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Solution of a Subclass of Lane-Emden Differential Equation by Variational Iteration Method

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Abstract
In this paper we apply He's variational iteration method to find out an appropriate solution to a class of singular differential equation under imposed conditions by introducing and inducting in a polynomial pro satisfying the given subject to conditions at the outset as selective function to the solution extracting process. As for as application part is concerned, Illustrative examples from the available literature when treated all over reveal and out show that the solution deduced by proposed method is exact and again polynomial. Overall, a successful produce of exact solutions by proposed process itself justify the effectiveness and efficiency of the method so very much.

Keywords: He's variational iteration method, Lane-Emden differential equation, exact solution, polynomial, Lagrange multiplier.

1. Introduction

The universe is filled with numerous scientific advances and full off due observations that had tempted and motivated to ponder on with outmost concern and curiosity all about. So happened realised seriously, considered accordingly, analysed with all effort either implicitly or explicitly to the field of multi-disciplinary sciences through modelling into suitable mathematical preposition like in the form of a singular second order differential equation endowed with known boundary or other subject to conditions. Further onwards for the sake of convenience and ascertaining definiteness to not only systematic characteristics but also to the corresponding proper solution out of these prolific existing thought provoking spectral problems of science thoroughly, the area of interest and investigation into such variety of subject matter is constricted and limited to only some of the phenomenon occurring in mathematical physics ,astrophysics, biological science of human physiology and chemical kinetics inter alia the theory of stellar structure , the thermal behaviour of a spherical cloud of a gas , the isothermal gas spheres, the thermionic emission of currents , the degeneration of white-dwarf of a star, the thermal distribution profile in a human head , the radial stress within a circular plane , the elastic pressure under normal pressure the oxygen tension in a spherical cell with Michaelis–Menton oxygen uptake kinetics, the reactants concentration in a chemical reactor, the radial stress on a rotationally symmetric shallow membrane cap, the temperature present in an anti-symmetric circular plate and many more like problems[3,4,5,9,11,12,13,18,25,40] .Thereupon the considered range of mathematically modelled problems may be affined to the a special class of Lane-Emden differential equation for apropos interpretation and comprehensive investigation. Let the Lane-Emden differential equation considered with composite imposed condition be

$$y''(x) + \frac{k}{n}y'(x) + f(x,y) = g(x)\ ,\ \ 0 < x \le 1 \tag{1.1}$$

Subject to conditions $y(0) = A\ \ ,\ y(1) = B$

or

$y'(0) = \ \ C\ \ ,\ y(1) = D$

Where f(x, y) is a real valued continuous function, g(x) $\in$ C [0 1] and A, B, C, D are real constant.
The parameter 'k' is a real number greater than or equal to one. If k =1 or k=2 the problem (1.1) reduces to cylindrical or spherical type by virtue of corroborating perspective symmetries properly. To begin with the mode of analytical realisation, as of now, we again opt to restrict the domain of class of differential equation (1.1) in the larger interest of finding any suitable, simple and effective methodology enabling some better and appropriate much needed solution to the corresponding accustomed and coherent subclass of problems. Subsequently as to the sufficient interest towards such class of problems a sincere attempt is made ahead via the technique of variational iteration method successfully.

Now consider a specific subclass of problems as follow

$$y''(x) + \frac{k}{x}y'(x) + a\,p(x)y^m(x) = q(x) \qquad 0 < x \leq 1 \;, m \in I^+ \tag{1.2}$$

Subject to conditions $y(0) = A \;, \; y(1) = B$

Or

$y(0) = C \;, \; y(0) = D$

where A, B, C and D are real constants and the parameter k≥1.

However, p(x) and q(x) are polynomials of suitable degree and 'a' is any real number. Solution to the class of problems (1.1) exists and is unique as well [15, 33, 38]. The point x=0 is a singular point of the problem may offer a peculiar behavior to the solution in the neighborhood of that point like out showing a rapid change, partly skeptical and chaotic towards some solution procuring process making one unable to understand about the behavior of the solution over there at 'x'(=0)equals to zero. However, due to singularity to the extreme of the solution domain any of the numerical scheme may again face convergence problem. However, the singular behavior could not impede the keen interest of researchers related to the field of study of such kind of thought provoking problems any more.

In recent past, with regard to finding the solution to the Lane-Emden equation so far several other methods like B-Spline method, Homotopy method, Finite element method, Lie group analysis, Modified Variational, iteration method, Adomian method, Modified Adomian Decomposition method, Multi-integral method, Differential method, Projection method, Legendre wavelets method, Taylors series method, Rational Chebyshev collocation method ,Pseudo spectral methods have had been discussed and applied gracefully [2,7,8,10,14,16,17,23,27,28,34,36,37,42,43].

The method under consideration that is to be put forward and proposed to be applied upon, is a method none other than the He's variational iteration method often ascribed to and eulogised for solving famous subtle and meticulous problems like Autonomous ordinary differential system, Nonlinear oscillations, Nonlinear relaxation phenomena in polycrystalline solids, Nonlinear thermo elasticity, Cubic nonlinear Schrodinger equation, Ion acoustic plasma wave, Nonlinear oscillators with discontinuities, non-Newtonian flows, Burger's and coupled Burger's equation, General Riccati equation, Multispecies Lotaka –Volterra equations, Rational solution of Toda lattice equation, Helmholtz equation ,Generalized KdV equation and Nonlinear differential equations of fractional order[1,6,20,21,29,30,31,32,35,39,40].

2. He's Variational Iteration Method (V I M)

Variational iteration method may be understood like simultaneous toning up of Lagrange multiplier and variational theory complimenting each other in unison. In as much as the type of such consequential mutual indiscrete coexistence happen to be deduced out of two different mathematical concepts altogether sometimes also referred to as modified Lagrange multiplier method previously put forward by Inokuti Sakine and Mura[19] and later on envisioned and improvised by Chinese mathematician J.H. He have outreached and surpassed a milestone for known to have solved plenty of challenging problems with perfection, accuracy and great efficiency.Wich is what that itself speaks the volumes of its ability to elicit solution out of a diversifying class of problems. In order to incorporate and treat on by this very method further on consider a general differential equation in operator form as

$Dy(x) = g(x) \quad ,x \in I \subseteq \mathbb{R}$ where D is the usual differential operator (2.11)

y(x) is twice continuously differentiable function on a domain and g(x) is real valued inhomogeneous term. The relation (2.1) may be decomposed as follows

$$Ly(x) + Ny(x) = g(x) \qquad x \in I \subseteq \mathbb{R} \tag{2.12}$$

where L and N are linear and nonlinear differential operators respectively.

The variation iteration method acquires high efficiency and real potential to the required process of finding solution systematically by successive generation of recursive relations of correctional functional with respect to (2.12) via variational theory. Observing the success and usefulness of the proposed variant on so many other intrigue solution desired problems it is expedient to introduce and treat the given class of problems similarly. It is important to note that the variation iteration method accumulates its inner efficiency and enough potential needed for the solution exhibiting procedure with regard to (2.12) is by virtue of successive generation of recursive correctional functional systematically with the help of well thought exotic concept of variational theory. Eventually, therefore for finding a just and acceptable solution to the class of the problems (1.1) we adhere to construct a sequence of integral equations also called as correctional functional to the problems (2.12) as follows

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s) ((L (y_n(s)) + N (\widetilde{y_n(s)}) - g (s)) ds \quad , n \geq 0 \tag{2.13}$$

Where $\lambda(s)$ is Lagrange multiplier determined optimally satisfying all stationary conditions after variational method is applied to (2.3).However, there exists one more important feature responsible for ease and utility of the proposed method realized so all over is the assumption and choice of considering the inconvenient highly nonlinear and complicated dependent variables as restricted variables so as to minimizing the magnitude of the undesired error creeping into the susceptible solution finding process of the general problem(1.1).The emblem aforementioned ' $\widetilde{y_n}$ ' is the restricted variation, which means $\delta\widetilde{y_n}$=0.Eventually, after ' λ' is determined , a proper and suitable selective function may it be a linear one or appropriately nonlinear with respect to (2.2) is assumed as an initial approximation for finding next successive iterative function by recursive sequence of correction functionals anticipating to satisfy the given boundary conditions. On few occasions it is witnessed that finally or preferably the limiting value (as $n \to \infty$) of sequential approximations incurred after due process of iteration leads to exact solution. However, to our class of problems we consider a polynomial pre satisfying either boundary condition or initial condition corresponding to the problem as selective function that is likely to produce well desired exact solution.

2.1Variational Method and generalized Lagrange Multiplier

In order to avert the inconvenience caused by the presence of singularity the model (1.2) is required to be treated by modifying the problem without changing the status of referred physical phenomenon.

Accordingly, the modified imposed value problem is

$$xy''(x) + ky'(x) + axp(x)y^m(x) = xq(x) \quad \text{for all ' x' belonging to } [0\ 1] \tag{2.11}$$

Thus the sequential correctional funtionals corresponding to (2.11) may be defined as follows

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\left(sy''(s) + ky'(s) + asp(s)y_n^m(s)\right) - sq(s)\,)\,ds \qquad n\geq 0 \tag{2.12}$$

where y_0(x) is the initial selective function and $y_n(x)$ is n^{th} iterate of the correctional functional. Now optimal value of λ(s) is identified naturally by taking variation with respect to y_n(x) and subject to restricted variation of unpleasing terms of $y_n(x)$ i.e. $\delta\widetilde{y_n}$(x) =0. Consequently to embark on the relation

$$\delta y_{n+1}(x) = \delta y_n(x) + \delta \int_0^x \lambda(s)(\,sy''(s) + ky'(s) + a\,s\,p(s)\widetilde{y_n^m(s)} - sq(s))ds \quad , \quad n\geq 0 \tag{2.13}$$

Further on by virtue of integrating ( ) by parts and subject to the restricted variation of y_n($i.e.\ \delta\widetilde{y_n}$=0) we have then for n $\geq$ 0

$$\delta y_{n+1}(x) = (1 + (r-1)\lambda(s) - s\lambda'(s))\ \delta y_n(x) + \delta(\lambda(s)\ sy_n'(s))\ |s=x + \int_0^x (s\lambda''(s) + (2-r)\lambda'(s))\ \delta y_n(s)ds,$$

This mathematical variational equation asserts that the stationary conditions are

$$1 + (r-1)\lambda(s) - s\lambda'(s) = 0 \tag{2.14}$$

$$\lambda(x) = 0 \tag{2.15}$$

$$s\lambda''(s) + (2-r)\lambda'(s) = 0 \tag{2.16}$$

The relations (2.14), (2.15) and (2.16) altogether implies that

$\lambda(s) \quad = \quad s^{k-1} \int_x^s \frac{1}{(\zeta^k)} d\zeta$

(2.17)

Henceforth, the sequence of correction functionals are now given by

$$y_{n+1}(x) = y_n(x) + \int_0^x s^{k-1}\left(\int_x^s \frac{1}{(\zeta^k)} d\zeta\right) (sy''(s) + ky'(s) + a\, s\, p(s) \widetilde{y_n^m(s)} - sq(s))ds \qquad n \geq 0 \qquad (2.18)$$

Moreover, the relation (2.18) can be rewritten as

$$y_{n+1}(x) = y_n(x) + \int_0^x \left(\int_x^s \frac{1}{(\zeta^k)} d\zeta\right) (s^k y''(s) + ks^{k-1} y'(s) + a\, s^k\, p(s) \widetilde{y_n^m(s)} - s^k q(s))ds \qquad n \geq 0 \qquad (2.19)$$

In view of (2.11), (2.18) and (2.19) it is plausible to observe that by variational theory the process of optimization self endeavors to multiply the relation (1.2) by 'k^{th} 'power of the variable 'x' to come to succor and modify the considered model problem in the neighborhood of singular point so that the proposed method becomes expedient and can be expedited elegantly to conclude about innate and cohesive grid scientific behavior of hitherto discussed well defined class of applied nature of problems.Again,we also observe that the execution process carried out this way facilitate to express all the derivative terms as a total differential of some function henceforth manifesting the differential equation like semi-exact. Which is why, we can visualize the Lagrange multiplier as semi integrating factor for our model problems.

Also clearly would it be deduced from (2.19) that the limit of the convergent iterative sequence $\{y_n\}_{n=1}^{\infty}$ satisfying given conditions is the desired exact solution to (1.1).

2.2 Convergence of Iterative Sequence

Essentially our prime motive in this section is to establish the convergence of the considered sequence of correctional functionals generated out after VIM is executed onto the class (1.1) with regard to establish (2.19) observe that

$y_{n+1}(x) = y_n(x) + \sum_{k=0}^{n-1}(y_{k+1}(x) - y_k(x))$ is the 'n^{th}' partial sum of the infinite series

$$y_0(x) + \sum_{k=0}^{\infty}(y_{k+1}(x) - y_k(x)) \qquad (2.21)$$

Then necessarily the convergence of infinite series (2.21) implies the convergence of intermediary iterative sequence $\{y_n(x)\}_{n=1}^{\infty}$ of partial sums of the auxiliary series (2.21) as well. Suppose $y_0(x)$ be the initial selective function of polynomial function consuming the given conditions in the problem. Then the first successive variational iterate is given by

$$y_1(x) = y_0(x) + \int_0^x \lambda(s)\left((s^k y_0'(s))' + s^k(a\,p(s)y_0^m - g(s)\right) ds \tag{2.22}$$

On integration by parts and erstwhile appliance of the proper stationary conditions we have

$$|y_1(x) - y_0(x)| = \left|\int_0^x (y_0'(s) + \lambda(s)s^k(ap(s)y_0^m - q(s))ds\right| \tag{2.23}$$

This implies that

$$|y_1(x) - y_0(x)| \le \int_0^x (|y_0'(s)| + |\lambda(s)|(|a||p(s)||y_0(s)|^m + |q(s)|)ds \tag{2.24}$$

Similarly, the relation (2.19) on carrying out similar simplifications and using stationary conditions, imply

$$|y_2(x) - y_1(x)| = \left|\int_0^x \lambda(s)s^k\, a\,p(s)(y_1^m(s) - y_0^m(s)\, ds\right|$$

or, $|y_2(x) - y_1(x)| \le \int_0^x (|a|\,|\lambda(s)p(s)|)(|(y_1^m(s) - y_0^m(s)|)\, ds$

$$\le \int_0^x |a|\,|\lambda(s)p(s)|)\,(|y_1(s) - y_0(s)|\left|\sum_{i=0}^{m-1} y_1^{m-i-1}(s)y_0^i(s)\right|\, ds$$

$$\le \int_0^x m|a||\lambda(s)p(s)||y_1(s) - y_0(s)|\,|y_1(s)|^{m-1}\, ds \tag{2.25}$$

And, above all

$$|y_{n+1}(x) - y_n(x)| = \left|\int_0^x as^k\lambda(s)\, p(s)(y_n^m(s) - y_{n-1}^m(s))\, ds\right|$$

or, $|y_{n+1}(x) - y_n(x)| \le \int_0^x |a|\,|\lambda(s)p(s)|\,(|y_n(s) - y_{n-1}(s)|)\left|\sum_{i=0}^{m-i-1} y_n^{m-i-1}(s)y_{n-1}^i(s)\right|\, ds$, $\forall\, n \ge 2$

$$\leq \int_0^x m|a| |\lambda(s)p(s)| |y_n(s) - y_{n-1}(s)| |y_n(s)|^{m-1} \, ds, \quad \forall n \in IN \quad , \; s \leq x \leq 1$$

(2.26)

Now, choose

$$M = \sup \left(|y_0'(s)| + |\lambda(s)| \left(|a| |p(s)| |y_0(s)|^m + |q(s)| \right), \; m|a| |\lambda(s)p(s)| |y_n(s)|^{m-1} \right) \quad (2.27)$$

$$\forall \quad 0 \leq s \leq x \leq 1, \quad n \in IN$$

Then, again observe and proceed to establish the inequality

$$|y_{n+1}(s) - y_n(s)| \leq \frac{M^{n+1} x^{n+1}}{n+1!} \qquad \forall n \in IN \qquad (2.28)$$

Obviously, relations (2.24), (2.25), (2.26) and (2.27) together imply that

$$|y_1(x) - y_0(x)| \leq \int_0^x M \, ds = Mx \qquad (2.29)$$

As well as, $|y_2(x) - y_1(x)| \leq M \int_0^x |(y_1(s)) - y_0(s))| ds$ (2.30)

$$s \leq x \leq 1$$

Using (2.29) in (2.30) we find that

$$|y_2(x) - y_1(x)| \leq M \int_0^x M s \, ds = \frac{M^2 x^2}{2}$$

$$s \leq x \leq 1$$

Thus, the statement (2.28) is true for natural number n=1

As usual, suppose that

$|y_n(s) - y_{n-1}(s)| \leq \frac{M^n x^n}{n!}$ holds for some, $n \in IN$ (2.31)

Then, again relations (2.24) and (2.27) altogether imply that

$$|y_{n+1}(x) - y_n(x)| \leq \int_0^x m|a| |\lambda(s)p(s)| \left(|y_n(s)|^{m-1} \right) |y_n(s) - y_{n-1}(s)| ds$$

$s \leq x \leq 1, n \in IN$

or, $|y_{n+1}(x) - y_n(x)| \leq m|a| \sup(|\lambda(s)||p(s)||y_n(s)|^{m-1} \int_0^x |y_n(s) - y_{n-1}(s)| ds$

$s \leq x \leq 1, n \in IN$

That implies by (2.27) and (2.31)

$|y_{n+1}(x) - y_n(x)| \leq M \int_0^x \frac{M^n s^n}{n!} ds = \frac{M^{n+1} x^{n+1}}{n+1!}$

Therefore, by Principle of Induction

$$|y_{n+1}(x) - y_n(x)| \leq \frac{M^{n+1} x^{n+1}}{n+1!} \quad \text{holds} \quad \forall x \in [0\ 1] \text{ and } \forall n \in IN \tag{2.32}$$

Now we claim that the series (2.21) converges both absolutely and uniformly for all $x \in [0\ 1]$ using (2.32)

$$\text{Since, } |y_0(x)| + \sum_{n=0}^{\infty} |y_{n+1}(x) - y_n(x)| \leq |y_0(x)| + \sum_{n=0}^{\infty} \frac{M^{n+1} x^{n+1}}{n+1!} = |y_0(x)| + (e^{Mx} - 1) \quad \forall x \in [01] \tag{2.33}$$

Therefore the series $y_0(x) + \sum_{k=0}^{\infty}(y_{k+1}(x) - y_k(x))$ converge uniformly $\forall x \in [01]$ and by virtue of (2.33) sequence of its partial sums $\{y_n(x)\}_{n=0}^{\infty}$ converges to solution function of the given class of problems.

3. Illustrative Problems

The proposed method is justified by successful implementation of VIM on some of the specific problems of linear and nonlinear type often referred, discussed and had been attempted to solve by other different methods in literature available so far.

3.1Example1:

Consider the following boundary value problem [21,34]

$$y''(x) + \frac{1}{x} y'(x) + y(x) = \frac{5}{4} + \frac{x^2}{16} \tag{3.11}$$

Subject to $y'(0) = 0$, $y(1) = \frac{17}{16}$

Solution: To solve (5.1) we construct correction functional as follows

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s) \left((xy_n'(s))' + sy(s) - s\left(\frac{5}{4}+\frac{s^2}{16}\right)\right) ds \qquad n \geq 0$$

Where 'λ(s)' is optimally identified Lagrange multiplier similar to (2.27).Then the first iterative solution is

$$y_1(x) = y_o(x) + \int_0^x \lambda(s) \left((sy_0'(s))' + s\,y_0(s) - s\left(\frac{5}{4}+\frac{s^2}{16}\right)\right) ds$$

Let y_0 (x) be the selective polynomial function satisfying the given boundary conditions. We may simply choose selective function as

$$y_0(x) = a + \left(\frac{17}{16} - a\right)x^2$$

Then the first iterate is as follows

$$y_1(x) = a + \left(\frac{17}{16} - a\right)x^2 + \int_0^x \lambda(s) \left((sy_0'(s))' + s\,y_0(s) - s\left(\frac{5}{4}+\frac{s^2}{16}\right)\right) ds$$

Now on performing simplifications, we get

$$y_1(x) = a + \left(\frac{17}{16} - a\right)x^2 + \frac{3}{4}(a-1)x^2 + \frac{1}{16}(a-1)x^4$$

Further onwards imposition of boundary condition on y_1(x) asserts that 'a=1'enabling, y_1 (x) = x^2 as the produced exact solution to the problem.

3.2Example2:

Consider the boundary value problem [7, 34,35]

$$-y''(x) - \frac{2}{x}y'(x) + (1-x^2)y(x) = x^4 - 2x^2 + 7 \qquad (3.21)$$

$y'(0) = 1$, $y(1) = 0$

Solution: The correction functional for the problem (3.21) is

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s) \left((-s^2 y_n')' - s^2(1-s^2)y_n(s) - (s^6 - 2s^4 + 7s^2)\right) ds \qquad (3.22)$$

'λ(s) 'is usual optimally identified Lagrange multiplier

Let $y_0(x) = a\,(1-x^2)$ as selective initial approximation function to induce successive first iterate as

$$y_1(x) = a\,(1-x^2) - \frac{7}{6}(a-1)s + \frac{1}{10}(1-a)x^4 - \frac{1}{42}(a-1)x^6$$

Since solution to (1.2) type of boundary value problems are unique, therefore upon matching the boundary condition we get 'a=1' rendering $y(x) = y_1(x) = (1-x^2)$ as the exact solution to (3.21)

3.3Example3:

Let the nonlinear boundary value problem [42]

$$y''(x) + \frac{4}{x}y'(x) + y^2(x) = x^6 + x^2 + 18\,x + 4 \quad (3.31)$$

y(0) =2, $y'(0) = 0$

Solution: The correctional functional with respect to (3.31) is given by

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\,((s^4 y_n')' + s^4 y_n^2(s) - s^4(s^6 + 4s^3 + 18s + 4))ds \quad \text{for} \quad n=0, 1\ldots \quad (3.32)$$

Let, $y_0(x) = y(x) = 2 + ax^2 + bx^3$ be the selective initial approximation function .Then by VIM,

First iterative approximate solution to (3.32) simplifies to

$$y_1(x) = 2 + x^3 - \frac{1}{7}ax^4 - \frac{1}{10}(b-1)x^5 - \frac{1}{54}a^2x^6 - \frac{1}{35}abx^7 - \frac{1}{88}(b^2-1)x^8 \quad (3.32)$$

Then on matching the given initial condition and applying unique feature of solution again implies that a=0 and b=1, exhibiting $y(x) = y_1(x) = 2 + x^3$, the exact solution to the problem.

However if we consider differential equation (3.31) along boundary conditions $y(0)$=2 and $y(1)$ =3 then (3.33) similarly provides exact solution to the boundary value problem as well.

3.4 Example4:

Consider the boundary value problem [17, 42]

$$y''(x)+\frac{2}{x}y'(x)+y^3(x)=6+x^6 \tag{3.41}$$

Subject to $y'(0)=0$, $y(1)=1$

Solution**:** The correctional functional for boundary value problem (3.41) is as follows

$$y_{n+1}(x)=y_n(x)+\int_0^x \lambda(s)\left((s^2y_n'(s))'+s^2y_n^3(s)-s^2(s^6+6)\right)ds \quad \text{for } n=0, 1, 2\ldots\ldots \tag{3.42}$$

Let $y(x)=y_0(x)=a+(1-a)x^2$ be the selective function satisfying the given boundary condition then

Then the first iterate by variation iteration method from (3.42) is given by

$$y_1(x)=a+(1-a)x^2-\frac{1}{6}a(a^2+6)x^2+\frac{3}{20}a^2(a-1)x^4-\frac{1}{14}a(a-1)^2x^6+\frac{1}{72}a(a^2-3(a-1))x^8$$

Now matching the conditions at the end points of the solution domain and using the fact that the solution to such boundary value problem are unique we get, $a=0$. Hence the method producing the exact solution

$$y(x)=y_1(x)=x^2 .$$

3.5Example5

Consider the problem[10,33]

$$y''(x)+\frac{8}{x}y'(x)+x\,y(x)=x^5-x^4+44x-30x \tag{3.51}$$

Subject to $y(0)=0$, $y'(0)=0$

Solution: If 'λ 'is the Lagrange multiplier then the first correctional functional is given by

$$y_1(x) = y_0(x) + \int_o^x \lambda(s)\left((s^8 y_0'(s))' + s^9\left(yo(s) - (s^5 - s^4 + 44s - 30s)\right)\right) ds$$

(3.52)

Let us consider $y(x) = y_0(x) = ax^3 + bx^4$ as selective function

Then on inserting the value of $y(x)$ in (3.52) we get

$$y_1(x) = -x^3 + bx^4 - \frac{1}{78}(a+1)x^6 - \frac{1}{98}(b-1)x^7 \quad (3.53)$$

Hence upon imposing the given initial condition in (3.53) and using the criterion of uniqueness of the solution we have a=-1 and b=1 felicitating y(x) = $-x^3 + x^4$.

Moreover, if we again consider the problem (3.51) along with condition $y(0)= y(1)=0$ then on the basis of similar logic on (3.53) we get an exact solution to the considered boundary value problem.

4. Conclusions

This is pertinent to note that He's variation iteration method applies successfully to a linear as well as to a nonlinear class of boundary or initial value problems of type (1.2). Frontier examples of relevance that have had occurred time and again and had been dealt by some other method of solution are taken and solved to focus and assert that a proper selection of selective function and henceforth imposition of boundary or initial condition as we please on iterative correctional function may lead to an exact solution. However sometimes necessity of uniqueness of solution is also assumed during solution maneuvering process.

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