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On the Solution of Incompressible Fluid Flow Equations: a Comparative Study

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Abstract

This paper studies and contrasts the performances of three iterative methods for computing the solution of large sparse linear systems arising in the numerical computations of incompressible Navier-Stokes equations. The emphasis is on the traditional Gauss-Seidel (GS) and Point Successive Over-relaxation (PSOR) algorithms as well as Krylov projection techniques such as Generalized Minimal Residual (GMRES). The performances of these three solvers for the second-order finite difference algebraic equations are comparatively studied by their application to solve a benchmark problem in Computational Fluid Dynamics (CFD). It is found that as the mesh size increases, GMRES gives the fastest convergence rate in terms of cpu time and number of iterations.

Keywords: viscous flows, Navier-Stokes equations, linear system, iterative, GMRES

1. Introduction

Consider the system of viscous incompressible flows governed by the unsteady Navier-Stokes equations, which are given (in non-dimensional form) as

$$\nabla . u = 0, \tag{1}$$

$$u_t + (u.\nabla)u = -\nabla p + \frac{1}{Re}\Delta u + f. \tag{2}$$

Our notation is standard: u is the fluid velocity; p is the pressure; Re is the Reynolds number and f represents a body force. The fluid is assumed to fill a two-dimensional square domain $\Omega(x,y) = [0,1] \times [0,1]$ with boundary $\partial\Omega = \Gamma_D$, where subscript D stands for Dirichlet. The flow is specified by the no-slip condition $u = g$ on Γ_D, where g is the velocity on the boundary.

In this paper we study the well known Stokes lid-driven cavity problem, i.e., the flow of an incompressible fluid within an enclosed square cavity driven by a sliding lid at constant speed (see Fig. 1). In our experiment the vertical velocity v is set to zero everywhere, whereas the horizontal velocity component u is set to unity on the lid, and is zero on the other boundaries. One interesting feature of this flow is that the pressure is singular at the top corners, i.e., where the imposed velocity is discontinuous. This flow problem has long served as a benchmark in the validation of two-dimensional numerical solution methods for incompressible flows (Ferziger & Peric 2002). It is of great interest in the CFD community since it mimics many aeronautical, environmental and industrial flows such the flow over structures in airfoils or the cooling flow over electronic devices (Povitsky 2001).

2. Procedure

We discretize $\Omega(x,y)$ with uniform and equal mesh sizes of dimension h and denote $N = 1/h$ as the number of grid points along x and y coordinates. Moreover, we use the index pair (i, j) to represent the mesh point (x_i, y_j), with $x_i = ih$ and $y_j = jh$ for $0 \leq \{i, j\} \leq N$. On such a uniform mesh, the numerical solution of (1)-(2) with second-order central-difference discretization introduces an explicit discretised Poisson-type equation for the pressure field with valid boundary conditions ($\partial p / \partial \Omega = 0$) as follows:

$$-4p_{i,j}^{n+1} + p_{i+1,j}^{n+1} + p_{i-1,j}^{n+1} + p_{i,j+1}^{n+1} + p_{i,j-1}^{n+1} = \frac{h}{\Delta t}[\bar{u}_{i,j}^{n+1} - \bar{u}_{i-1,j}^{n+1} + \bar{v}_{i,j}^{n+1} - \bar{v}_{i,j-1}^{n+1}], \tag{3}$$

where $\bar{u}$ and $\bar{v}$ are intermediate velocity components obtained by a Hodge Decomposition of (1)-(2), Δt is the time step and the superscript $n+1$ is the time level (Moreno & Ramaswamy 1997). Such type of Pressure-Poisson equation (PPE) is obtained when a projection scheme such as SIMPLE, PISO or Fractional Step Method is employed.

In literature, it has been reported that in finding a numerical solution for (1)-(2), the major computational cost is spent in the calculation part which involves (3) (Patankar 1980; Johnston & Liu 2004,). In the sequel, our aim is to compare the performances of some iterative methods to solve (3). Following the conventional process, (3) can be written as the linear system

$$Ax = b, \tag{4}$$

where the coefficient matrix A is block tri-diagonal (Fig. 2). The elements of x consist of the pressure at each point on the grid and b is obtained explicitly from the right hand side of (3).

3. Experiments

We compare the performances of three iterative methods to solve the linear system (4). The linear solvers considered here are the Gauss-Seidel, Point Successive Over Relaxation and the Krylov projection Generalized Minimal Residual (Saad & Schultz 1986) methods. The objective is to study their convergence rate and cpu time.

The convergence criteria for determining steady state is based on the difference between two successive iterates; whereby the procedure for finding the fluid velocity u is stopped when this difference, measured by L_1 norm is less than a prescribed tolerance (*tol*). The numerical experiments are performed for uniform grids with 12×12, 16×16, 20×20, 30×30 and 40×40 points for $Re = 100, 400 \text{ and } 1000$. Tables (1)-(3) show the number of iterations required (k) and the computational time (cpu) in seconds for each solver to reach steady state.

It can be observed from these tables that the GMRES method is more efficient than the GS and PSOR methods. For low Reynolds number (100 and 400), when the mesh size is above 30×30, the PSOR method is about 1.2 times faster than the GS method, while the GMRES method is more than 2.5 times faster than the PSOR method. For higher Reynolds number (1000), the GS and PSOR methods are comparable in terms of computational time required to reach steady state. However, GMRES gains over GS and PSOR by more than 1.3 times as from the 30×30 grids. It is interesting to note that for all cases studied, the speed-up of GMRES for the fine mesh is much more efficient than for the coarse mesh.

Fig. 3 displays the velocity vector and pressure fields of the lid-driven cavity flow at $Re = 100$ calculated on a 22×22 grid. It can be observed that the moving lid creates a strong vortex and a sequence of weaker vortices in the lower two corners of the cavity.

4. Conclusions

In this paper, we demonstrated that for the numerical solution of incompressible flows, the use of GMRES for solving the linear PPE is much more efficient than the traditional GS and PSOR schemes. In practice, one always uses a fine mesh to obtain a better accuracy of the required solution. Following the conventional process, we observed that for large grid size GMRES converges faster than other methods. Moreover, the use of GMRES provides very accurate numerical results by using very little cpu time and virtual storage.

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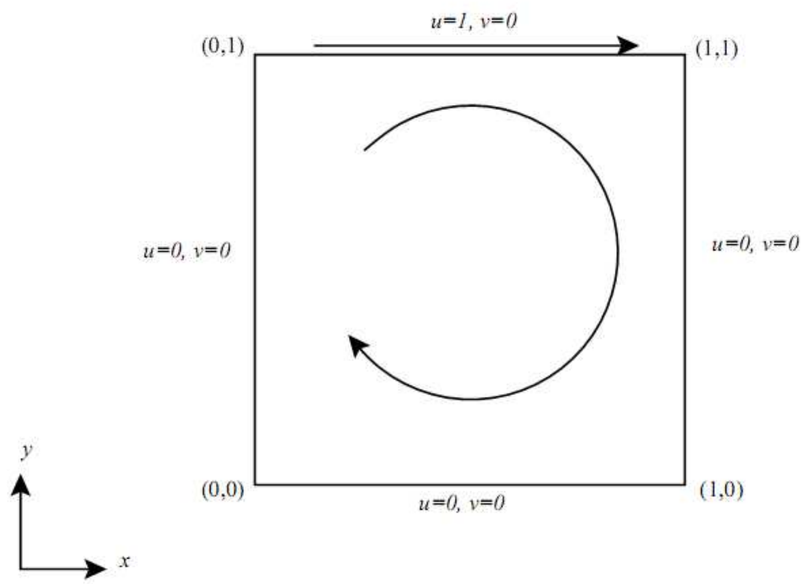


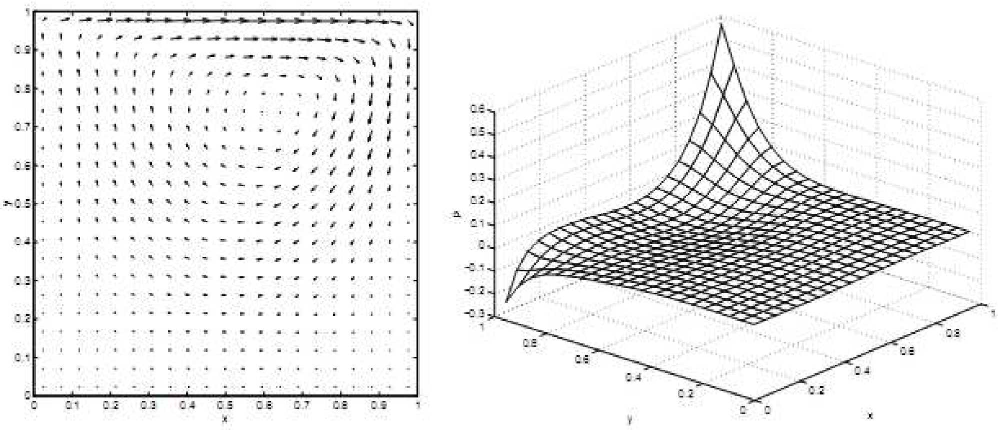
Figure 1. Geometry of the lid-driven cavity problem in the domain $\Omega(x, y)$

$$A = \begin{bmatrix} A_0 & I & & & & \\ I & A_1 & I & & & \\ & I & A_1 & I & & \\ & & \ddots & \ddots & \ddots & \\ & & & I & A_1 & I \\ & & & & I & A_0 \end{bmatrix}$$

with

$$\mathbb{R}^{N\times N} \ni A_0 = \begin{bmatrix} -2 & 1 & & & & \\ 1 & -3 & 1 & & & \\ & 1 & -3 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -3 & 1 \\ & & & & 1 & -2 \end{bmatrix} \text{ and } \mathbb{R}^{N\times N} \ni A_1 = \begin{bmatrix} -3 & 1 & & & & \\ 1 & -4 & 1 & & & \\ & 1 & -4 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -4 & 1 \\ & & & & 1 & -3 \end{bmatrix}.$$

Figure 2. The coefficient matrix A of the linear system (4)



(a) Flow field (b) Pressure field

Figure 3. Flow and pressure fields at $Re = 100$ calculated on a 22×22 grid

Table 1. Performance of three iterative methods for PPE equation (Re = 100, tol = 10^{-6})

Grids	GS		PSOR		GMRES	
	k	cpu	*k*	cpu	*k*	cpu
12×12	310	5.5	317	5.0	310	9.5
16×16	433	20.4	444	17.7	434	19.4
20×20	562	57.5	574	49.6	562	38.4
30×30	1023	420.3	1032	362.7	1025	173.1
40×40	1862	1946.7	1869	1687.6	1863	557.6

Table 2. Performance of three iterative methods for PPE equation (Re = 400, tol = 10^{-6})

Grids	GS		PSOR		GMRES	
	k	cpu	*k*	cpu	*k*	Cpu
12×12	733	8.6	727	7.7	723	21.5
16×16	1011	31.9	1004	28.1	1003	42.2
20×20	1305	91.6	1300	81.6	1305	80.2
30×30	2065	617.9	2048	545.5	2046	306.3
40×40	2794	2375.5	2788	2117.8	2790	771.4

Table 3. Performance of three iterative methods for PPE equation (*Re* = 1000, *tol* = 10^{-6})

Grids	GS		PSOR		GMRES	
	k	cpu	*k*	cpu	*k*	Cpu
12×12	1249	13.0	1270	11.9	1247	37.5
16×16	1712	44.4	1705	40.8	1705	72.2
20×20	2245	118.0	2280	110.9	2231	133.0
30×30	3701	721.0	3640	679.9	3643	503.1
40×40	5327	2819.9	5231	2748.5	5225	1323.6