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The Existence and Uniqueness Solution to the Development of the Diffusion Equation by Using Arbitrary Function

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Abstract:

The main objective of this paper is to consider the development of the diffusion in R^3 by using any arbitrary function a(t),in which the existence and the uniqueness theorem of the solution have been proved.

1- Introduction:

Rabab Ahamed.Shanab.et al.in [1],extend the work of Kalita et al [1], in which they solve the steady convection-diffusion equation with variable coefficients on non-uniform grid.The approach is based on using fourth order taylor series expansion to approximate the derivatives appearing in the convection diffusion equation.Then the original convection-diffusion equation is used again to replace the resulting higher order derivative terms,which leads to a higher order scheme on a compact stencil of nineteen grid points.the effectiveness of this method is seen from the fact that it can handle the singularity perturbed problems by employing a flexible discretized grid that can be adapted to the singularity in the domain.Four difficult test cases are chosen to demonstrate the accuracy of the present scheme.

In [2],the diffusion of suddenly occurring local high temperature in homogeneous half-infinite space is studied in the cases of one, two and three-dimensional half space.

In [3],the combination of the time-parallel “parallel full approximation scheme in space and time” with a parallel multigrid method in space, resulting in a mesh-based solver for the three-dimensional heat equation with a uniquely high degree of efficient concurrency.

In [4],the anthors are concerned with the numerical solution of a two-dimensional space-time fractional differential equations used to model the dynamic properties of complex systems governed by anomalous diffusion.The space-time fractional anomalous diffusion equation is defined by replacing the second order space derivatives and the first order time derivatives with Riesz and Caputo operators,respectively.

In [5],the anthors are introduce Fourier spectral methods as an attractive and easy-to-code alternative for the integration of fractional-in-space reaction diffusion equations.The main advantages of the proposed schemes is that they yield a fully diagonal representation of the fractional operator,with increased accuracy and efficiency when compared to low-order counterparts, and a completely straightforward extension for two and three spatial dimensions. The transmission of linguistic change within a speech community is characterized by incrementation within a faithfully reproduced pattern characteristic of the family tree model, while diffusion across communities shows weakening of the original pattern and a loss of structural features, It is proposed that this is the result of the difference between the learning abilities of children and adults,Evidence is drawn from two studies of geographic diffusion.

In [6],the space-time neutron diffusion equations with multi-group of delayed neutrons are a couple of the nonlinear partial differential equations, The finite difference method is used to reduce the partial differential equations into ordinary differential equations.This ordinary differential equations are rewritten in a matrix form.

In [7] and [8],the atmospheric air pollution turbulent fluxes have been assumed to be proportional to the mean concentration gradient.This assumption, along with the equation of continuity, leads to the advection-diffusion equation.Also many models simulating air pollution dispersion are based upon the solution (numerical or analytical) of the advection-diffusion equation assuming turbulence parameterization for realistic physical scenarios.

In [9],the anthors proved that the temperature distribution in the limit one – dimensional rod with time – averaged sources of heat is the uniform asymptotic approximation of the temperature distribution in the initial problem in an arbitrary sub domain of the plane rod and in an arbitrary time interval , which are located at a positive distance from the ends of the rod and the initial time instance , respectively , of course ,the temperature in the one – dimensional rod , which is a function of the longitudinal coordinate x and the time t , is identified with a function of (x,y,t),which is independent of the transversal coordinate y of the plane rod.

In [10],the anthors are obtained an asymptotic expansion, containing regular boundary corner functions in the small parameter, for the solution of a second order partial differential equation, they are constructed the asymptotic expansion $u_n(x, t, \varepsilon)$ for the modified problem and prove that it is the unique solution.also,they have proved that the solution is valid uniformly in the considered domain,and the asymptotic approximation is within $O(\varepsilon^{n+1})$.

The development of the wave equation with some conditions is considered and proved that the existence and the uniqueness of solution by using the reflection method, in [11].

In [12],a modification of an initial – boundary – value problem in the critical case for the heat – conduction equation in a thin domain have been considered in which they are justify asymptotic expansions of the solution of the problems with respect to a small parameter ε>0,also they proved that the solution is uniform in the domain and the asymptotic approximation is within $O(\varepsilon^{n+1})$.

In [13],the asymptotic first – order solution of a partial differential equation with small parameter have been constrncted,and they have proven that the solution is unique and uniform in the domain.

In this work,we study Cauchy problem for the development of the diffusion equation in R^3depending on arbitrary function a(t), also we give some applications.

2- Statement of the problem:

Let us consider the Cauchy problem for the diffusion equation in R^3

$$u_t = a(t)\Delta u = a(t)\left(u_{xx} + u_{yy} + u_{zz}\right),\ t > 0$$
$$u(P, 0) = \emptyset \qquad (2.1)$$

where P = (x,y,z) ∈ R^3 and $\emptyset(P)$ and a(t) are a given function so this problem is agen evalization to the Cauchy problem with constant coefficients considered in [14].

Proposition 2.1.

We start first with the following proposition suppose u_1(x,t), u_2(y,t) and u_3(z,t) are solutions of the one-dimensional diffusion equation $u_t - a(t)u_{ss}$=0,where s ∈ {x,y,z}.
then u(x,y,z,t) = a(t)u_1(x,t) u_2(y,t) u_3(z,t) is a solution of $u_t - a(t)\Delta u = 0$ in R^3.

Proof : If

$u(x, y, z, t) = a(t)u_1(x,t)u_2(y,t)u_3(z,t)$ is a solution,then

$$\begin{aligned}
u_t &= a(t)u_1[u_2u_{3t} + u_3u_{2t}] + u_2u_3[a(t)u_{1t} + u_1\acute{a}(t)] \\
&= a(t)u_1u_2u_{3t} + a(t)u_1u_3u_{2t} + u_2u_3a(t)u_{1t} + u_2u_3u_1\acute{a}(t) \\
&= u_2u_3a(t)u_{1t} + a(t)u_1u_3u_{2t} + a(t)u_1u_2u_{3t} + u_2u_3u_1\acute{a}(t) \\
&= a(t)\ [u_{1t}u_2u_3 + u_1u_{2t}u_3 + u_1u_2u_{3t}] + [u_1u_2u_3]\acute{a}(t) \\
&= a(t)[u_{1xx}u_2u_3 + u_1u_{2yy}u_3 + u_1u_2u_{3zz}] + [u_1u_2u_3]\acute{a}(t) \\
&= a(t)\Delta[u_1u_2u_3] + [u_1u_2u_3]\acute{a}(t) \\
&= a(t)\Delta u + [u_1u_2u_3]\acute{a}(t) \qquad \blacksquare
\end{aligned}$$

As it is known from the theory of partial differential equations.

$$G(x,t) = \frac{1}{\sqrt{a(t)}\sqrt{\pi}} e^{\frac{-x^2}{a(t)}}$$

is a fundamental solution of the diffusion equation $u_t - a(t)u_{xx} = 0$

Hence by proposition (2.1) the function:

$G_3(P,t) = G(x,t)G(y,t)G(z,t)$

$$G_3(P,t) = \frac{1}{\sqrt{a(t)}\sqrt{\pi}} e^{\frac{-x^2}{a(t)}} \times \frac{1}{\sqrt{a(t)}\sqrt{\pi}} e^{\frac{-y^2}{a(t)}} \times \frac{1}{\sqrt{a(t)}\sqrt{\pi}} e^{\frac{-z^2}{a(t)}}$$

$$= \frac{1}{\sqrt{(a(t)\pi)^3}} e^{\frac{-(x^2+y^2+z^2)}{a(t)}}$$

$$= \frac{1}{\sqrt{(a(t)\pi)^3}} e^{\frac{-|P|^2}{a(t)}}, \text{ where } P=(x,y,z)$$

is a solution of

$$u_t - a(t)\Delta u = 0, P \in R^3, t > 0 \qquad (2.2)$$

$G_3(P,t)$ is again called the fundamental solution of (2.2).observe that:

$$\int_{R^3} G_3\ (P,t)\ dP = \int_{-\infty}^{\infty} G(x,t)dx \int_{-\infty}^{\infty} G(y,t)dy \int_{-\infty}^{\infty} G(z,t)\,dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{a(t)}\sqrt{\pi}} e^{\frac{-x^2}{a(t)}} dx \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{a(t)}\sqrt{\pi}} e^{\frac{-y^2}{a(t)}} dy \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{a(t)}\sqrt{\pi}} e^{\frac{-z}{a(t)}} dz$$

$$= \frac{\sqrt{a(t)\pi}}{\sqrt{a(t)\pi}} \times \frac{\sqrt{a(t)\pi}}{\sqrt{a(t)\pi}} \times \frac{\sqrt{a(t)\pi}}{\sqrt{a(t)\pi}} = 1 \qquad (2.3)$$

The case when the initial data $\emptyset(P)$ is a function with separable variables will be considered,as:

$$\emptyset(P) = \varphi(x)\psi(y)\sigma(z) \qquad (2.4)$$

where $\varphi(x), \psi(y)$and$\sigma(z)$ are integrable functions about the variables x,y,z .

Proposition 2.2.

Suppose that $\emptyset(P)$ is a function with separable variables (2.4) , where φ, ψ and σ are bounded and continuous function.Then:

$$u(P,t) = \int_{R^3} G_3\,(P-Q,t)\emptyset(Q)dQ \qquad (2.5)$$

With $Q = (\xi, \eta, \varsigma) \in R^3$ is a solution of eq.(2.4)

Proof :

Separating the surface integral yields to

$u(P,t) = \int_{R^3} G_3(P-Q,t)\emptyset(Q)dQ$

$= \int_{-\infty}^{\infty} G(x-\xi,t)\varphi(\xi)d\xi \times \int_{-\infty}^{\infty} G(y-\eta,t)\psi(\eta)d\eta \times \int_{-\infty}^{\infty} G(z-\varsigma,t)\sigma(\varsigma)d\varsigma$

$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{a(t)}\sqrt{\pi}} e^{\frac{-(x-\xi)^2}{a(t)}} d\xi \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{a(t)}\sqrt{\pi}} e^{\frac{-(y-\eta)^2}{a(t)}} d\eta \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{a(t)}\sqrt{\pi}} e^{\frac{-(Z-\varsigma)^2}{a(t)}} d\varsigma$

$= \frac{\sqrt{a(t)\pi}}{\sqrt{a(t)\pi}} \times \frac{\sqrt{a(t)\pi}}{\sqrt{a(t)\pi}} \times \frac{\sqrt{a(t)\pi}}{\sqrt{a(t)\pi}} = 1$

$= u_1(x,t)u_2(y,t)u_3(z,t)$

By using theorem (4.7) [14], and proposition (2.1) it follows that:

$$u_t - a(t)\Delta u = 0 \text{ for } (P,t) \in R^3 \times (0,\infty)$$

and

$\lim_{t\to 0+} u(P,t) = \lim_{t\to 0+} u_1(x,t) \lim_{t\to 0+} u_2(y,t) \lim_{t\to 0+} u_3(z,t)$, Let a(t)= t

Then:

$$\lim_{t\to 0+} u_1(x,t) = \frac{1}{\sqrt{a(t)}\sqrt{\pi}} e^{\frac{-x^2}{a(t)}}$$

$$\lim_{t\to 0} u_1(x,t) = \frac{1}{\sqrt{t}\sqrt{\pi}} e^{\frac{-x^2}{t}} = \frac{1}{0} e^{\frac{-x^2}{0}} = \infty . e^{-\infty} = \infty . 0 = 0$$

$$\text{Lim}_{t\to 0+} u_2(y,t) = \frac{1}{\sqrt{a(t)}\sqrt{\pi}} e^{\frac{-y^2}{a(t)}}$$

$$\lim_{t\to 0} u_2(y,t) = \frac{1}{\sqrt{t}\sqrt{\pi}} e^{\frac{-y^2}{t}} = \frac{1}{0} e^{\frac{-y^2}{0}} = \infty . e^{-\infty} = \infty . 0 = 0$$

$$\text{Lim}_{t\to 0+} u_3(z,t) = \frac{1}{\sqrt{a(t)}\sqrt{\pi}} e^{\frac{-z^2}{a(t)}}$$

$$\lim_{t\to 0} u_3(z,t) = \frac{1}{\sqrt{t}\sqrt{\pi}} e^{\frac{-z^2}{t}} = \frac{1}{0} e^{\frac{-z^2}{0}} = \infty . e^{-\infty} = \infty . 0 = 0$$

Then:

$$\lim_{t\to 0+} u(P,t) = \lim_{t\to 0+} u_1(x,t) \lim_{t\to 0+} u_2(y,t) \lim_{t\to 0+} u_3(z,t)$$

$$= \varphi(x)\psi(y)\sigma(z) = \phi(P) \qquad \blacksquare$$

Proposition (2.2) can be extended for any initial data which is a finite linear combination of functions with separable variables of the form:

$$\emptyset_n(P) = \sum_{k=1}^{n} c_k\, \varphi_k(x)\, \psi_k(y) \sigma_k(z) \tag{2.6}$$

Let us show that any continuous and bounded function on R^3can be uniformly approximated by functions of type (2.6) on bounded domains.Now if f is a bounded function on [0,1],then we may define:

$$B_{n(x)} = \sum_{k=0}^{n} \binom{n}{a_k(t)} f\left(\frac{a_k(t)}{n}\right) x^{a_k(t)} (1-x)^{n-a_k(t)}$$

Theorem 2.1. (Bernstein):

Let $f(x) \in C[0,1]$. Then $B_n(x) \to f(x)$ uniformly for $x \in [0,1]$ as $n \to +\infty$.

Proposition 2.3.

Let $\emptyset(P) \in C([0,1]^3)$, $|\emptyset(P)| \le M$ and $\varepsilon > 0$. there exists a function with separable variables $\emptyset_n(P) \in C([0,1]^3)$,such that: $|\emptyset_n(P)| \le M$ and $|\emptyset(P) - \emptyset_n(P)| < \varepsilon$ if $P \in [0,1]^3$.

Proof :

Let $\varepsilon > 0$. then by theorem (2.1)there exists n, such that:

$$\left|\emptyset(x,y,z) - \sum_{k=0}^{n} \binom{n}{a_k(t)} \emptyset\left(\frac{a_k(t)}{n}, y, z\right) x^{a_k(t)} (1-x)^{n-a_k(t)}\right| < \frac{\varepsilon}{2}$$

for (x,y,z)$\in [0,1]^3$.

By the same way there exists n_k , such that:

$$\left|\emptyset\left(\frac{a_k(t)}{n}, y, z\right) - \sum_{m=0}^{n_k} \binom{n_k}{m} \emptyset\left(\frac{a_k(t)}{n}, \frac{m}{n_k}, z\right) y^m (1-y)^{n_k-m}\right| < \frac{\varepsilon}{2} .$$

for (y,z) $\in [0,1]^2$.

Let $\emptyset_n(x,y,z) = \sum_{k=0}^{n} \sum_{m=0}^{n_k} \binom{n}{a_k(t)} \binom{n_k}{m}$

$$\emptyset\left(\frac{a_k(t)}{n}, \frac{m}{n_k}, z\right) x^{a_k(t)} (1-x)^{n-a_k(t)} y^m (1-y)^{n_k-m}$$

We have that $\emptyset_n(x,y,z) \in C([0,1]^3)$ is a function with separable variables and

$|\emptyset(x,y,z)-\emptyset_n(x,y,z)| \le \left|\emptyset(x,y,z)-\sum_{k=0}^{n}\binom{n}{a_k(t)}\emptyset\left(\frac{a_k(t)}{n},y,z\right)x^{a_k(t)}(1-x)^{n-a_k(t)}\right| + \sum_{k=0}^{n}\binom{n}{a_k(t)}x^{a_k(t)}(1-x)^{n-a_k(t)}\left|\emptyset\left(\frac{a_k(t)}{n},y,z\right)-\sum_{m=0}^{n_k}\binom{n_k}{m}\emptyset\left(\frac{a_k(t)}{n},\frac{m}{n_k},z\right)y^m(1-y)^{n_k-m}\right| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}\sum_{k=0}^{n}\binom{n}{a_k(t)}x^{a_k(t)}(1-x)^{n-a_k(t)} = \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

By the construction of $\emptyset_n(x,y,z)$ we have $|\emptyset_n(x,y,z)| \le M$ ■

It is remark able that if we let R> 0, they rescaling the variables (x,y,z) with $\left(\frac{x+R}{2R},\frac{y+R}{2R},\frac{z+R}{2R}\right)$ and one can prove that every function $\phi(x,y,z) \in C([-R,R]^3)$ can be uniformly approximated by a function $\emptyset_n(x,y,z) \in C([-R,R]^3)$ with separable variables.

Theorem 2.2.

Suppose $\phi(P) \in C(R^3) \cap L^\infty(R^3)$. then the function

$$u(P,t) = \int_{R^3} G_3(P-Q,t)\phi(Q)dQ$$

is a solution of the diffusion equation (2.2) on R^3 and

$$\lim_{t\to 0+} u(P,0) = \emptyset(P) \qquad (2.7)$$

on a uniformly bounded subset of R^3

Proof :

By Proposition (2.2) it follows that u(P,t) satisfies (2.2)

Let us show that (2.7) holds. Suppose ε > 0 and $B \subset R^3$ is a bounded set.Making the change of variables $Q = P - 2\sqrt{a(t)t}P'$,we have

$$u(p,t) = \frac{1}{\sqrt{\pi^3}}\int_{R^3} e^{-|P'|}\phi\left(P-2\sqrt{a(t)t}P'\right)dP' , \quad P=(x,y,z) \qquad (2.8)$$

where $\acute{P}=(p,q,r) \in R^3$. Let $|\emptyset(P)| \le M$, $P \in R^3$ and denote

$a(t)_R = [-R, R^3]$, $\tilde{a}(t)_R = R^3 \backslash a(t)_R$

There exist R> 0 and $\phi_n(P) \in C([-R,R]^3)$ with separable variables such that:

$$\frac{1}{\sqrt{\pi^3}}\int_{\tilde{a}(t)_R} e^{-|P'|}dP' < \frac{\varepsilon}{8M} \text{ and } B \subset [-R,R]^3 \qquad (2.9)$$

$$\rightarrow\left(\frac{1}{\sqrt{\pi^3}}\int_{\tilde{a}(t)_R} e^{-|p|^2}dp < \frac{\varepsilon}{8M},\ \frac{1}{\sqrt{\pi^3}}\int_{\tilde{a}(t)_R} e^{-|q|^2}dq < \frac{\varepsilon}{8M},\right.$$

$$\left.\rightarrow\frac{1}{\sqrt{\pi^3}}\int_{\tilde{a}(t)_R} e^{-|r|^2}dr < \frac{\varepsilon}{8M}\right)$$

$$\rightarrow|\phi(P)-\phi_n(P)| < \frac{\varepsilon}{4} \text{ if } P \in [-R,R]^3 = \qquad (2.10)$$

$$\rightarrow\left(|\phi(x)-\phi_n(x)| < \frac{\varepsilon}{4} \text{ if } x \in [-R,R]^3,\right.$$

$$\rightarrow|\phi(y)-\phi_n(y)| < \frac{\varepsilon}{4}\ if\ y \in [-R,R]^3,$$

$$\left.\rightarrow|\phi(z)-\phi_n(z)| < \frac{\varepsilon}{4}\ if\ z \in [-R,R]^3\right)$$

By continuity of $\phi(P)$ there exists δ >0 such that if t ∈ (0,δ) ,then:

$$max_{\acute{P}\in a(t)_R}\left|\emptyset\left(P-2\sqrt{a(t)t\acute{P}}\right)-\emptyset(P)\right| < \frac{\varepsilon}{4}\ for\ P \in a(t)_R = \qquad (2.11)$$

$$\left(\left|max_{p\in a(t)_R}\left|\emptyset\left(x-2\sqrt{a(t)tp}\right)-\emptyset(x)\right| < \frac{\varepsilon}{4}\ for\ x \in a(t)_R, max_{q\in a(t)_R}\left|\emptyset\left(y-2\sqrt{a(t)tq}\right)-\emptyset(y)\right| < \frac{\varepsilon}{4}\ for\ y \in a(t)_R, max_{r\in a(t)_R}\left|\emptyset\left(z-2\sqrt{a(t)tr}\right)-\emptyset(z)\right| < \frac{\varepsilon}{4}\ for\ z \in a(t)_R\right|\right) < \frac{\varepsilon}{4}\ for\ P \in a(t)_R$$

Finally for $P \in B \subset a(t)_R$ and t ∈ (0,δ), by (2.9),(2.10) and (2.11), we have

$$|u(P,t)-\emptyset(P)| \le \int_{R^3} G_3(P-Q,t)\,|\phi(Q)-\phi_n(P)|\,dQ + |\phi_n(P)-\phi(P)|$$

$=\frac{1}{\sqrt{\pi^3}}\int_{R^3} e^{-|\acute{P}|^2}\left|\emptyset\left(P-2\sqrt{a(t)t\acute{P}}\right)-\emptyset_n(P)\right|d\acute{P}+|\emptyset_n(P)-\emptyset(P)|$ $=(\frac{1}{\sqrt{\pi^3}}\int_{R^3} e^{-|p|^2}\left|\emptyset\left(x-2\sqrt{a(t)tp}\right)-\emptyset_n(x)\right|dp+|\phi_n(x)-\phi(x)|,\frac{1}{\sqrt{\pi^3}}\int_{R^3} e^{-|q|^2}\left|\emptyset\left(y-2\sqrt{a(t)tq}\right)-\emptyset_n(y)\right|dq+$

$|\phi_n(y)-\phi(y)|,\frac{1}{\sqrt{\pi^3}}\int_{R^3} e^{-|r|^2}\left|\emptyset(z-2\sqrt{a(t)tr}-\emptyset_n(z)\right|dr+|\phi_n(z)-\phi(z)|)$

$\leq\frac{2M}{\sqrt{\pi^3}}\int_{\tilde{a}(t)_R} e^{-|\acute{P}|^2}d\acute{P}+\frac{1}{\sqrt{\pi^3}}\int_{a(t)_R} e^{-|\acute{P}|^2}\left|\emptyset(P-2\sqrt{a(t)t\acute{P}})\right|-\emptyset(P)d\acute{P}$

$=(\frac{2M}{\sqrt{\pi^3}}\int_{\tilde{a}(t)_R} e^{-|p|^2}dp+\frac{1}{\sqrt{\pi^3}}\int_{a(t)_R} e^{-|p|^2}\left|\emptyset(x-2\sqrt{a(t)tp}\right|-$

$\phi(x)dp,\frac{2M}{\sqrt{\pi^3}}\int_{\tilde{a}(t)_{CR}} e^{-|q|^2}dq+\frac{1}{\sqrt{\pi^3}}\int_{a(t)_R} e^{-|q|^2}\left|\emptyset(y-2\sqrt{a(t)tq}\right|-\phi(y)dq,\frac{2M}{\sqrt{\pi^3}}\int_{\tilde{a}(t)_R} e^{-|r|^2}dr+$

$\frac{1}{\sqrt{\pi^3}}\int_{a(t)_R} e^{-|r|^2}\left|\emptyset(z-2\sqrt{a(t)tr}\right|-\phi(z)dr)+\frac{1}{\sqrt{\pi^3}}\int_{\tilde{a}(t)_R} e^{-|\acute{P}|^2}|\emptyset(P)-\emptyset_n(P)|\,d\acute{P}+|\phi_n(P)-\phi(P)|=$

$(\frac{1}{\sqrt{\pi^3}}\int_{\tilde{a}(t)_R} e^{-|p|^2}|\emptyset(x)-\emptyset_n(x)|\,dp+|\emptyset_n(x)-\emptyset(x)|,$

$\frac{1}{\sqrt{\pi^3}}\int_{\tilde{a}(t)_R} e^{-|q|^2}|\emptyset(y)-\emptyset_n(y)|\,dq+|\emptyset_n(y)-\phi(y)|,$

$\frac{1}{\sqrt{\pi^3}}\int_{\acute{a}(t)_R} e^{-|r|^2}|\emptyset(z)-\emptyset_n(z)|\,dr+|\phi_n(z)-\phi(z)|)$

$<2M.\frac{\varepsilon}{8M}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}=\varepsilon$

Which completes the proof . ■

3- Study some application about diffusion equation:

In this section, some real life problems will be considered as an illustrative examples in order to show the validity of the results of this paper.

Problem 3.1.

To solve the diffusion equation with constant dissipation:

$$\begin{cases} u_t-a(t)u_{xx}+bu=0, & for\ \ 0<x<10 \\ u(x,0)=\phi(x), \end{cases}$$

where b > 0 is a constant.

Solution:

Suppose a(t) $=t^2$, also b = 2, make the change of variables u(x,t) $=e^{-bt}v(x,t)$

Let u = $e^{-2t}v(x,t)$.then:

$u_t=-2e^{-2t}$v(x,t) + $e^{-2t}v_t$(x,t) and $t^2u_{xx}=t^2e^{-2t}v_{xx}$(x,t)

Substituting these into the PDE for u we get:

$u_t-t^2v_{xx}=0$. the initial condition for m is

v(x,0) = $e^{-2(0)}$ u(x,0) = ϕ(x) the solution for u is

u(x,t) = $e^{-2t}\int_0^{10} S(x-y,t)\,\phi(y)dy$

u(x,t)= $e^{-2t}\int_0^{10} Y\,\phi(y)dy$

where Y = $S(x-y,t)$

u(x,t) = 50 e^{-2t} Yϕ

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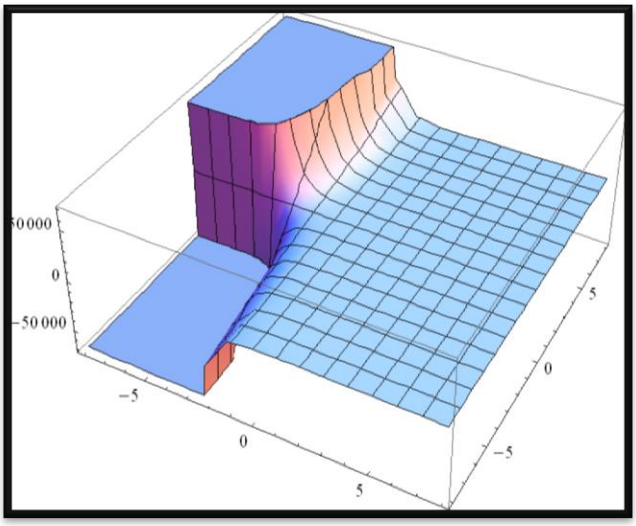


Fig(3.1)
The salution of problem (3.1)

Problem 3.2.

To solve $u_t = a(t)u_{xx}$, $u(x,0) = 0$, $u(0,t) = 1$ on the half-line $0 < x < \infty$.

Solution:

Let u be the solution of the problem and let v =u-1 then v satisfies $v_t = a(t)v_{xx}$, v(0,t) = u(0,t) – 1 = 0,
v(x,0) = u(x,0) – 1= -1 , let a(t)= $t^2 + t$
this a standard IBVP with the Dirichlet boundary condition. The Solution is v(x,t) =

$$\frac{1}{\sqrt{a(t)}\sqrt{\pi}}\int_0^\infty \left[e^{\frac{-(x-y)^2}{a(t)}} - e^{\frac{-(x+y)^2}{a(t)}}\right](-1)dy$$

$$= \frac{1}{\sqrt{t^2+t}\sqrt{\pi}}\int_0^\infty \left[e^{\frac{-(x-y)^2}{\sqrt{t^2+t}}} - e^{\frac{-(x+y)^2}{\sqrt{t^2+t}}}\right](-1)dy$$

Let (x-y) /$\sqrt{t^2+t} = p$
and (x+y)/ $\sqrt{t^2+t}$ =q then the result becomes

$$v(x,t)= \frac{1}{\sqrt{\pi}}[\int_{\frac{x}{\sqrt{t^2+t}}}^{-\infty} e^{-p^2}dp + \frac{1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{t^2+t}}}^{\infty} e^{-q^2}dq]$$

$$= \frac{1}{\sqrt{\pi}}[-\int_{-\infty}^{\frac{x}{\sqrt{t^2+t}}} e^{-p^2}dp + \int_{\frac{x}{\sqrt{t^2+t}}}^{\infty} e^{-q^2}dq]$$

$$= \frac{-1}{\sqrt{\pi}}\int_{-\infty}^{\frac{x}{\sqrt{t^2+t}}} e^{-p^2}dp + \frac{1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{t^2+t}}}^{\infty} e^{-q^2}dq$$

$$= \frac{1}{2}\left(-1 - Erf\left[\frac{x}{\sqrt{t(1+t)}}\right]\right) + \frac{1}{2}Erfc[\frac{x}{\sqrt{t(1+t)}}]$$

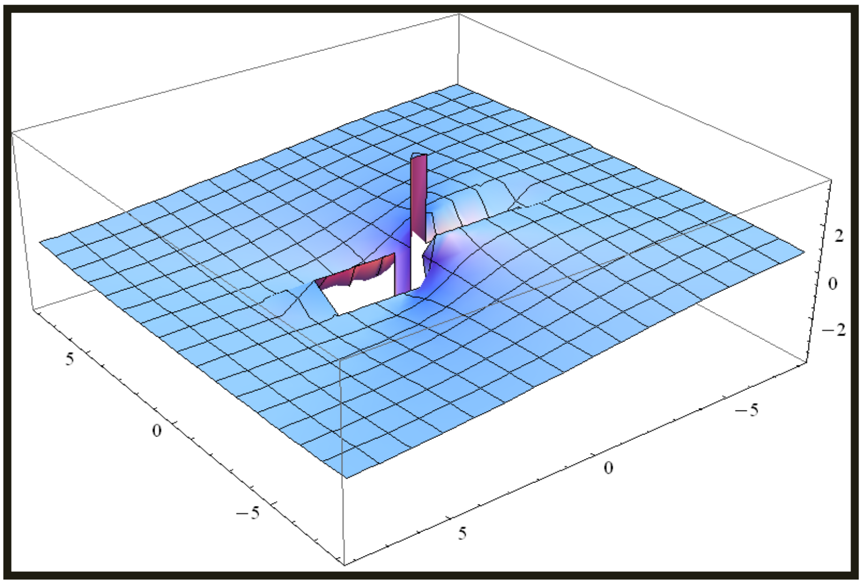


Fig (3.2)
The solution of problem (3.2)

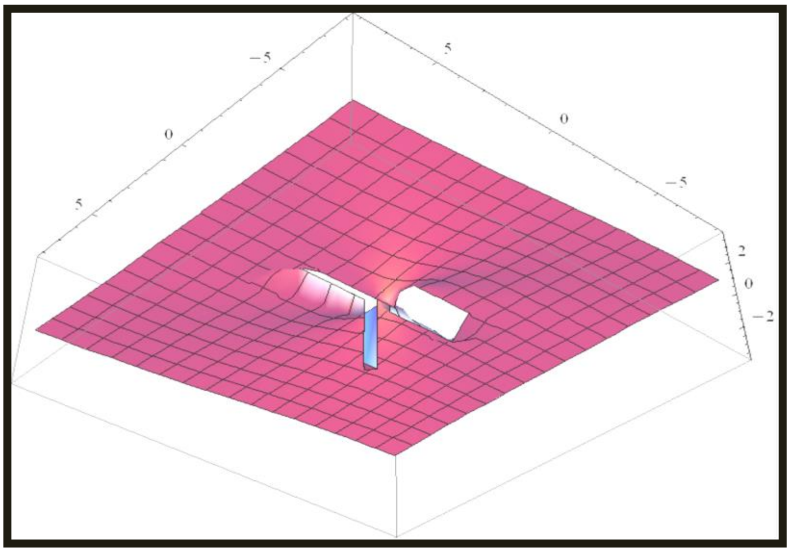


Fig (3.3)
The solution of problem (3.2)

4- <u>Fourier Method for the Diffusion Equation in Higher Dimensions:</u>

In this section, the Fourier method to the diffusion equation will be applied. which means to consider.

$$u_t = a(t)\Delta u$$

in $\Omega \times (0,\infty)$, where $\Omega \subset R^2$ is a bounded domain with standard initial and boundary conditions on $\partial \Omega$. Such BVPs are as follows:

$u_t = a(t)\Delta u \ in \ \Omega \times (0,\infty)$,
$u(x,y,0) = \phi(x,y)$ $(x,y) \in \Omega$,
$u(x,y,t) = 0$ on $\partial\Omega \times [0,\infty)$

$$\frac{\partial u}{\partial n}(x,y,t) = 0 \quad on \quad \partial\Omega \times [0,\infty)$$
$$u_n(x,y,t) + \sigma u(x,y,t) = 0$$
$$\frac{\partial u}{\partial n}(x,\mathrm{y},t) + \sigma u(x,y,t) = 0 \quad on \quad \partial\Omega \times [0,\infty). \qquad (4.1)$$

Separating variables

$$u(x,y,t) = \Phi(x,y)T(t)$$

and substituting into (4.1) we see that Φ and T must satisfy

$$a(t)\Delta u \quad \text{then } a(t) = \Delta\phi(x,y)T(t)$$
$$a(t)\Delta u(x,y,t) = \Delta\phi(x,y)T(t)$$
$$\phi(x,y)\acute{T}(t) = \Delta\phi(x,t)T(t)$$
$$\frac{\phi(x,y)\acute{T}(t)}{T(t)} = \frac{\Delta\phi(x,y)}{\phi(x,y)} = -\lambda$$
$$\frac{\acute{T}(t)}{a(t)T(t)} = \frac{\Delta\Phi(X,Y)}{\Phi(X,Y)} = -\lambda$$

Where λ is constant . this leads to the eigenvalue problem for the Laplacia

$$-\Delta\Phi = \lambda\Phi \text{ in } \Omega, \text{ Suppose } \lambda = 3,$$

With boundary condition

$$\Phi = 0 \quad \text{on} \quad \partial\Omega \qquad (4.2)$$
$$\frac{\partial\Phi}{\partial n} = 0 \quad on\ \partial\Omega \qquad (4.3)$$
$$\frac{\partial\Phi}{\partial n} + \sigma\Phi = 0 \quad on\ \partial\Omega \qquad (4.4)$$

It can be shown that for each one of the boundary conditions (4.2)-(4.4) there is an infinite sequence of eigenvalues

$$3_n \longrightarrow \infty \qquad as \quad n \longrightarrow \infty$$

and an infinite set of orthogonal eigenfunctions which is complete. Denote by Φ_n the eigenfunction corresponding to 3_n with the understanding that not all of 3_n are distinct. Solving the ODE for T(t)

$$\acute{T}(\mathrm{t}) + a(t)3_nT(t) = 0$$

We fined

$$T(t) = a_n e^{-a(t)3_n t}$$

We are looking for a solution of the form

$$u(x,y,t) = \sum_{n=1}^{\infty} A_n e^{-a_k(t)3_n t}\Phi_n(x,y)\,, \qquad (4.5)$$

which satisfies the initial condition if

$$\phi(x,y) = \sum_{n=1}^{\infty} A_n\Phi_n(x,y).$$

By the orthogonality of (Φ_n) it follows that

$$A_n = \frac{\int\int_{\Omega}\phi(x,y)\Phi_n(x,y)dxdy}{\int\int_{\Omega}\Phi_n^2(x,y)\,dxdy}. \qquad (4.6)$$

If we suppose $\phi(x,y) \in L^2(\Omega)$ it can be shown that the series (4.5) is convergent for $t > 0$ and $u(x,y,t) \longrightarrow \phi(x,y)$ **as** $t \longrightarrow 0$ in the mean-square sense in Ω.

5- <u>Statement of the Problem Using Fourier Method:</u>

This problem may be stated and solved as follows.

Problem 5.1.

To solve the equation $u_t - a(t)\Delta u = 0$, $t > 0$ by using Fourier method Where $a(t) = t^2 + 3$, $f(x) = 2x^2 + x$, $0 < x < L$ and $L = 4$ then $0 < x < 4$

Solution:

$$u(x,t) = \frac{1}{\sqrt{a(t)\pi}}\sum_{n=1}^{10} B_n e^{\frac{-x^2}{a(t)}} = \frac{1}{\sqrt{(t^2+3)\pi}}\sum_{n=1}^{10} B_n \boldsymbol{e}^{\frac{-x^2}{t^2+3}}$$

$B_n = \frac{2}{l}\int_0^l f(x)\, sin\frac{n\pi x}{l}\, dx = \frac{2}{4}\int_0^4 (2x^2 + x)\, sin\frac{n\pi x}{4}\, dx$

$B_n = \frac{76}{3} sin\frac{n\pi x}{4}$

u(x,t) = $\frac{1}{\sqrt{(t^2+3)\pi}}\sum_{n=1}^{10}\frac{76}{3} sin\frac{n\pi x}{4}\; e^{\frac{-x^2}{t^2+3}} \longrightarrow$ u(x,t) = $\frac{760 e^{\frac{-x^2}{3+t^2}} sin\frac{n\pi x}{4}}{3\sqrt{(3+t^2)\pi}}$

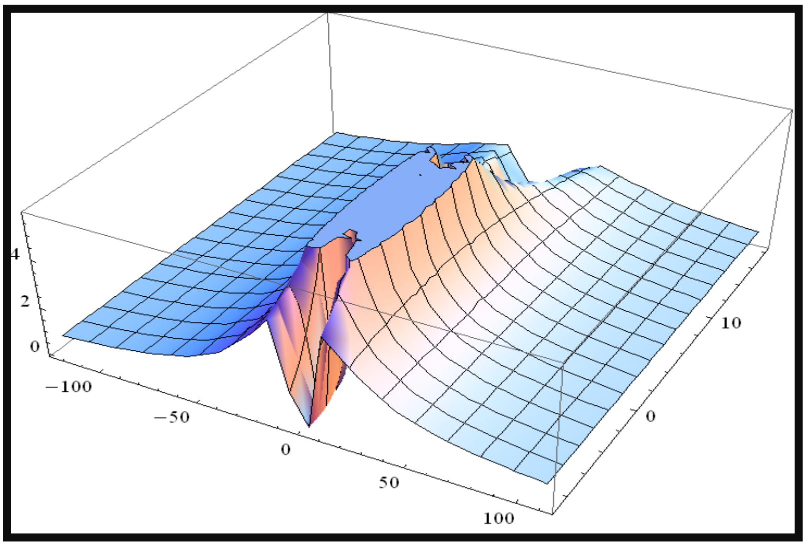


Fig (5.1)
The solution of problem (5.1)

Problem 5.2.

To solve the equation $u_t - a(t)\Delta u = 0$, t > 0 by using Fourier method where a(t) = 3 + $e^{(t+5)^2}$, f(y) = $(5 + y)^2$, 0 < y < L and L = 6 then 0 < y < 6

Solution:

u(y,t) = $\frac{1}{\sqrt{a(t)\pi}}\sum_{n=1}^{12} A_n e^{\frac{-y^2}{a(t)}}$

u(y,t) = $\frac{1}{\sqrt{(3\,+\,e^{(t+5)^2})\pi}}\sum_{n=1}^{12} A_n e^{\frac{-y^2}{(3\,+\,e^{(t+5)^2})}}$

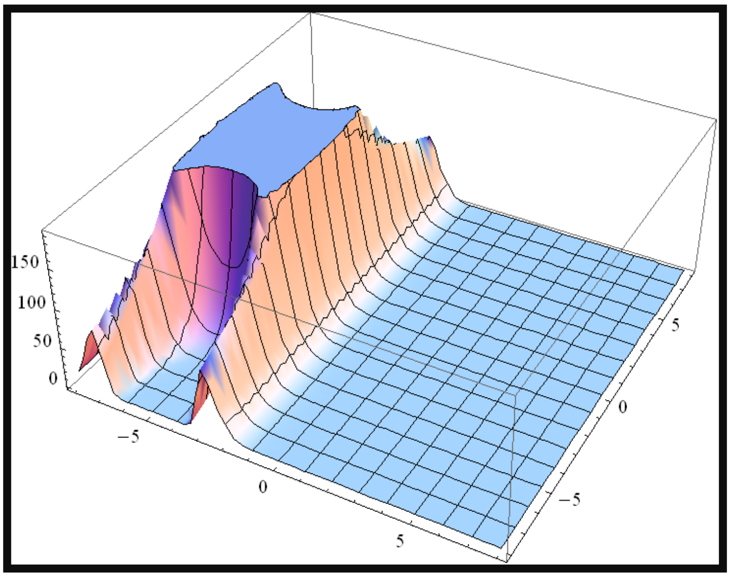
$A_n = \frac{2}{l}\int_0^l f(y)\, cos\frac{n\pi y}{l}\, dy$

$A_n = \frac{2}{6}\int_0^6 (5 + y)^2\, cos\frac{n\pi y}{6}\, dy$

$= 134\, cos\frac{n\pi y}{6}$

u(y,t)= $\frac{1}{\sqrt{\left(3\,+\,e^{(t+5)^2}\right)\pi}}\sum_{n=1}^{12} 134\, cos\frac{n\pi y}{6}\; e^{\frac{-y^2}{\left(3+e^{(t+5)^2}\right)}}$

u(y,t)= $\frac{1608 e^{\frac{-y^2}{3+e^{(5+t)^2}}} cos\frac{n\pi y}{6}}{\sqrt{(3+e^{(5+t)^2})\pi}}$



Fig(5.2)
The solution of problem (5.2)

Problem 5.3.

To solve the equation $u_t - a(t)\Delta u = 0$,t > 0 by using Fourier method where a(t) = $5 + \frac{2t^7}{12}$, f(z) = $4z + z^2$, 0 < z < L and L = 3 , 0 < z < 3

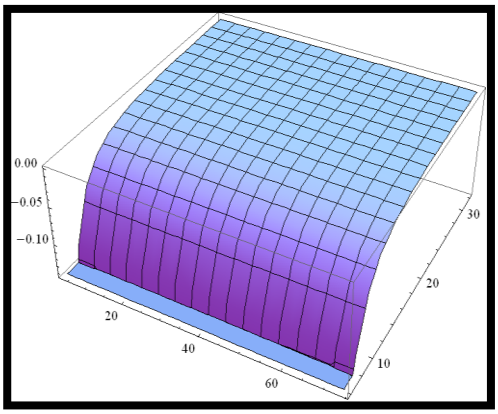
Solution:

$$u(z,t) = \frac{1}{\sqrt{a(t)\pi}} \sum_{n=1}^{6} A_n e^{\frac{-z^2}{a(t)}} = \frac{1}{\sqrt{(5+\frac{2t^7}{12})\pi}} \sum_{n=1}^{6} A_n e^{\frac{-z^2}{(5+\frac{2t^7}{12})}}$$

$$A_n = \frac{2}{l}\int_0^l f(z) \cos\frac{n\pi z}{l} dz$$

$$A_n = \frac{2}{3}\int_0^3 (4z + z^2) \cos\frac{n\pi z}{3} dz = 18 \cos\frac{n\pi z}{3}$$

$$u(z,t) = \frac{1}{\sqrt{(5+\frac{2t^7}{12})\pi}} \sum_{n=1}^{6} 18 \cos\frac{n\pi z}{3}\ e^{\frac{-z^2}{(5+\frac{2t^7}{12})}} \to u(z,t) = \frac{108 e^{\frac{-z^2}{5+\frac{t^7}{6}}} \cos\frac{n\pi z}{3}}{\sqrt{(5+\frac{t^7}{6})\pi}}$$



Fig(5.3)
The solution of problem (5.3)

References:

[1] Rabab A. S, Laila F. S and Salwa A.M, Non-uniform HOC scheme for the 3D convection– Diffusion Equation , Applied and Computational Mathematics,Vol.2, No. 3, pp.64-77, 2013 .

[2] Grigore,Cividjian and Dumitru Broscareanu,Thermal Field Diffusion in one,Two and three-Dimensional Half space,Serbian Journal of Electrical Engineering,Vol.8, No.3,pp.229-244,2011.

[3] Robert Speck,DanielRuprecht ,matthew Emmett Bolten and Rolf Krause,A Space-Time Parallel Solver for The three-Dimensional Heat Equation, Julich Supercomputin Centre, Forschungszentrum Julich,Germany ,2013.

[4] Necati Ozdemir , Derya Avcı and Beyza Billur Iskender, The Numerical Solutions of a Two-Dimensional Space-Time Riesz - Caputo Fractional Diffusion Equation, An International Journal of Optimization and Control: Theories and Applications,Vol.1, No. 1, pp.17-26, 2011.

[5] Alfonso Bueno-Orovio,David Kay and Kevin Burrage, Fourier Spectral Methods for Fractional-In-Spareaction Diffusion Equations,Preprint Submitted to Journal of Computational Physics,2012.

[6] William Labov, University of Pennsylvania Transmission and Diffusion, 2007.

[7] Abdallah A. Nahla, Faisal A. Al-Malki and Mahmoud Rokaya, Numerical Techniques for the Neutron Diffusion Equations in the Nuclear Reactors, Adv. Studies Theor. Phys, Vol. 6, No.14,pp. 649 – 664,2012.

[8] Daniela Buske, Marco Tullio Vilhena, Tiziano Tirabassi and Bardo Bodmann,Air Pollution Steady-State Advection-Diffusion Equation:The General Three-Dimensional Solution, Journal of Environmental Protection,pp.1124-1134, 2012.

[9] V. B. Levenshtam and H. D. Abood , Asymptotic Integration of The Problem on Heat Distribution in A Thin Rod Wirth Rapidly Varying Sources of Heat , Journal of Mathematical Science. Vol.129 , No.1 , 2005.

[10] Hayder Jabber Abood and Ali Abbas Jabir , Asymptotic Approximation of the Second Order Partial Differential Equation by Using Many Functions , British Journal of Science ,Vol.5, 2012.

[11] Hayder Jabber Abood and Ahmed Hadi Hussain , The Existence Solution to the Development Wave Equation With Arbitrary Conditions, Mathematical Theory and Modeling, Vol.3,No.4, 2013.

[12] Hayder Jabber Abood, On the Formation of an Asymptotic Solution of the Heat Equation With Small Parameter, European Journal of Scientific Research, Vol.56, No.4 ,pp.471-481, 2011.

[13] Hayder Jabber Abood, Estimation of Partial Differential Equations Depending on a Small Parameter , American Journal of Scientific Research , pp.32-39, 2011.

[14] Ioannis P Stavroulakis Stepan A Tersian, Partial Differential Equations, pp.255,2003.