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# A Focus on a Common Fixed Point Theorem using Weakly Compatible Mappings

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**Abstract**

The purpose of this paper is to present a common fixed point theorem in a metric space which generalizes the result of Bijendra Singh and M.S.Chauhan using the weaker conditions such as Weakly compatible mappings and Associated sequence in place of compatibility and completeness of the metric space. More over the condition of continuity of any one of the mapping is being dropped.

**Keywords:** Fixed point, Self maps, , compatible mappings, weakly compatible mappings , associated sequence.

## 1 . Introduction

G.Jungck [1] introduced the concept of compatible maps which is weaker than weakly commuting maps. After wards Jungck and Rhoades[4] defined weaker class of maps known as weakly compatible maps.

### 1.1 Definitions and Preliminaries

#### 1.1.1 Compatible mappings

Two self maps S and T of a metric space (X,d) are said to be compatible mappings if $\lim_{n\to\infty} d(STx_n,TSx_n)=0$, whenever $<x_n>$ is a sequence in X such that $\lim_{n\to\infty} Sx_n=\lim_{n\to\infty} Tx_n= t$ for some $t\in X$.

#### 1.1. 2 Weakly compatible

Two self maps S and T of a metric space (X,d) are said to be weakly compatible if they commute at their coincidence point. i.e if Su=Tu for some u∈X then STu=TSu.

It is clear that every compatible pair is weakly compatible but its converse need not be true.

To know the relation among commutativity,compatibility and weakly compatibility refer some of the research papers like Jungck,Pant,B.Fisher,Srinivas and others.

Bijendra Singh and M.S.Chauhan proved the following theorem.

1.1.3 **Theorem (A):** Let A,B,S and T be self mappings from a complete metric space (X,d) into itself satisfying the following conditions

$A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$ …… .(1.1.4)

One of A,B,S or T is continuous ……………(1.1.5)

$$[d(Ax,By)]^2 \le k_1[d(Ax,Sx)d(By,Ty)+d(By,Sx)d(Ax,Ty)] + k_2[d(Ax,Sx)d(Ax,Ty)+d(By,Ty)d(By,Sx)] \quad \text{…… (1.1.6)}$$

where $0 \le k_1 + 2k_2 < 1, k_1, k_2 \ge 0$

The pairs (A,S) and (B,T) are compatible on X ……….(1.1.7)

Further if

X is a complete metric space …..(1.1.8)

Then A,B,S and T have a unique common fixed point in X.

Now we generalize the theorem using is weakly compatible mappings and Associated Sequence.

**1.1.9 Associated Sequence:** Suppose AB,S and T are self maps of a metric space (X,d) satisfying the condition (1.1.4). Then for an arbitrary $x_0 \in X$ such that $Ax_0 = Tx_1$ and for this point $x_1$, there exist a point $x_2$ in X such that $Bx_1 = Sx_2$ and so on. Proceeding in the similar manner, we can define a sequence $<y_n>$ in X such that $y_{2n}=Ax_{2n}= Tx_{2n+1}$ and $y_{2n+1}=Bx_{2n+2} = Sx_{2n+1}$ for $n \ge 0$. We shall call this sequence as an "*Associated sequence of* $x_0$ "relative to the four self maps A,B,S and T.

Now we prove a lemma which plays an important role in our main Theorem.

**1.1.10 Lemma:** Let A,B,S and T be self mappings from a complete metric space (X,d) into itself satisfying the conditions (1.1.4) and (1.1.6) Then the associated sequence $\{y_n\}$ relative to four self maps is a Cauchy sequence in X.

**Proof:** From the conditions (1.1.4),(1.1.6) and from the definition of associated sequence we have

 

$$[d(y_{2n+1},y_{2n})]^2 = [d(Ax_{2n},Bx_{2n-1})]^2$$
$$\le k_1[d(Ax_{2n},Sx_{2n})\ d(Bx_{2n-1},Tx_{2n-1})+d(Bx_{2n-1},Sx_{2n})\ d(Ax_{2n},Tx_{2n-1}]$$
$$+k_2[d(Ax_{2n},Sx_{2n})\ d(Ax_{2n},Tx_{2n-1})+d(Bx_{2n-1},Tx_{2n-1})\ d(Bx_{2n-1}Sx_{2n})]$$

$$= k_1[d(y_{2n+1},y_{2n})\ d(y_{2n},y_{2n-1})+0]$$
$$+k_2[d(y_{2n+1},y_{2n})\ d(y_{2n+1},y_{2n-1})+0]$$

This implies

$$d(y_{2n+1},y_{2n}) \le k_1\ d(y_{2n},y_{2n-1})+k_2[d(y_{2n+1},y_{2n})+d(y_{2n},y_{2n-1})]$$
$$d(y_{2n+1},y_{2n}) \le h\ d(y_{2n},y_{2n-1})$$
$$where\ h=\frac{k_1+k_2}{1-k_2}<1$$

$For\ every\ \text{int}\ eger\ p>0, we\ get$

$$d(y_n,y_{n+p}) \le d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+.........+d(y_{n+p-1},y_{n+p})$$
$$\le h^n d(y_0,y_1)+h^{n+1}d(y_0,y_1)+..........+h^{n+p-1}d(y_0,y_1)$$
$$\le \left(h^n+h^{n+1}+..........+h^{n+p-1}\right)d(y_0,y_1)$$
$$\le h^n\left(1+h+h^2+..........+h^{p-1}\right)d(y_0,y_1)$$

Since h<1, , $h^n \to 0$ as $n\to\infty$, so that $d(y_n,y_{n+})\to 0$. This shows that the sequence $\{y_n\}$ is a Cauchy sequence in X and since X is a complete metric space, it converges to a limit, say $z\in X$.

The converse of the Lemma is not true, that is A,B,S and T are self maps of a metric space (X,d) satisfying (1.1.4) and (1.1.6), even if for $x_0\in X$ and for associated sequence of $x_0$ converges, the metric space (X,d) need not be complete. The following example establishes this.

**1.1.11. Example:** Let X=(-1,1) with d(x,y)=$|x-y|$

$$Ax=Bx=\begin{cases}\frac{1}{8} & if\ -1\le x<\frac{1}{10}\\ \frac{1}{10} & if\ \frac{1}{10}\le x<1\end{cases} \qquad Sx=Tx=\begin{cases}\frac{1}{8} & if\ -1<x<\frac{1}{10}\\ \frac{1}{5}-x & if\ \frac{1}{10}\le x<1\end{cases}$$

Then A(X) =B(X)= $\left\{\frac{1}{8},\frac{1}{10}\right\}$ while S(X) =T(X) = $\left\{\frac{1}{8}\cup\left[\frac{1}{5},\frac{-4}{5}\right)\right\}$ so that the conditions $A(X)\subset T(X)$ and $B(X)\subset S(X)$ are satisfied .Clearly (X,d) is not a complete metric space. It is easy to prove that the associated sequence $Ax_0,Bx_1Ax_2,Bx_3,..,Ax_{2n},Bx_{2n+1}....,$ converges to the point $\frac{1}{8}$ if . $-1\le x<\frac{1}{10}$ and converges to the point $\frac{1}{10}$ if $\frac{1}{10}\le x<1$,in both the cases it converges to a point in X, but X is not a complete metric space.

Now we generalize the above Theorem (A) in the following form.

**1.1.12. Theorem (B):** Let A,B,S and T are self maps of a metric space (X,d) satisfying the conditions (1.1.4), (1.1.6) and

the pairs (A,S) and (B,T) are weakly compatible ……(1.1.13)

Further

The associated sequence relative to four self maps AB, S and T such that the sequence $Ax_0,Bx_1Ax_2,Bx_3,..,Ax_{2n},Bx_{2n+1},.....$ converges to $z\in X$. as $n\to\infty$. ……..(1.1.14)

Then A,B,S and T have a unique common fixed point z in X.

**Proof:** From the condition (1.3), $B(X)\subset S(X)$ implies there exists $u\in X$ such that z=Su. We prove Au=Su.

Now consider

$$\begin{aligned}[d(Au,Bx_{2n+1})]^2 \le\ & k_1[d(Au,Su)\ d(Bx_{2n+1},Tx_{2n+1})+d(Bx_{2n+1},Su)\ d(Au,Tx_{2n+1})] \\ & +k_2[d(Au,Su)\ d(Au,Tx_{2n+1})+d(Bx_{2n+1},Tx_{2n+1})\ d(Bx_{2n+1},Su)]\end{aligned}$$

$$letting\ \ n\to\infty, Bx_{2n+1},Tx_{2n+1}\to z, then\ \ we\ get$$

$$[d(Au,z)]^2\ \le k_2\ [d(Au,z)]^2$$

$$[d(Au,z)]^2\,(1-k_2)\le 0$$

Since $0\le k_1+2k_2<1,\ where\ \ k_1,k_2\ge 0$ ,we get d(Au,z)=0,this implies Au=z.Thus Au=Su=z.

Since the pair (A,S) is weakly compatible and Au=Su=z implies ASu=SAu or Az=Sz.

Since $A(X)\subset T(X)$ implies there exists $v\in X$ such that z=Tv. We prove Bv=Tv.

Consider

$$[d(Au,Bv)]^2 \le k_1[d(Au,Su)\ d(Bv,Tv)+d(Bv,Su)\ d(Au,Tv)]$$
$$+k_2[d(Au,Su)\ d(Au,Tv)+d(Bv,Tv)\ d(Bv,Su)]$$

$\sin ce\ Au = Su = z\ and\ z = Tv\ we\ get$

$$[d(z,Bv)]^2 \le k_2[d(z,Bv)]^2$$
$$[d(z,Bv)]^2(1-k_2) \le 0 \quad and \quad \sin ce\ 0 \le k_1 + 2k_2 < 1,\ k_1, k_2 \ge 0$$

$we\ get\ d(z,Bv) = 0\ or\ Bv = z.\ Thus\ Bv = Tv = z.$

Again since the pair (B,T) is weakly compatible and Bv=Tv=z implies BTv=TBv or Bz=Tz.

Now consider

$$[d(Az,z)]^2 = [d(Az,Bv)]^2$$
$$\le k_1[d(Az,Sz)\ d(Bv,Tv)+d(Bv,Sz)\ d(Az,Tv)]$$
$$+k_2[d(Az,Sz)\ d(Au,Tv)+d(Bv,Tv)\ d(Bv,Su)]$$

$\sin ce\ Az = Sz\ and\ Bv = Tv = z,\ we\ get$

$$[d(Az,z)]^2 \le k_1[d(Az,z)]^2$$
$$[d(Az,z)]^2(1-k_1) \le 0 \quad and \quad \sin ce\ 0 \le k_1 + 2k_2 < 1,\ k_1, k_2 \ge 0$$

$we\ get\ d(Az,z) = 0\ or\ Az = z.\ Thus\ Az = Sz = z.$

Again consider

$$[d(z,Bz)]^2 = [d(Au,Bz)]^2$$
$$\le k_1[d(Au,Su)\ d(Bz,Tz)+d(Bz,Su)\ d(Au,Tz)]$$
$$+k_2[d(Au,Su)\ d(Au,Tz)+d(Bz,Tz)\ d(Bz,Su)]$$

$\sin ce\ Au = Su = Bv = Tv = z,\ Az = Sz = z\ and\ Bz = Tz,\ we\ get$

$$[d(z,Bz)]^2 \le k_1[d(z,Bz)]^2$$
$$[d(z,Bz)]^2(1-k_1) \le 0 \quad and \quad \sin ce\ 0 \le k_1 + 2k_2 < 1,\ k_1, k_2 \ge 0$$

$we\ get\ d(Bz,z) = 0\ or\ Bz = z.\ Thus\ Bz = Tz = z.$

Since Az=Bz=Sz=Tz=z,we get z in a common fixed point of A,B,S and T.The uniqueness of the fixed point can be easily proved.

**1.1.15. Remark** From the example given above, clearly the pairs (A,S) and (B,T) are weakly compatible

as they commute at coincident points $\frac{1}{8}$ and $\frac{1}{10}$. But the pairs (A,S) and (B,T) are not compatible

For this, take a sequence $x_n=\left(\frac{1}{10}+\frac{1}{10^n}\right)$ for n≥1, then $\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Sx_n=\frac{1}{10}$ and $\lim_{n\to\infty} ASx_n=\frac{1}{8}$

also $\lim_{n\to\infty} SAx_n=\frac{1}{10}$. So that $\lim_{n\to\infty} d(ASx_n,SAx_n)\neq 0$. Also note that none of the mappings are

continuous and the rational inequality holds for the values of $0\leq k_1+2k_2<1,\ where\ \ k_1,k_2\geq 0$.

Clearly $\frac{1}{10}$ is the unique common fixed point of A,B,S and T.

**1.1.16 Remark** Theorem (B) is a generalization of Theorem (A) by virtue of the weaker conditions such as weakly compatibility of the pairs (A,S) and (B,T) in place of compatibility of (A,S) and (B,T); The continuity of any one of the mappings is being dropped and the convergence of associated sequence relative to four self maps A,B,S and T in place the complete metric space

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