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Common Fixed Points of Weakly Reciprocally Continuous Maps using a Gauge Function

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Abstract

The aim of the present paper is to obtain a common fixed point theorem by employing the recently introduced notion of weak reciprocal continuity. We demonstrate that weak reciprocal continuity ensures the existence of fixed points under contractive conditions which otherwise do not ensure the existence of fixed points. Our result generalize and extend several well-known fixed point theorems due to Boyd and Wong (1969), Jungck(1976), Pant (1994) and Pathak et al (1997).

Keywords: Fixed point theorems, compatible maps, A-compatible maps, T-compatible maps, reciprocal continuity, weak reciprocal continuity

1. Introduction

The question of continuity of contractive maps in general and of continuity at fixed points in particular emerged with the publication of two research papers by Kannan (1968, 1969) in 1968 and 1969 respectively. These two papers generated unprecedented interest in the fixed point theory of contractive maps which, in turn, resulted in vigorous research activity on the existence of fixed points of contractive maps and the question of continuity of contractive maps at their fixed points turned into an open question.

In (1998), 30 years after Kannan's celebrated papers, Pant (1998) introduced the notion of reciprocal continuity for a pair of mappings and as an application of this concept obtained the first fixed point theorem, in which the common fixed point was a point of discontinuity.

Definition 1.1

Two self mappings A and T of a metric space (X, d) are defined to be reciprocally continuous iff $lim_n ATx_n = At$ and $lim_n TAx_n = Tt$, whenever $\{x_n\}$ is a sequence in X such that $lim_n Ax_n = lim_n Tx_n = t$ for some t in X.

The notion of reciprocal continuity has been employed by many researchers in diverse settings to establish fixed point theorems which admit discontinuity at the fixed point. Imdad *et al* (2009) used this concept in the setting of non-self mappings. Singh and Mishra (2002) have used reciprocal continuity to establish general fixed point theorems for hybrid pairs of single valued and multi-valued maps. P.Balasubramaniam *et al* (2002) extended the study of reciprocal continuity to fuzzy metric spaces. Suneel Kumar *et al* (2008) studied this concept in the setting of probabilistic metric space. S. Murlishankar *et al* (2009) established a common fixed point theorem in an intuitionistic fuzzy metric space using contractive condition of integral type. Chugh *et al* (2003) and Kumar *et al* (2002) have, in the setting of metric spaces, obtained interesting fixed point theorems which do not force the map to be continuous at the fixed point.

The notion of reciprocal continuity is mainly applicable to compatible mapping satisfying contractive conditions. To widen the scope of the study of fixed points from the class of compatible mappings satisfying contractive conditions to a wider class including compatible as well as noncompatible mappings satisfying contractive, nonexpansive or Lipschitz type condition Pant *et al*(2011) generalized the notion of reciprocal continuity by introducing the new concept of weak reciprocal continuity as follows:

Definition 1.2
Two self mappings A and T of a metric space (X, d) are defined to be weak reciprocally continuous iff $lim_n ATx_n = At$ or $lim_n TAx_n = Tt$, whenever $\{x_n\}$ is a sequence in X such that $lim_n Ax_n = lim_n Tx_n = t$ for some t in X.
Jungck(1986) generalized the notion of weakly commuting maps by introducing the concept of compatible maps.
Definition 1.3
Two self mappings A and T of a metric space (X, d) are compatible iff $lim_n\, d(ATx_n, TAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $lim_n Ax_n = lim_n Tx_n = t$ for some t in X.

In (1993), Jungck et al (1993) further generalized the concept of weakly commuting mappings by introducing the notion of compatible of type (A).

Definition 1.4

Two self mappings A and T of a metric space (X, d) are compatible of type (A) iff $lim_n\, d(AAx_n, TAx_n) = 0$ and $lim_n\, d(ATx_n, TTx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $lim_n Ax_n = lim_n Tx_n = t$ for some t in X.

In (1998), Pathak et al (1998) generalized the notion of compatibility of type (A) by introducing the two analogous notions of compatibility, i.e., A-compatible and T-compatible.

Definition1.5

Two self mappings A and T of a metric space (X, d) are A-compatible iff $lim_n\, d(ATx_n, TTx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $lim_n Ax_n = lim_n Tx_n = t$ for some t in X.

Definition 1.6

Two self mappings A and T of a metric space (X, d) are T-compatible iff $lim_n d(AAx_n, TAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $lim_n Ax_n = lim_n Tx_n = t$ for some t in X.

It may be noted that the notions compatible and A-comaptible (or T-compatible, compatible mappings of type *(A)*) are independent to each other. If both A and T are continuous, then all the analogous notions of compatibility including compatibility are equivalent to each other.

As an application of weak reciprocal continuity we prove a common fixed point theorem for a $\emptyset$ –contractive condition that extend the scope of the study of common fixed point theorems from the class of compatible or analogous compatible continuous mappings to a wider class of mappings which also includes discontinuous mappings.

1. Main Results

Theorem 2.1
Let A and T be weakly reciprocally continuous self mappings of a complete metric space (X, d) satisfying
(i) $AX \subset TX$
(ii) $d(Ax, Ay) \leq \emptyset\,(max\{ d(Tx, Ty), d(Ax, Tx), d(Ay,Ty), [d(Ax, Ty) + d(Ay, Tx)]/2 \})$,
where $\emptyset : R_+ \rightarrow R_+$ denotes an upper semi continuous function such that $\emptyset$ (t) < t for each $t > 0$.
If A and T are either compatible or A-compatible or T-compatible then A and T have a unique common fixed point.

Proof.

Let x_0 be any point in X. Define sequences $\{x_n\}$ and $\{y_n\}$ in X given by the rule

$y_n = Ax_n = Tx_{n+1}$.

This can be done since $AX \subset TX$. Then using (ii) we obtain

$$d(y_n, y_{n+1}) = d(Ax_n, Ax_{n+1}) \le \phi(max\{d(Tx_n, Tx_{n+1}), d(Ax_n, Tx_n), d(Ax_{n+1}, Tx_{n+1}), [d(Ax_n, Tx_{n+1}) + d(Ax_{n+1}, Tx_n)]/2\}).$$
$$= \phi(d(y_{n-1},y_n) < d(y_{n-1},y_n).$$

Thus $d(y_n, y_{n+1}) \le (d(y_{n-1}, y_n) < d(y_{n-1},y_n)$.

(2.1)

Similarly, $d(y_{n-1}, y_n) \le (d(y_{n-2}, y_{n-1}) < d(y_{n-2}, y_{n-1})$.

(2.2)

Thus we see that $\{d(y_n,y_{n+1})\}$ isstrictly decreasing sequence of positive numbers and hence tends to a limit $r \ge 0$.

Suppose $r > 0$, then relation (2.1) on making $n \to \infty$ and in view of upper semi continuity of ϕ yields $r \le \phi(r) < r$, a contradiction. Hence $r = lim_{n \to\infty} d(y_n, y_{n+1}) = 0$. We claim $\{y_n\}$ is a Cauchy sequence. Suppose it is not. Then there exists an $\varepsilon > 0$ and a subsequence $\{y_{ni}\}$ of $\{y_n\}$ such that $d(y_{ni}, y_{ni+1}) > 2\varepsilon$. Since $lim_n d(y_n, y_{n+1}) = 0$,there exists integers m_i satisfying $n_i < m_i < n_{i+1}$ such that

$d(y_{ni}, y_{mi}) \ge \varepsilon$. If not then

$d(y_{ni}, y_{ni+1}) \le d(y_{ni}, y_{ni+1} - 1) + d(y_{ni+1} - 1, y_{ni+1}) < \varepsilon + d(y_{ni+1} - 1, y_{ni+1}) < 2\varepsilon$,

a contradiction. If m_i be the smallest integer such that $d(y_{ni}, y_{mi}) > \varepsilon$, then

$$\varepsilon \le d(y_{ni}, y_{mi}) < d(y_{ni}, y_{mi-2}) + d(y_{mi-2}, y_{mi-1}) + d(y_{mi-1}, y_{mi})$$
$$< \varepsilon + d(y_{mi-2}, y_{mi-1}) + d(y_{mi-1}, y_{mi}),$$

that is, there exists integer m_i satisfying $n_i < m_i < n_{i+1}$ such that

$$d(y_{ni}, y_{mi}) \ge \varepsilon \text{ and } lim_n\, d(y_{ni}, y_{mi}) = \varepsilon, \quad (2.3)$$

without loss of generality we can assume that n_i is odd and m_i is even. Now by virtue of (1), we have

$$d(y_{ni+1}, y_{mi+1}) \le \phi\ ((d(y_{ni}, y_{mi}) + d(y_{ni}, y_{ni+1})).$$

Now on letting $n_i \to \infty$ and in view of (2.3) and upper semi continuity of ϕ, the above relation yields $\varepsilon \le \phi(\varepsilon) < \varepsilon$, a contradiction. Hence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a point t in X such that $y_n \to u$ as $n \to \infty$. Moreover,

$$y_n = Ax_n = Tx_{n+1} \to u. \quad (2.4)$$

Suppose that A and T are compatible mappings. Now, weak reciprocal continuity of A and T implies that $lim_n ATx_n = Au$ or $lim_n TAx_n = Tu$. Let $lim_n TAx_n = Tu$. Then compatibility of A and T yields $lim_n d(ATx_n, TAx_n) = 0$, i.e., $lim_n ATx_n = Tu$. By virtue of (2.4) this yields $lim_n ATx_{n+1} = lim_n AAx_n = Tu$. If $Au \ne Tu$ then using (ii) we get $d(Au, AAx_n) \le \phi\ (max\{d(Tu, TAx_n), d(Au, Tu), d(AAx_n, TAx_n), [d(Au, TAx_n) + d(AAx_n, Tu)]/2\}$. On letting $n \to \infty$ we get $d(Au, Tu) \le \phi\ (d(Au, Tu)) < d(Au, Tu)$, a contradiction. Hence $Au = Tu$. Again compatibility of A and T implies commutativity at coincidence points. Hence $ATu = TAu = AAu = TTu$. Further, if $Au \ne AAu$ then in view of (ii), we get $d(Au, AAu) \le \phi\ (max\{d(Tu, TAu), d(Au, Tu), d(AAu, TAu), [d(Au, TAu) + d(AAu, Tu)]/2\} = \phi(d(Au, AAu)) < d(Au, AAu)$, a contradiction. Hence $Au = AAu = TAu$. Therefore, $Au = Tu$ is a common fixed point of A and T.

Next suppose that $lim_n ATx_n = Au$. Then $AX \subset TX$ implies that $Au = Tv$ for some v in X and $lim_n ATx_n = Au = Tv$. Compatibility of A and T implies, $lim_n TAx_n = Tv$. By virtue of (2.4) this also yields $lim_n ATx_{n+1} = lim_n AAx_n = Tv$. If $Av \ne Tv$ then using (ii) we get $d(Av, AAx_n) \le \phi\ (max\{d(Tv, TAx_n), d(Av, Tv), d(AAx_n, TAx_n), [d(Av, TAx_n) + d(AAx_n, Tv)]/2\}$. On *letting* $n \to \infty$ we get $d(Av, Tv) \le \phi\ (d(Av, Tv)) < d(Av, Tv)$, a contradiction. Hence $Av = Tv$. Compatibility of A and T implies commutativity at coincidence points. Hence $ATv = TAv = AAv = TTv$. Further, if $Av \ne AAv$ then in view of (ii), we get $d(Av, AAv) \le \phi\ (d(Av, AAv)) < d(Av, AAv)$, a contradiction. Hence $Av = AAv = TAv$. Therefore, $Av = Tv$ is a common fixed point of A and T.

Now, suppose that A and T are A-compatible mappings. Weak reciprocal continuity of A and T implies that $lim_n ATx_n = Au$ or $lim_n TAx_n = Tu$. Let $lim_n TAx_n = Tu$. Then A-compatibility of A and T yields $lim_n d(ATx_n, TTx_n) = 0$. By virtue of $\lim_n TTx_{n+1} = lim_n TAx_n = Tu$ this yields $lim_n ATx_n = Tu$. If $Au \neq Tu$ then using (ii) we get $d(Au, ATx_n) \leq \emptyset(max\{d(Tu, TTx_n), d(Au, Tu), d(ATx_n, TTx_n), [d(Au, TTx_n) + d(ATx_n, Tu)]/2\}$. On letting $n \to \infty$ we get $d(Au, Tu) \leq \emptyset(d(Au, Tu)) < d(Au, Tu)$, a contradiction. Hence $Au=Tu$. Again A- compatibility implies commutativity at coincidence points. Hence $ATu=TAu=AAu=TTu$. Further, if $Au \neq AAu$ then in view of (ii), we get $d(Au, AAu) \leq \emptyset(max\{d(Tu, TAu), d(Au, Tu), d(AAu, TAu), [d(Au, TAu) + d(AAu, Tu)]/2\} = \emptyset(d(Au, AAu)) < d(Au, AAu)$, a contradiction. Hence $Au = AAu = TAu$. Therefore, $Au = Tu$ is a common fixed point of A and T.

Next suppose *that* $lim_n ATx_n = Au$. Then $AX \subset TX$ implies that $Au=Tv$ for some v in X and $lim_n ATx_n = Au = Tv$. A-Compatibility of A and T implies, $lim_n TTx_n = Tv$. If $Av \neq Tv$ then using (ii) we get $d(Av, ATx_n) \leq \emptyset(max\{d(Tv, TTx_n), d(Av, Tv), d(ATx_n, TTx_n), [d(Av, TTx_n) + d(ATx_n, Tv)]/2\}$. On letting $n \to \infty$ we get $d(Av, Tv) \leq \emptyset(d(Av, Tv)) < d(Av, Tv)$, a contradiction. Hence $Av=Tv$. A-Compatibility implies commutativity at coincidence points. Hence $ATv=TAv=AAv=TTv$. Further, if $Av \neq AAv$ then in view of (ii), we get $d(Av, AAv) \leq \emptyset(d(Av, AAv)) < d(Av, AAv)$, a contradiction. Hence $Av = AAv = TAv$. Therefore, $Av = Tv$ is a common fixed point of A and T.

Finally, suppose that A and T are T-compatible mappings. Now, weak reciprocal continuity of A and T implies that $lim_n ATx_n = Au$ or $lim_n TAx_n = Tu$. Let $lim_n TAx_n = Tu$. Then T-compatibility of A and T yields $lim_n d(TAx_n, AAx_n) = 0$, *i.e.*, $lim_n AAx_n = Tu$. If $Au \neq Tu$ then using (ii) we *get* $d(Au, AAx_n) \leq \emptyset(max\{d(Tu, TAx_n), d(Au, Tu), d(AAx_n, TAx_n), [d(Au, TAx_n) + d(AAx_n, Tu)]/2\}$. On letting $n \to \infty$ we get $d(Au, Tu) \leq \emptyset(d(Au, Tu)) < d(Au, Tu)$, a contradiction. Hence $Au=Tu$. Again T- compatibility implies commutativity at coincidence points. Hence $ATu=TAu=AAu=TTu$. Further, if $Au \neq AAu$ then in view of (ii), we get $d(Au, AAu) \leq \emptyset(max\{d(Tu, TAu), d(Au, Tu), d(AAu, TA), [d(Au, TAu) + d(AAu, Tu)]/2\} = \emptyset(d(Au, AAu)) < d(Au, AAu)$, a contradiction. Hence $Au = AAu = TAu$. Therefore, $Au = Tu$ is a common fixed point of A and T.

Next suppose *that* $lim_n ATx_n = Au$. Then $AX \subset TX$ implies that $Au=Tv$ for some v in X and $lim_n ATx_n = Au = Tv$. By virtue of (2.4) this yields $lim_n ATx_{n+1} = lim_n AAx_n = Au = Tv$. T-Compatibility of A and T implies, $lim_n TAx_n = Tv$. If $Av \neq Tv$ then using (ii) we get $d(Av, AAx_n) \leq \emptyset(max\{d(Tv, TAx_n), d(Av, Tv), d(AAx_n, TAx_n), [d(Av, TAx_n) + d(AAx_n, Tv)]/2\}$. On letting $n \to \infty$ we get $d(Av, Tv) \leq \emptyset(d(Av, Tv)) < d(Av, Tv)$, a contradiction. Hence $Av=Tv$. T-Compatibility implies commutativity at coincidence points. Hence $ATv=TAv=AAv=TTv$. Further, if $Av \neq AAv$ then in view of (ii), we get $d(Av, AAv) \leq \emptyset(d(Av, AAv)) < d(Av, AAv)$, a contradiction. Hence $Av = AAv = TAv$. Therefore, $Av = Tv$ is a common fixed point of A and T.
Uniqueness of the common fixed point theorem follows easily in each of the three cases.
We now furnish an example to illustrate Theorems 2.1.

Example 2.1
Let $X = [2, 20]$ and d be the usual metric on X. Define A and $T: X \to X$ as follows
$Ax = 2$ if $x = 2$ or $x > 5$, $Ax = 6$ if $2 < x \leq 5$,
$Tx = 2$, $Tx = 12$ if $2 < x \leq 5$, $Tx = x - 3$ if $x > 5$.
Then A and T satisfy all the conditions of Theorem 2.1 and have a common fixed point at $x = 2$. It can be verified in this example that the mappings A and T are T-compatible. It can also be noted that A and T are weakly reciprocally continuous. To see this, let $\{x_n\}$ be a sequence in X such that $fx_n \to t$, $gx_n \to t$ for some t. Then $t=2$ and either $\{x_n\}=2$ for each n or $\{x_n\}=5+\varepsilon_n$ where $\varepsilon \to 0$ as $n \to \infty$. If $x_n=2$ for each n, $ATx_n \to 2 = A2$ and $TAx_n \to 2 = T2$. If $x_n = 5+\varepsilon_n$, then $Ax_n \to 2$, $Tx_n \to 2$, $ATx_n \to 6 \neq A2$ and $TAx_n \to 2 = T2$. Thus, $\lim_n TAx_n = T2$ but $lim_n ATx_n \neq A2$. Hence A and T are weakly reciprocally continuous. It is also obvious that A and T are not reciprocally continuous mappings.

Remark 2.1
Theorem 2.1 contains proper generalizations of many important fixed point theorems, we mention only those due to Boyd and Wong (1969), Jungck (1976), Pant (1994) and Pathak et al (1997).

As a direct consequence of the above theorem we get the following corollary.

Corollary 2.1
Let A and T be reciprocally continuous self mappings of a complete metric space (X, d) satisfying
(i) $AX \subset TX$
(ii) $d(Ax, Ay) \leq \emptyset(\max\{ d(Tx, Ty), d(Ax, Tx), d(Ay,Ty), [d(Ax, Ty) + d(Ay, Tx)]/2 \})$,
where, $\emptyset: R_+ \to R_+$ denotes an upper semi continuous function such that $\emptyset(t) < t$ for each $t > 0$.
If A and T are either compatible or A-compatible or T-compatible then A and T have a unique common fixed point.

If we let $\emptyset(t) = kt, o \leq k < 1$, then we get the following corollaries:

Corollary 2.2
Let A and T be weakly reciprocally continuous self mappings of a complete metric space (X, d) satisfying
(i) $AX \subset TX$
(ii) $d(Ax, Ay) \leq k(\max\{ d(Tx, Ty), d(Ax, Tx), d(Ay,Ty), [d(Ax, Ty) + d(Ay, Tx)]/2 \}), o \leq k < 1$,
If A and T are either compatible or A-compatible or T-compatible then A and T have a unique common fixed point.

Corollary 2.3
Let A and T be reciprocally continuous self mappings of a complete metric space (X, d) satisfying
(i) $AX \subset TX$
(ii) $d(Ax, Ay) \leq k(\max\{ d(Tx, Ty), d(Ax, Tx), d(Ay,Ty), [d(Ax, Ty) + d(Ay, Tx)]/2 \}), o \leq k < 1$,
If A and T are either compatible or A-compatible or T-compatible then A and T have a unique common fixed point.

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