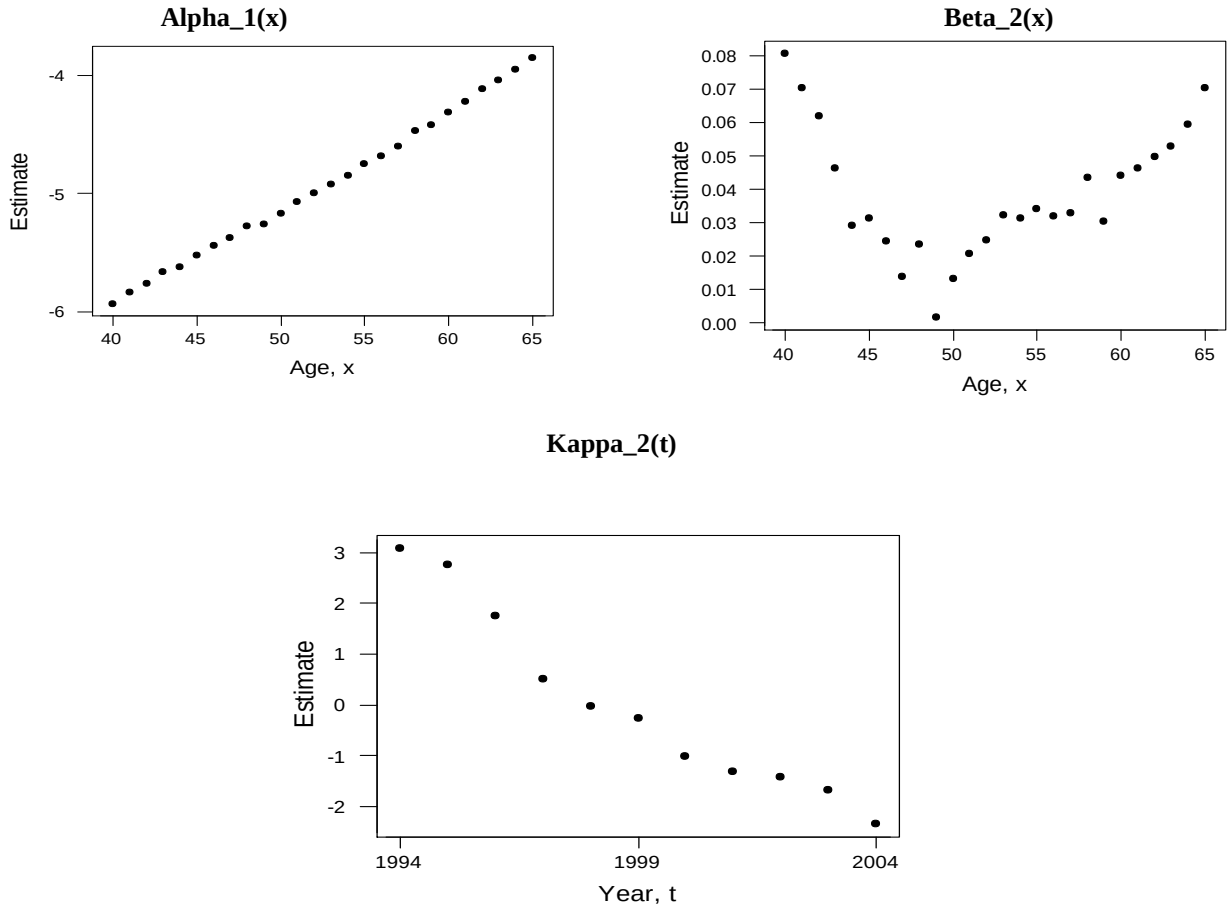

Fig. 1: Parameter Estimate for Model M1

Appendix 2: Parameter Estimate for Model M2

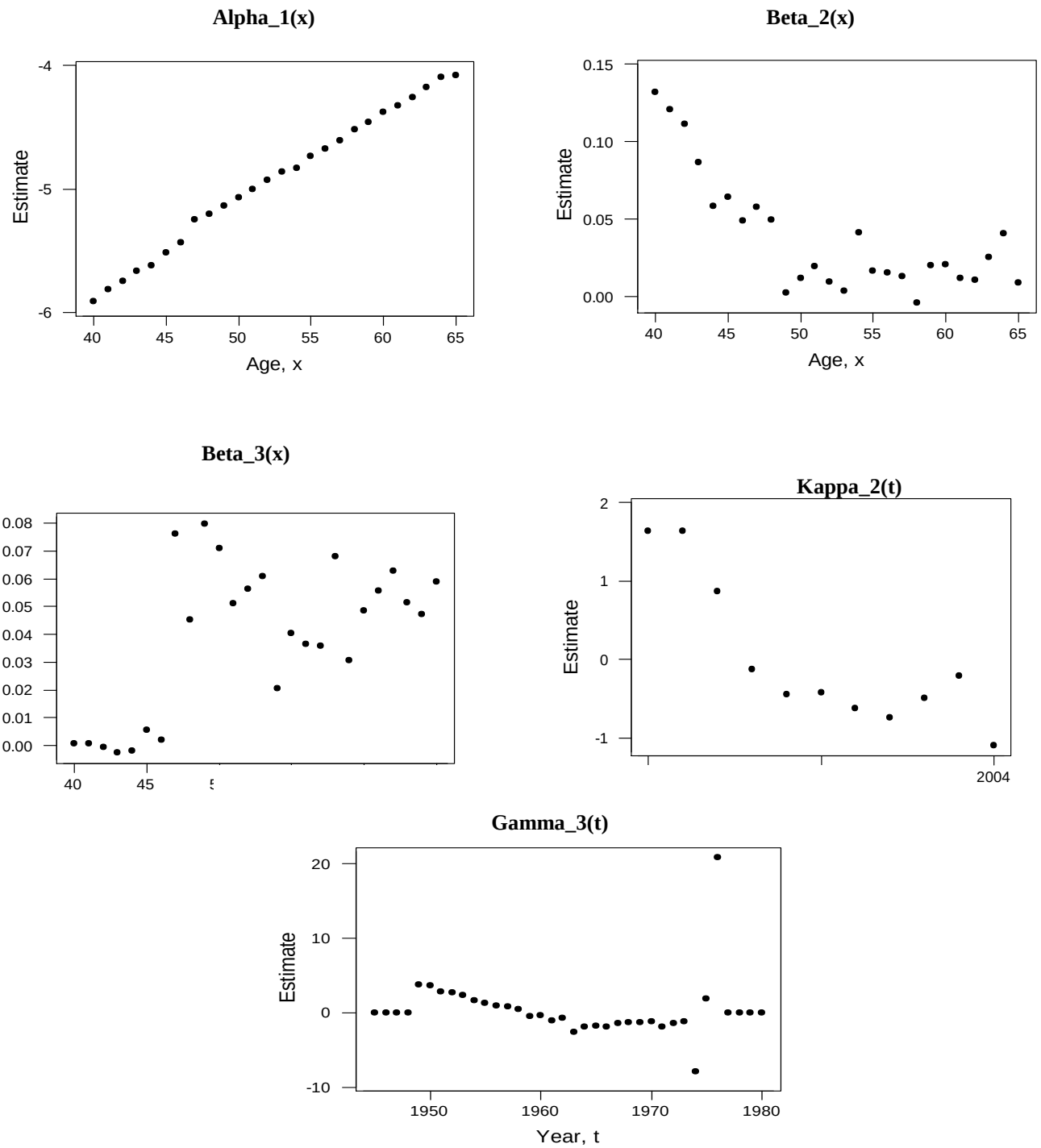

Fig. 2: Parameter Estimate for Model M1

Appendix 3

Model	Maximum Log-Likelihood	Effective Number of Parameters	BIC (Rank)
M1	-2315.84	61	-2486.13(2)
M2	-1671.88	121	-2009.68(1)

Table 1: Maximum likelihood, effective number of parameters estimated, and Bayes Information Criterion (BIC) for each model

Appendix 4: **Parameter Estimate of model 1**

Age (x)	$\alpha_1(x)$	$\beta_2(x)$
40	-5.937042853	0.080622726
41	-5.832821137	0.070231279
42	-5.760835316	0.061703447
43	-5.664341717	0.046331111
44	-5.619190022	0.029260349
45	-5.528015331	0.031185781
46	-5.441756056	0.024557437
47	-5.375378933	0.013803064
48	-5.274069819	0.023450762
49	-5.259912296	0.001616123
50	-5.167240022	0.013201032
51	-5.074705715	0.0206006
52	-4.996847258	0.024783167
53	-4.922039409	0.032162274
54	-4.852327599	0.031430612
55	-4.754379387	0.034225793
56	-4.682924938	0.031836655
57	-4.605436091	0.032808223
58	-4.477155788	0.043577064
59	-4.427597978	0.030443467
60	-4.313642237	0.044129187
61	-4.225618755	0.046220154
62	-4.121278258	0.049671558
63	-4.042966069	0.052734988
64	-3.953832987	0.059234277
65	-3.854388809	0.070178872

Model 1

Year (t)	γ_c
1994	3.079477024
1995	2.739722685
1996	1.749328401
1997	0.498474844
1998	-0.036705991
1999	-0.260030904
2000	-1.012452624
2001	-1.308463477
2002	-1.415588747
2003	-1.683642571
2004	-2.350118639

Model 2

Year (t)	γ_c
1994	1.628738445
1995	1.635743837
1996	0.871444639
1997	-0.120958091
1998	-0.441538865
1999	-0.419992009
2000	-0.616423163
2001	-0.741391836
2002	-0.488667499
2003	-0.211894212
2004	-1.095061247

Appendix 5: **Parameter Estimate Model 2**

Age (x)	$\alpha_1(x)$	$\beta_2(x)$	$\beta_3(x)$
40	-5.908303692	0.131624617	0.000884748
41	-5.813137796	0.12080494	0.000671676
42	-5.74938941	0.111320644	-0.000569007
43	-5.661971894	0.086599849	-0.002586968
44	-5.618820612	0.058280378	-0.001701373
45	-5.518406098	0.064433736	0.005614764
46	-5.437848195	0.048954141	0.002068801
47	-5.252309633	0.058019162	0.075951201
48	-5.202643297	0.049638789	0.04532135
49	-5.135564502	0.002480762	0.079708326
50	-5.066928149	0.011995112	0.070670779
51	-5.006004835	0.019437026	0.051055738
52	-4.9292125	0.009703233	0.056368388
53	-4.860674347	0.00407842	0.060915211
54	-4.836233333	0.041141518	0.020493053
55	-4.733568489	0.016757601	0.040347454
56	-4.675278618	0.01566199	0.036315232
57	-4.611548761	0.013476646	0.035817291
58	-4.522660116	-0.003800249	0.067962919
59	-4.457691118	0.02017791	0.030581779
60	-4.381610718	0.020645279	0.048407888
61	-4.324550438	0.01213218	0.055655417
62	-4.261203374	0.010688066	0.06279266
63	-4.180327153	0.025631833	0.05135103
64	-4.10006123	0.04083316	0.047182665
65	-4.08080195	0.009283257	0.058718981

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