Characterizations and estimations of Size Biased Generalized Rayleigh Distribution

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ABSTRACT

Since the widely using of the weighted distribution in many fields of real life such various areas including medicine, ecology, reliability, and so on , then we try to shed light and record our contribution in this field thru the research. In this paper, a new class of size-biased Generalized Rayleigh distribution is defined. A size-biased Generalized Rayleigh distribution, a particular case of weighted Generalized Rayleigh distribution, taking the weights as the variate values has been defined. The power and logarithmic moments of this family is defined. Some important theorems of SBGRD has been derived and studied. A new moment estimation method of parameters of SBGRD using its characterization is presented. In brief, this paper consists of presentation of general review of important properties of the new distribution. Bayes estimators of Size biased Generalized Rayleigh distribution (SBGRD), that stems from an extension of Jeffery's prior (Al-Kutubi [13]) with a new loss function (Al-Bayyati [12]). We are proposing four different types of estimator. Under squared error loss function, there are two estimators formed by using Jaffrey prior and an extension of Jaffrey's prior. The two remaining estimators are derived using the same Jeffrey's prior and extension of Jeffrey's prior under a new loss function. These methods are compared by using mean square error through simulation study with varying sample sizes.

Keywords: Generalized Rayleigh distribution, Size biased generalized Rayleigh distribution, Logarithmic moment, squared error loss function, Al-Bayatti's loss function.

1. Introduction

The generalized Rayleigh distribution (GRD) is considered to be a very useful life distribution. Rayleigh distribution is an important distribution in statistics and operations research. It is applied in several areas such as health, agriculture, biology, and other sciences. The probability distribution of Generalized Rayleigh distribution is given as:

$$f(x;\theta,k)=\frac{k}{\theta^{\frac{1}{k}}\Gamma\left(\frac{1}{k}\right)}\exp\left(-\frac{x^k}{\theta}\right) \qquad for\ \ x\geq 0, k\, and\, \theta>0 \tag{1.1}$$

$$=0, otherwise$$

Surles and Padgett [4] introduced two-parameter Burr Type X distribution and correctly named as the generalized Rayleigh distribution. The two-parameter generalized Rayleigh distribution is a particular member of the generalized Weibull distribution, originally proposed by Mudholkar and Srivastava [7]. Several aspects of the one-parameter (scale parameter equals one) generalized Rayleigh distribution were studied by Sartawi and Abu-Salih [5], and Raqab [6]. proposed another generalization of Rayleigh distribution. Ahmed et al (2013) estimates the parameter of Rayleigh distribution. It presents a flexible family in the varieties of shapes and is suitable for modeling data with different types of hazard rate function: increasing, decreasing and upside down bathtub shape (UBT). The Generalized Rayleigh distribution includes several other distributions as special or limiting cases, such as gamma, Weibull and exponential distributions. Its mean and variance are given by:

$$\mu = \frac{\theta^{\frac{1}{k}}\Gamma\left(\frac{2}{k}\right)}{\Gamma\left(\frac{1}{k}\right)} \tag{1.2}$$

$$\mu_2' = \frac{\theta^{\frac{2}{k}}\Gamma\left(\frac{3}{k}\right)}{\Gamma\left(\frac{1}{k}\right)}$$

And $$\mu_2 = \frac{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)}{\Gamma\left(\frac{1}{k}\right)} - \left[\frac{\theta^{\frac{1}{k}}\Gamma\left(\frac{2}{k}\right)}{\Gamma\left(\frac{1}{k}\right)}\right]^2 \tag{1.3}$$

Prentice [8] resolved the convergence problem using a nonlinear transformation of GG model. However, despite its long history and growing use in various applications, the GG family and its properties has-been remarkably presented in different papers. Ahmed. *et al* [9] introduced a new moment method of estimation of parameters of the Size biased generalized gamma distribution using its characterization. Their characterization is used to derive the expectation and the variance of $V_n^{\ 2}$ and then the new estimators for the two parameters of size-biased Generalized Rayleigh distribution are proposed.

2. SIZE-BIASED GENERALIZED RAYLEIGH DISTRIBUTION.

Size biased distributions are a special case of the more general form known as weighted distributions..First introduced by Fisher [1] to model ascertainment bias, these were later formalized in a unifying theory by Rao [2]. These distributions arise in practice when observations from a sample are recorded with unequal probability and provide unifying approach for the problems when the observations fall in the non –experimental, non – replicated and non –random categories. A size biased generalized Rayleigh distribution (SBGRD) is obtained by applying the weights x^c, where c =1 to the weighted Generalized Rayleigh distribution. A size biased generalized Rayleigh distribution (SBGRD) is obtained by applying the weights x^c, where c =1 to the weighted Generalized Rayleigh distribution.

We have from relation (1.1) and (1.2)

$$\int_0^\infty x\, f(x;\theta,k)dx = \frac{\theta^{\frac{1}{k}}\Gamma\left(\frac{2}{k}\right)}{\Gamma\left(\frac{1}{k}\right)}$$

This gives the size-biased generalized Rayleigh distribution (SBGRD) as:

$$f_S(x;\theta,k) = \frac{kx}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)}\exp\left(-\frac{x^k}{\theta}\right) \qquad for\ \ x \ge 0, k\, and\, \theta > 0$$
$$= 0, otherwise \tag{2.1}$$

The CDF of the Size biased generalized Rayleigh distribution is given by:

$$F(x;\theta,k)=\frac{\gamma\left(\frac{2}{k},x\right)}{\Gamma\left(\frac{2}{k}\right)} \tag{2.2}$$

Special cases:

The distribution like the Size-biased exponential distributions as a special case when k = 1, then the probability density function is given as:

$$f_S(x;\theta)=\frac{x}{\theta^2}\exp\left(-\frac{x}{\theta}\right) \quad for\ \ x\geq 0, k>0$$
$$=0, otherwise$$

The distribution like the Size-biased Rayleigh distribution as a special case, when k=2 then the probability density function is given as:

$$f_S(x;\theta)=\frac{2x}{\theta}\exp\left(-\frac{x^2}{\theta}\right) \quad for\ \ x\geq 0,\theta>0$$
$$=0, otherwise$$

Hazard functions

The hazard function for the Size biased generalized Rayleigh distribution is given as:

$$h_s(x;\theta,k)=\frac{f(x;\theta,,k)}{1-F(x;\theta,,k)}$$

$$h(x;\theta,k)=\frac{kxe^{-\frac{x^k}{\theta}}}{\theta^{\frac{2}{k}}\left[\Gamma\left(\frac{2}{k}\right)-\gamma\left(\frac{2}{k},x\right)\right]} \tag{2.3}$$

The reverse hazard function for the Size biased generalized Rayleigh distribution is given as:

$$h_{rv}(x;\theta,k)=\frac{f(x;\theta,k)}{F(x;\theta,k)}$$

$$h(x;\theta,k)=\frac{kxe^{-\frac{x^k}{\theta}}}{\theta^{\frac{2}{k}}\left[\gamma\left(\frac{2}{k},x\right)\right]} \tag{2.4}$$

Theorem1: Let $f(x;\theta,k)$ be a twice differentiable probability density function of a continuous random variable X. Define $n(x;\theta,k)=-\frac{f'(x;\theta,k)}{f(x,\theta,k)}$,where $f'(x;\theta,k)$ is the first derivative of $f(x;\theta,k)$ with respect to x. Furthermore, suppose that the first derivative of $n(x;\theta,k)$ exist.

a) If $n'(x;\theta,k)<0,$ for all x > 0, then the hazard function is monotonically decreasing.

b) If $n'(x;\theta,k)>0,$ for all x > 0, then the hazard function is monotonically increasing.

c) Suppose there exist x_0 such that $n'(x;\theta,k)<0,\ for\ all\ 0<x<x_0,\ n'(x_0;\theta,k)=0$

And $n'(x;\theta,k)>0,\ for\ all\ 0>x_0$. In addition, $\lim_{x\to 0} f(x)=\infty$, then the hazard function is upside down bathtub shape.

Proof: Using equation (2.1), the derivative of the $f(x;\theta,k)$ is given by:

$$f'(x;\theta,k)=\frac{k}{\theta^{\frac{2}{k}+1}\Gamma\left(\frac{2}{k}\right)}e^{\frac{-x^k}{\theta}}\left[kx^k+\theta\right]$$

Therefore, $n(x;\theta,k)=-\frac{f'(x;\theta,k)}{f(x,\theta,k)}$

$$n(x;\theta,k)=-\frac{kx^k-\theta}{\theta x}$$

The derivative of $n(x;\theta,k)$ is given as:

And $$n'(x;\theta,k)=\frac{k(k-1)x^{k-2}}{\theta}+\frac{1}{x^2} \tag{2.5}$$

Collory:

a) If $k\geq 1$, then $n'(x;\theta,k)>0$, for all x > 0, then the hazard function is monotonically increasing.

b) If $k<1$, then $n'(x;\theta,k)<0$, then the hazard function is monotonically decreasing.

c) If $0<k<1$, then the hazard function is upside down bathtub shape.

Structural properties: The structural properties of SBGRD can be obtained by as:

$$\mu_r' = \frac{1}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)}\Gamma\left(\frac{r+2}{k}\right).\theta^{\frac{r+2}{k}} \tag{2.6}$$

$$\mu = Mean=\frac{\Gamma\left(\frac{3}{k}\right)}{\Gamma\left(\frac{2}{k}\right)}.\theta^{\frac{1}{k}} \tag{2.7}$$

$$\mu_2 = \frac{\theta^{\frac{2}{k}}}{\Gamma\left(\frac{2}{k}\right)}\left[\Gamma\left(\frac{4}{k}\right)-\frac{\Gamma^2\left(\frac{3}{k}\right)}{\Gamma\left(\frac{2}{k}\right)}\right] \tag{2.8}$$

$$CV = \frac{\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)-\Gamma^2\left(\frac{3}{k}\right)}{\Gamma\left(\frac{3}{k}\right)} \tag{2.9}$$

The moment generating function of Size-biased generalized Rayleigh distribution is obtained as:

$$E(e^{tx^k})=\int_0^\infty e^{tx^k} f_s(x;\theta,k)dx$$

$$E(e^{tx^k})=\frac{1}{(1-\theta t)^{\frac{2}{k}}} \tag{2.10}$$

The Characteristic function of Size-biased generalized Rayleigh distribution is obtained as:

$$E(e^{itx^k}) = \int_0^\infty e^{itx^k} f_s(x;\theta,k)dx$$

$$E(e^{itx^k}) = \frac{1}{(1-\theta it)^{\frac{2}{k}}} \qquad (2.11)$$

The Shannon's entropy of Size-biased generalized Rayleigh distribution is given as:

$$H[f_s(x;\theta,k)] = \frac{\log\theta}{k} - \log k + \log\Gamma\left(\frac{2}{k}\right) + [k-1]\frac{\Gamma\left(\frac{2}{k}+1\right)}{k\Gamma\left(\frac{2}{k}\right)} \qquad (2.12)$$

3. NEW MOMENT ESTIMATOR OF THE SIZE-BIASED GENERALIZED RAYLEIGH DISTRIBUTION

Note that Hwang .T and Huang. P [3] have obtained more general characterizations with the independence of sample coefficient of variation V_n with sample mean $\overline{X}_n$ as one of its special cases when random samples are drawn from the generalized gamma distribution. Their characterization is used to derive the expectation and the variance of $V_n^{\,2}$ and then the new estimators for the three parameters of size-biased generalized Rayleigh distribution are proposed. For deriving new moment estimators of three parameters of the size-biased generalized Rayleigh distribution, we need the following theorem obtained by using the similar approach of Hwang .T and Huang .P (Theorems of 2006).

Theorem 3.1. Let $n \geq 3$ and let $X_1, X_2, X_3 ... X_n$ be a n positive identical independently distributed random variables having a probability density function $f(x)$.Then the independence of the sample mean $\overline{X}_n$ and the sample coefficient of variation $V_n = \frac{S_n}{\overline{X}_n}$ is equivalent to that $f(x)$ is a size-biased generalized Rayleigh distribution where S_n is the sample standard deviation.

The next theorem is easy to prove and need to derive the expectation and the variance of $V_n^{\,2} = \left(\frac{S_n}{\overline{X}_n}\right)^2$, where $\overline{X}_n$ and S_n are respectively the sample mean and the sample standard deviation.

Theorem 3.2: Let $n \geq 3$ and let $X_1, X_2, X_3 ... X_n$ be a n positive identical independently distributed random samples drawn from a population having a size-biased generalized Rayleigh distribution

$$f_S(x;\theta,k) = \frac{kx}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)}\exp\left(-\frac{x^k}{\theta}\right) \qquad for\ x \geq 0, k\ and\ \theta > 0$$

$$= 0, otherwise$$

Then $$E(S_n^{\,2}) = \frac{\theta^{\frac{2}{k}}\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right) - \Gamma^2\left(\frac{3}{k}\right)\right]}{\Gamma^2\left(\frac{2}{k}\right)}$$

Proof: Here, $E\left(X^{m}\right)=\dfrac{1}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)}\Gamma\left(\dfrac{m+2}{k}\right).\theta^{\frac{m+2}{k}}$

$$E\left(\overline{X}_{n}\right)=\frac{\Gamma\left(\frac{3}{k}\right)\theta^{\frac{1}{k}}}{\Gamma\left(\frac{2}{k}\right)}$$

$$E(\overline{X}_{n}^{\ 2})=\frac{\theta^{\frac{2}{k}}\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)+(n-1)\theta^{\frac{2}{k}}\Gamma^{2}\left(\frac{3}{k}\right)\right]}{n\Gamma^{2}\left(\frac{2}{k}\right)}$$

And $E(S_{n}^{\ 2})=\dfrac{\theta^{\frac{2}{k}}\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)-\Gamma^{2}\left(\frac{3}{k}\right)\right]}{\Gamma^{2}\left(\frac{2}{k}\right)}$ (3.1)

Where $\overline{X}_{n}$ and $S_{n}^{\ 2}$ are respectively their sample mean and sample variance.

Theorem 3.3: Let $n\geq 3$ and let $X_{1},X_{2},X_{3}...X_{n}$ be a n positive identical independently distributed random samples drawn from a population having a size-biased generalized Rayleigh distribution

$$f_{S}(x;\theta,k)=\frac{kx}{\theta^{\frac{2}{k}}\Gamma\left(\frac{2}{k}\right)}\exp\left(-\frac{x^{k}}{\theta}\right) \qquad for\ \ x\geq 0, k\ and\ \theta>0$$

$$=0, otherwise$$

Then $E\left(\dfrac{S_{n}^{\ 2}}{\overline{X}^{2}_{n}}\right)=\dfrac{n\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)-\Gamma^{2}\left(\frac{3}{k}\right)\right]}{\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)+(n-1)\Gamma^{2}\left(\frac{3}{k}\right)}$

Where $\overline{X}_{n}$ and $S_{n}^{\ 2}$ are respectively their sample mean and sample variance.

Proof: By theorem 3.1, we have

$$E(S_{n}^{\ 2})=E\left(\frac{S_{n}^{\ 2}}{\overline{X}_{n}^{\ 2}}.\overline{X}_{n}^{\ 2}\right)=E\left(\frac{S_{n}^{\ 2}}{\overline{X}_{n}^{\ 2}}\right)..E\left(\overline{X}_{n}^{\ 2}\right)$$

And hence $E\left(\dfrac{S_{n}^{\ 2}}{\overline{X}_{n}^{\ 2}}\right)=\dfrac{E\left(S_{n}^{\ 2}\right)}{E\left(\overline{X}_{n}^{\ 2}\right)}$

Applying theorem 3.2 to the above identity yields that

$$E\left(\frac{S_n^{\,2}}{\overline{X}^2{}_n}\right)=\frac{n\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)-\Gamma^2\left(\frac{3}{k}\right)\right]}{\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)+(n-1)\Gamma^2\left(\frac{3}{k}\right)} \tag{3.2}$$

Thus.3.3 is established.

Theorem 3.4: Let $n \geq 3$ and let $X_1, X_2, X_3 ... X_n$ be a n positive identical independently distributed random samples drawn from a population having a size-biased generalized Rayleigh distribution

$$E(S_n^{\,2})=\frac{\theta^{\frac{2}{k}}\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)-\Gamma^2\left(\frac{3}{k}\right)\right]}{\Gamma^2\left(\frac{2}{k}\right)}$$

$$E\left(\frac{S_n^{\,2}}{\overline{X}_n^{\,2}}\right)=\frac{n\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)-\Gamma^2\left(\frac{3}{k}\right)\right]}{\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)+(n-1)\Gamma^2\left(\frac{3}{k}\right)}$$

Furthermore, if SBGR distribution, we have

$$\frac{\sigma^2}{\mu^2}=\frac{\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)\right]}{\Gamma^2\left(\frac{3}{k}\right)}-1 \tag{3.3}$$

And it can be show that

$$E\left(\frac{S_n^{\,2}}{\overline{X}_n^{\,2}}\right)\rightarrow\frac{\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right)\right]}{\Gamma^2\left(\frac{3}{k}\right)}-1 \tag{3.4}$$

Comparing above two equations, we have

Note that $E\left(\frac{S_n^{\,2}}{\overline{X}_n^{\,2}}\right)\rightarrow\frac{\sigma^2}{\mu^2}$ as $n\rightarrow\infty$ and that this limit is the square of the population coefficient of variation. Thus, $\frac{S_n^{\,2}}{\overline{X}_n^{\,2}}$ is an asymptotically unbiased estimator of the square of the population coefficient of variation. Therefore, Then we can solve numerically via moment method the below equations for estimating of SBGR parameters

$$\frac{\sum_{i=1}^{n} x_i}{n}=\frac{\Gamma\left(\frac{3}{k}\right)\theta^{\frac{1}{k}}}{\Gamma\left(\frac{2}{k}\right)} \tag{3.5}$$

$$\frac{S_n^{\ 2}}{n\,\overline{X}_n^{\ 2}} = \frac{\left[\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right) - \Gamma^2\left(\frac{3}{k}\right)\right]}{\Gamma\left(\frac{4}{k}\right)\Gamma\left(\frac{2}{k}\right) + (n-1)\Gamma^2\left(\frac{3}{k}\right)} \tag{3.6}$$

4. Estimation of parameters

In this section, we discuss the various estimation methods for size biased Generalized Rayleigh distribution and verifying their efficiencies.

4.1 Methods of Moments

Replacing sample moments with population moments, we get

$$\overline{X} = \frac{\Gamma\left(\frac{3}{k}\right)}{\Gamma\left(\frac{2}{k}\right)}.\theta^{\frac{1}{k}} \tag{4.1}$$

$$S^2 = \frac{\theta^{\frac{2}{k}}}{\Gamma\left(\frac{2}{k}\right)}\left[\Gamma\left(\frac{4}{k}\right) - \frac{\Gamma^2\left(\frac{3}{k}\right)}{\Gamma\left(\frac{2}{k}\right)}\right] \tag{4.2}$$

From above two equations, we get

$$\frac{S^2}{\overline{X}^4} = \frac{\Gamma^3\left(\frac{2}{k}\right)\Gamma\left(\frac{4}{k}\right) - \Gamma^2\left(\frac{2}{k}\right)\Gamma^2\left(\frac{3}{k}\right)}{\Gamma^2\left(\frac{3}{k}\right)} \tag{4.3}$$

Solving above equation for k, we get the estimate for k and substituting that value in equation (4.1), we get the estimate of θ.

4.2 Method of Maximum Likelihood estimator.

Maximum likelihood estimation has been the most widely used method for estimating the parameters of the Size biased generalized Rayleigh distribution. Let $x_1, x_2, x_3.......x_n$ be a random sample from the size biased generalized Rayleigh distribution, and then the corresponding likelihood function is given as

$$L(X;\theta,k) = \frac{k^n}{\theta^{\frac{2n}{k}}\Gamma^n\left(\frac{2}{k}\right)}\prod_{i=1}^{n} x_i \exp\left(\frac{-\sum_{i=1}^{n} x_i^{\ k}}{\theta}\right) \tag{4.4}$$

The log-likelihood function is given as:

$$\log L(x;\theta,k) = n\log k - \frac{2n\log\theta}{k} - n\log\Gamma\left(\frac{2}{k}\right) + \sum_{i=1}^{n}\log x_i - \frac{\sum_{i=1}^{n} x_i^{\ k}}{\theta} \tag{4.5}$$

Now, we obtain the normal equations, we get

$$\frac{2n}{\theta k} + \frac{\sum_{i=1}^{n} x_i^{\ k}}{\theta^2} = 0 \tag{4.6}$$

$$\frac{n}{k}+\frac{2n\log\theta}{k^2}-2n\frac{\Gamma'\left(\frac{2}{k}\right)}{\Gamma\left(\frac{2}{k}\right)}-\frac{\sum_{i=1}^{n}x_i^{\,k}\log\sum_{i=1}^{n}x_i}{\theta}=0 \qquad (4.7)$$

After solving equation (4.6), we have

$$\hat{\theta}=\frac{k\sum_{i=1}^{n}x_i^{\,k}}{2n} \qquad (4.8)$$

Substitute the value of $\hat{\theta}$ in equation (4.7), we get the estimate of k .

4.3 BAYESIAN ANALYSIS OF SIZE-BIASED GENERALIZED RAYLEGH DISTRIBUTION

Bayesian analysis is an important approach to statistics, which formally seeks use of prior information and Bayes Theorem provides the formal basis for using this information. In this approach, parameters are treated as random variables and data is treated fixed. Ghafoor et al [10],Ahmed et al (2007) and Rahul et al [11] have discussed the application of Bayesian methods**.** An important requisite in Bayesian estimation is the appropriate choice of prior(s) for the parameters. However, Bayesian analysts have pointed out that there is no clear cut way from which one can conclude that one prior is better than the other. Very often, priors are chosen according to ones subjective knowledge and beliefs. However, if one has adequate information about the parameter(s) one should use informative prior(s), otherwise it is preferable to use non informative prior(s).

4.3.1 Parameter estimation under squared error loss function.

In this section, two different prior distributions are used for estimating the parameter of the size biased generalized Rayleigh distribution namely; Jeffery's prior and extension of Jeffrey's prior information.

Bayes estimation of parameter of size biased generalized Rayleigh distribution under Jeffrey's prior.

Consider there are n recorded values $\underline{x}=(x_1,\ldots x_n)$ from (2.1). We consider the extended Jeffrey's prior as: $g(\theta)\propto\sqrt{[I(\theta)]}$

Where $[I(\theta)]=-nE\left[\frac{\partial^2\log f(x;\theta,k)}{\partial\theta^2}\right]$ is the Fisher's information matrix. For the model (2.1),

$$g(\theta)=k\sqrt{\frac{1}{\theta}}$$

Then the joint probability density function is given by:

$$f(\underline{x},\theta)=L(x;\theta)g(\theta)$$

$$f(\underline{x},\theta)=\frac{k^n}{\theta^{\frac{2n}{k}+\frac{1}{2}}\Gamma^n\left(\frac{2}{k}\right)}\prod_{i=1}^{n}x_i\exp\left(\frac{-\sum_{i=1}^{n}x_i^{\,k}}{\theta}\right)$$

And the corresponding marginal PDF of $\underline{x}=(x_1,\ldots x_n)$ is obtained as:

$$p(\underline{x})=\frac{k^n}{\Gamma^n\left(\frac{2}{k}\right)}\prod_{i=1}^{n}x_i\frac{\Gamma\left(\frac{2n}{k}+\frac{3}{2}\right)}{\left[\left(\sum_{i=1}^{n}x_i^{\,k}\right)\right]^{\frac{2n}{k}+\frac{3}{2}}} \qquad (4.9)$$

The posterior PDF of θ has the following form

$$\pi_1(\theta/\underline{x}) = \frac{\left[\sum_{i=1}^{n} x_i^{\,k}\right]^{\frac{2n}{k}+\frac{3}{2}}}{\Gamma\left(\frac{2n}{k}+\frac{3}{2}\right)} \exp\left(\frac{-\sum_{i=1}^{n} x_i^{\,k}}{\theta}\right)\left(\frac{1}{\theta}\right)^{\frac{2n}{k}+\frac{1}{2}} \qquad (4.10)$$

By using a squared error loss function $L(\hat{\theta},\theta) = c(\hat{\theta}-\theta)^2$ for some constant c, the risk function is:

$$R(\hat{\theta}) = \int_0^\infty c(\hat{\theta}-\theta)^2 \pi_1(\theta/\underline{x})d\theta$$

$$R(\hat{\theta}) = c\hat{\theta}^2 + \frac{c\Gamma\left(\frac{2n}{k}-\frac{1}{2}\right)}{\Gamma\left(\frac{2n}{k}+\frac{3}{2}\right)}\left(\sum_{i=1}^{n} x_i^{\,k}\right)^2 - \frac{2c\hat{\theta}\Gamma\left(\frac{2n}{k}+\frac{1}{2}\right)}{\Gamma\left(\frac{2n}{k}+\frac{3}{2}\right)}\sum_{i=1}^{k} x_i^{\,k}$$

Now $\frac{\partial R(\hat{\theta})}{\partial\theta} = 0$, Then the Bayes estimator is

$$\hat{\theta}_1 = \frac{\Gamma\left(\frac{2n}{k}+\frac{1}{2}\right)}{\Gamma\left(\frac{2n}{k}+\frac{3}{2}\right)}\sum_{i=1}^{k} x_i^{\,k} \qquad (4.11)$$

Bayes estimation of parameter of size biased generalized Rayleigh distribution using extension of Jeffrey's prior.

We consider the extended Jeffrey's prior are given as: $g(\theta) \propto [I(\theta)]^{c_1}; c_1 \varepsilon R^+$

Where $[I(\theta)] = -nE\left[\frac{\partial^2 \log f(x;\theta,k)}{\partial\theta^2}\right]$ is the Fisher's information matrix. For the model (2.1),

$$g(\theta) = k\left[\frac{1}{\theta}\right]^{c_1}$$

Then the joint probability density function is given by:

$$f(\underline{x},\theta) = L(x;\theta)g(\theta)$$

$$f(\underline{x},\theta) = \frac{k^n}{\theta^{\frac{2n}{k}+c_1}\Gamma^n\left(\frac{2}{k}\right)}\prod_{i=1}^{n} x_i \exp\left(\frac{-\sum_{i=1}^{n} x_i^{\,k}}{\theta}\right)$$

And the corresponding marginal PDF of $\underline{x} = (x_1,\ldots x_n)$ is obtained as:

$$p(\underline{x}) = \frac{k^n}{\Gamma^n\left(\frac{2}{k}\right)}\prod_{i=1}^{n} x_i \frac{\Gamma\left(\frac{2n}{k}+c_1+1\right)}{\left[\left(\sum_{i=1}^{n} x_i^{\,k}\right)\right]^{\frac{2n}{k}+c_1+1}} \qquad (4.12)$$

The posterior PDF of θ has the following form

$$\pi_2(\theta/\underline{x}) = \frac{\left[\sum_{i=1}^{n} x_i^k\right]^{\frac{2n}{k}+c_1+1}}{\Gamma\left(\frac{2n}{k}+c_1+1\right)} \exp\left(\frac{-\sum_{i=1}^{n} x_i^k}{\theta}\right)\left(\frac{1}{\theta}\right)^{\frac{2n}{k}+c_1} \tag{4.13}$$

By using a squared error loss function $L(\hat{\theta},\theta) = c(\hat{\theta}-\theta)^2$ for some constant c, the risk function is:

$$R(\hat{\theta}) = \int_0^\infty c(\hat{\theta}-\theta)^2 \pi_1(\theta/\underline{x})d\theta$$

$$R(\hat{\theta}) = c\hat{\theta}^2 + \frac{c\Gamma\left(\frac{2n}{k}+c_1-1\right)}{\Gamma\left(\frac{2n}{k}+c_1+1\right)}\left(\sum_{i=1}^{n} x_i^k\right)^2 - \frac{2c\hat{\theta}\Gamma\left(\frac{2n}{k}+c_1\right)}{\Gamma\left(\frac{2n}{k}+c_1+1\right)}\sum_{i=1}^{k} x_i^k$$

Now $\frac{\partial R(\hat{\theta})}{\partial \theta} = 0$, Then the Bayes estimator is

$$\hat{\theta}_2 = \frac{\Gamma\left(\frac{2n}{k}+c_1\right)}{\Gamma\left(\frac{2n}{k}+c_1+1\right)}\sum_{i=1}^{k} x_i^k \tag{4.14}$$

Remark 1: Replacing c_1= 1/2 in (4.14), we get the Bayes estimator under squared error loss function with Jeffrey's prior which is same as (4.11). By Replacing c_1= 3/2 in (4.14), we get the Hartigan's prior. By Replacing c_1= 0 in (4.14), thus we get uniform prior.

4.3.2 Parameter estimation under a new loss function.

This section uses a new loss function introduced by Al-Bayyati [12]. Employing this loss function, we obtain Bayes estimators using Jeffrey's and extension of Jeffrey's prior information.

Al-Bayyati introduced a new loss function of the form:

$$l_A(\hat{\theta},\theta) = \theta^{c_2}(\hat{\theta}-\theta)^2; c_2 \varepsilon R. \tag{4.15}$$

Here, this loss function is used to obtain the estimator of the parameter of the size biased generalized Rayleigh distribution.

Bayes estimation of parameter of size biased generalized Rayleigh distribution under Jeffrey's prior.

By using the loss function in the form given in (4.15), we obtained the following risk function:

$$R(\hat{\theta}) = \int_0^\infty \theta^{c_2}(\hat{\theta}-\theta)^2 \pi_1(\theta/\underline{x})d\theta$$

$$R(\hat{\theta}) = \hat{\theta}^2 \frac{\Gamma\left(\frac{2n}{k}+c_2+\frac{3}{2}\right)}{\Gamma\left(\frac{2n}{k}+\frac{3}{2}\right)}\left[\frac{1}{\sum_{i=1}^{n} x_i^k}\right]^{c_2} + \frac{\Gamma\left(\frac{2n}{k}+c_2-\frac{1}{2}\right)}{\Gamma\left(\frac{2n}{k}+\frac{3}{2}\right)}\left(\frac{1}{\sum_{i=1}^{k} x_i^k}\right)^{c_2+2} - \frac{2\hat{\theta}\Gamma\left(\frac{2n}{k}+c_2+\frac{1}{2}\right)}{\Gamma\left(\frac{2n}{k}+\frac{3}{2}\right)}\left[\frac{1}{\sum_{i=1}^{n} x_i^k}\right]^{c_2-1}$$

Now $\frac{\partial R(\hat{\theta})}{\partial \theta} = 0$, Then the Bayes estimator is

$$\hat{\theta}_3 = \frac{\Gamma\left(\frac{2n}{k} + \frac{1}{2} + c_2\right)}{\Gamma\left(\frac{2n}{k} + \frac{3}{2} + c_2\right)} \left(\sum_{i=1}^{k} x_i^{\,k}\right)^{c_2 - 1} \tag{4.16}$$

Remark 2: Replacing $c_2 = 0$ in (4.16), we get the Bayes estimator under squared error loss function with Jeffrey's prior which is same as (4.11). By Replacing $c_2 = 1$ in (4.16), we get the Hartigan's prior. By Replacing $c_2 = -1/2$ in (4.16), thus we get uniform prior.

Bayes estimation of parameter of size biased generalized Rayleigh distribution using extension of Jeffrey's prior.

By using the loss function in the form given in (4.15), we obtained the following risk function:

$$R(\hat{\theta}) = \int_0^\infty \theta^{c_2} \left(\hat{\theta} - \theta\right)^2 \pi_1(\theta / \underline{x}) d\theta$$

$$R(\hat{\theta}) = \hat{\theta}^2 \frac{\Gamma\left(\frac{2n}{k} + c_2 + c_1 + 1\right)}{\Gamma\left(\frac{2n}{k} + c_1 + 1\right)} \left[\frac{1}{\sum_{i=1}^{n} x_i^{\,k}}\right]^{c_2} + \frac{\Gamma\left(\frac{2n}{k} + c_2 + c_1 - 1\right)}{\Gamma\left(\frac{2n}{k} + c_1 + 1\right)} \left(\frac{1}{\sum_{i=1}^{k} x_i^{\,k}}\right)^{c_2 + 2} - \frac{2\hat{\theta}\Gamma\left(\frac{2n}{k} + c_2 + c_1\right)}{\Gamma\left(\frac{2n}{k} + c_1 + 1\right)} \left[\frac{1}{\sum_{i=1}^{n} x_i^{\,k}}\right]^{c_2 - 1}$$

Now $\frac{\partial R(\hat{\theta})}{\partial \theta} = 0$, Then the Bayes estimator is

$$\hat{\theta}_4 = \frac{\Gamma\left(\frac{2n}{k} + c_1 + c_2\right)}{\Gamma\left(\frac{2n}{k} + c_1 + c_2 + 1\right)} \left(\sum_{i=1}^{k} x_i^{\,k}\right) \tag{4.17}$$

Remark 3: Replacing $c_1 = 1/2$ and $c_2 = 0$ in (4.17), we get the Bayes estimator under squared error loss function with Jeffrey's prior which is same as (4.11). By Replacing $c_1 = 3/2$ and $c_2 = 0$ in (4.17), we get the Hartigan's prior.By Replacing $c_1 = 0$ and $c_2 = 0$ in (4.17), thus we get uniform prior.

5 Simulation Study of Size biased Generalized Rayleigh distribution

In our simulation study, we chose a sample size of n=25, 50 and 75 to represent small, medium and large data set. The scale parameter is estimated for Size biased Generalized Rayleigh distribution by the methods of Maximum Likelihood and Bayesian using Jeffrey's & extension of Jeffrey's prior methods. For the scale parameter we have considered $\theta = 0.5$ and 1.0. The values of Jeffrey's extension were $c_1 = 0.5$, 1.0, 1.5 and 2.0. The value for the loss parameter $c_2 = 0$ and ± 1.0. This was iterated 5000 times and the scale parameter for each method was calculated. A simulation study was conducted using R-software to examine and compare the performance of the estimates for different sample sizes with different values for the Extension of Jeffrey's' prior and the loss functions. The results are presented in tables for different selections of the parameters and c extension of Jeffrey's prior.

Table 5.1Structural properties of Size biased Generalized Rayleigh distribution

n	θ	k	Mean	variance	S.D	C.V
25	0.5	1.0	1.3057878	0.0013881	0.0372575	0.028532
	1.0	1.5	1.310549	0.0182512	0.1350972	0.103084
50	0.5	1.0	0.3054362	0.0011856	0.0344329	0.112733
	1.0	1.5	0.3199525	0.0407701	0.2019163	0.631082
75	0.5	1.0	1.3256778	1.425e-05	0.0037708	0.002844
	1.0	1.5	1.310549	0.0319192	0.1786595	0.136324

Table 5.2: Shannon's Entropy, AIC and BIC criteria of Size-biased Generalized Rayleigh Distribution

N	θ	k	Shannon's Entropy	AIC	BIC
25	0.5	1.0	3.414851	174.7425	177.1803
	1.0	1.5	4.724048	240.2024	242.6402
50	0.5	1.0	1.879161	191.9161	195.7401
	1.0	1.5	2.362021	240.2021	244.0261
75	0.5	1.0	1.365678	208.8517	213.4867
	1.0	1.5	1.181019	181.1529	185.7878

From the above table 5.2, we can conclude that the Size-biased Generalized Rayleigh Distribution have the smallest AIC and BIC values when sample size is 75 and scale parameter is 1.0 and shape parameter k=1.5.

Table 5.3 Mean Squared Error for ($\hat{\theta}$) under Jeffrey's prior

n	θ	k	θ_{ML}	θ_{SL}	θ_{NL}		
					C2=-1.0	C2=-0	C2=1.0
25	0.5	1.0	0.4184437	0.02261071	0.02053641	0.02261071	0.02469849
	1.0	1.5	0.35385413	0.35744087	0.35031408	0.35744087	0.36473563
50	0.5	1.0	0.3912145	0.01621413	0.01527433	0.01621413	0.01716265
	1.0	1.0	0.3218592	0.3243453	0.3193622	0.3243453	0.3292847
75	0.5	1.0	0.3897367	0.01572284	0.01517348	0.01572284	0.16547846
	1.0	1.0	0.2638086	0.2654711	0.2621403	0.2654711	0.2687786

Table 5.4: Mean Squared Error for ($\hat{\theta}$) under extension of Jeffrey's prior

n	θ	k	C_1	θ_{ML}	θ_{SL}	θ_{NL}		
						C2=-1.0	C2=0	C2=1.0
25	0.5	1.0	0.5	0.41844371	0.02261071	0.02053641	0.02261071	0.02469849
			1.0	0.41844371	0.02365331	0.02157146	0.02365331	0.02574551
			1.5	0.41844372	0.02469849	0.02261071	0.02469849	0.02679371
			2.0	0.41844371	0.02574551	0.02365331	0.02574551	0.02784246
	1.0	1.0	0.5	0.35385413	0.35744087	0.35031408	0.35744087	0.36473563
			1.0	0.35385413	0.36106952	0.35385413	0.36106952	0.36843503
			1.5	0.35385413	0.36473563	0.35744087	0.36473563	0.37216383
			2.0	0.35385413	0.36843503	0.36106952	0.36843503	0.37591841
50	0.5	1.0	0.5	0.39121450	0.01621413	0.01527433	0.01621413	0.01716265
			1.0	0.39121450	0.01668737	0.01574307	0.01668737	0.01763985
			1.5	0.39121450	0.01716265	0.01621413	0.01716265	0.01811884
			2.0	0.39121450	0.01763985	0.01668737	0.01763985	0.0185995
	1.0	1.0	0.5	0.33218592	0.3243453	0.3193622	0.3243453	0.3292847
			1.0	0.33218592	0.3268205	0.3218592	0.3268205	0.3317378
			1.5	0.33218592	0.3292847	0.3243453	0.3292847	0.3341798
			2.0	0.33218592	0.3317378	0.3268205	0.3317378	0.3366108
75	0.5	1.0	0.5	0.38973671	0.01572284	0.01517348	0.01572284	0.01654786
			1.0	0.38973671	0.0163256	0.01544774	0.0153256	0.01653488
			1.5	0.38973671	0.01507789	0.01468892	0.01507789	0.01569146
			2.0	0.38973671	0.01604798	0.01543174	0.01604798	0.01666804
	1.0	1.0	0.5	0.26547112	0.2654711	0.2621403	0.26547112	0.2687786
			1.0	0.26547112	0.2671278	0.2638086	0.2671278	0.2704236
			1.5	0.26547112	0.2687786	0.2654711	0.2687786	0.2720628
			2.0	0.26547112	0.2704236	0.2671278	0.2704236	0.2736962

ML= Maximum Likelihood, SL=Squared Error Loss Function, NL= New Loss Function,

In table 5.3, Bayes'estimation with New Loss function under Jeffrey's prior provides the smallest values in most cases especially when loss parameter C_2 is -1. Similarly, in table 5.4, Bayes' estimation with New Loss function under extension of Jeffrey's prior provides the smallest values in most cases especially when loss parameter C_2 is - 1 whether the extension of Jeffrey's prior is 0.5, 1.0, 1.5 or 2.0. Moreover, when the sample size increases from 25 to 75, the MSE decreases quite significantly.

6. Conclusion

In this research article we have introduced a new class of Size biased Generalized Rayleigh distribution. Various structural and characterizing properties of this new model has been derived and studied. From the simulation study, it has been observed that when the sample size increases from 25 to 75, the Shannon's entropy, AIC and BIC values decreases quite significantly. We have primarily studied the Bayes estimator of the parameter of the Rayleigh distribution under the extended Jeffrey's prior assuming two different loss functions. The extended Jeffrey's prior gives the opportunity of covering wide spectrum of priors to get Bayes estimates of the parameter - particular cases of which are Jeffrey's prior and Hartigan's prior.We have also addressed the problem of Bayesian estimation for the Size biased Generalized Rayleigh distribution, under asymmetric and symmetric loss functions and that of Maximum Likelihood Estimation. From the results, we observe that in most cases, Bayesian Estimator under New Loss function (Al-Bayyati's Loss function) has the smallest Mean Squared Error values for both prior's i.e, Jeffrey's and an extension of Jeffrey's prior information.

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