

# Solution of Linear and Nonlinear Partial Differential Equations Using Mixture of Elzaki Transform and the Projected Differential Transform Method

Tarig. M. Elzaki<sup>1\*</sup> & Eman M. A. Hilal<sup>2</sup>

1. Mathematics Department, Faculty of Sciences and Arts-Alkamil, King Abdulaziz University,  
Jeddah-Saudi Arabia.

Mathematics Department, Faculty of Sciences, Sudan University of Sciences and Technology-Sudan.

2. Mathematics Department, Faculty of Sciences for Girles King Abdulaziz University  
Jeddah-Saudi Arabia

\* E-mail of the corresponding author: [Tarig.alzaki@gmail.com](mailto:Tarig.alzaki@gmail.com)

## Abstract

The aim of this study is to solve some linear and nonlinear partial differential equations using the new integral transform "Elzaki transform" and projected differential transform method. The nonlinear terms can be handled by using of projected differential transform method; this method is more efficient and easy to handle such partial differential equations in comparison to other methods. The results show the efficiency and validation of this method.

Keywords: Elzaki transform, projected differential transform method, nonlinear partial differential equations.

## 1. Introduction

Many problems of physical interest are described by linear and nonlinear partial differential equations with initial or boundary conditions, these problems are fundamental importance in science and technology especially in engineering. Some valuable contributions have already been made to showing differential equations such as Laplace transform method [Islam, Yasir Khan, Naeem Faraz and Francis Austin (2010), Lokenath Debnath and Bhatta (2006)], Sumudu transform [A.Kilicman and H.E.Gadain (2009), (2010)], differential transform method etc. Elzaki transform method is very effective tool for solve linear partial differential equations [Tarig Elzaki & Salih Elzaki (2011)]. In this study we use the projected differential transform method [Nuran Guzel and Muhammet Nurulay (2008), Shin- Hsiang Chang , Ling Chang (2008) ] to decompose the nonlinear terms, this means that we can use both Elzaki transform and projected differential transform methods to solve many nonlinear partial differential equations.

### 1.1. Elzaki Transform:

The basic definitions of modified of Sumudu transform or Elzaki transform is defined as follows, Elzaki transform of the function  $f(t)$  is

$$E [f (t)] = v \int_0^{\infty} f (t) e^{-\frac{t}{v}} dt, \quad t > 0$$

Tarig M. Elzaki and Salth M. Elzaki in [2011], showed the modified of Sumudu transform [Kilicman & ELtayeb (2010)] or Elzaki transform was applied to partial differential equations, ordinary differential equations, system of ordinary and partial differential equations and integral equations.

In this paper, we combined Elzaki transform and projected differential transform methods to solve linear and nonlinear partial differential equations.

To obtain Elzaki transform of partial derivative we use integration by parts, and then we have:

$$E \left[ \frac{\partial f (x, t)}{\partial t} \right] = \frac{1}{v} T (x, v) - v f (x, 0) \qquad E \left[ \frac{\partial^2 f (x, t)}{\partial t^2} \right] = \frac{1}{v^2} T (x, v) - f (x, 0) - v \frac{\partial f (x, 0)}{\partial t}$$

$$E \left[ \frac{\partial f (x, t)}{\partial x} \right] = \frac{d}{dx} [T (x, v)] \qquad E \left[ \frac{\partial^2 f (x, t)}{\partial x^2} \right] = \frac{d^2}{dx^2} [T (x, v)]$$

**Proof:**

To obtain ELzaki transform of partial derivatives we use integration by parts as follows:

$$E \left[ \frac{\partial f}{\partial t} (x, t) \right] = \int_0^{\infty} v \frac{\partial f}{\partial t} e^{-\frac{t}{v}} dt = \lim_{p \rightarrow \infty} \int_0^p v e^{-\frac{t}{v}} \frac{\partial f}{\partial t} dt = \lim_{p \rightarrow \infty} \left\{ \left[ v e^{-\frac{t}{v}} f (x, t) \right]_0^p - \int_0^p e^{-\frac{t}{v}} f (x, t) dt \right\}$$

$$= \frac{T (x, v)}{v} - v f (x, 0)$$

We assume that  $f$  is piecewise continuous and it is of exponential order.

Now  $E \left[ \frac{\partial f}{\partial x} \right] = \int_0^{\infty} v e^{-\frac{t}{v}} \frac{\partial f (x, t)}{\partial x} dt = \frac{\partial}{\partial x} \int_0^{\infty} v e^{-\frac{t}{v}} f (x, t) dt$ , using the Leibnitz rule to find:

$$E \left[ \frac{\partial f}{\partial x} \right] = \frac{d}{dx} [T (x, v)]$$

By the same method we find:  $E \left[ \frac{\partial^2 f}{\partial x^2} \right] = \frac{d^2}{dx^2} [T (x, v)]$

To find:  $E \left[ \frac{\partial^2 f}{\partial t^2} (x, t) \right]$

Let  $\frac{\partial f}{\partial t} = g$ , then we have:

$$E \left[ \frac{\partial^2 f}{\partial t^2} (x, t) \right] = E \left[ \frac{\partial g (x, t)}{\partial t} \right] = E \left[ \frac{g (x, t)}{v} \right] - v g (x, 0)$$

$$E \left[ \frac{\partial^2 f}{\partial t^2} (x, t) \right] = \frac{1}{v^2} T(x, v) - f(x, 0) - v \frac{\partial f}{\partial t} (x, 0)$$

We can easily extend this result to the nth partial derivative by using mathematical induction.

### 1.2. Projected Differential Transform Methods:

In this section we introduce the projected differential transform method which is modified of the differential transform method.

*Definition:*

The basic definition of projected differential transform method of function  $f(x_1, x_2, \dots, x_n)$  is defined as

$$f(x_1, x_2, \dots, x_{n-1}, k) = \frac{1}{k!} \left[ \frac{\partial^k f(x_1, x_2, \dots, x_n)}{\partial x_n^k} \right]_{x_n=0} \quad (1)$$

Such that  $f(x_1, x_2, \dots, x_n)$  is the original function and  $f(x_1, x_2, \dots, x_{n-1}, k)$  is projected transform function.

And the differential inverse transform of  $f(x_1, x_2, \dots, x_{n-1}, k)$  is defined as

$$f(x_1, x_2, \dots, x_n) = \sum_{k=0}^{\infty} f(x_1, x_2, \dots, x_{n-1}, k) (x - x_0)^k \quad (2)$$

The fundamental theorems of the projected differential transform are

**Theorems:**

(1) If  $z(x_1, x_2, \dots, x_n) = u(x_1, x_2, \dots, x_n) \pm v(x_1, x_2, \dots, x_n)$

Then  $z(x_1, x_2, \dots, x_{n-1}, k) = u(x_1, x_2, \dots, x_{n-1}, k) \pm v(x_1, x_2, \dots, x_{n-1}, k)$

(2) If  $z(x_1, x_2, \dots, x_n) = cu(x_1, x_2, \dots, x_n)$

Then  $z(x_1, x_2, \dots, x_{n-1}, k) = cu(x_1, x_2, \dots, x_{n-1}, k)$

(3) If  $z(x_1, x_2, \dots, x_n) = \frac{du(x_1, x_2, \dots, x_n)}{dx_n}$

Then  $z(x_1, x_2, \dots, x_{n-1}, k) = (k+1)u(x_1, x_2, \dots, x_{n-1}, k+1)$

(4) If  $z(x_1, x_2, \dots, x_n) = \frac{d^n u(x_1, x_2, \dots, x_n)}{dx_n^n}$

Then  $z(x_1, x_2, \dots, x_{n-1}, k) = \frac{(k+n)!}{k!} u(x_1, x_2, \dots, x_{n-1}, k+n)$

(5) If  $z(x_1, x_2, \dots, x_n) = u(x_1, x_2, \dots, x_n) v(x_1, x_2, \dots, x_n)$

Then  $z(x_1, x_2, \dots, x_{n-1}, k) = \sum_{m=0}^k u(x_1, x_2, \dots, x_{n-1}, m) v(x_1, x_2, \dots, x_{n-1}, k-m)$

(6) If  $z(x_1, x_2, \dots, x_n) = u_1(x_1, x_2, \dots, x_n) u_2(x_1, x_2, \dots, x_n) \dots u_n(x_1, x_2, \dots, x_n)$  Then

$$z(x_1, x_2, \dots, x_{n-1}, k) = \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} u_1(x_1, x_2, \dots, x_{n-1}, k_1) u_2(x_1, x_2, \dots, x_{n-1}, k_2 - k_1) \times \dots u_{n-1}(x_1, x_2, \dots, x_{n-1}, k_{n-1} - k_{n-2}) u_n(x_1, x_2, \dots, x_{n-1}, k - k_{n-1})$$

(7) If  $z(x_1, x_2, \dots, x_n) = x_1^{q_1} x_2^{q_2} \dots x_n^{q_n}$

Then  $z(x_1, x_2, \dots, x_{n-1}, k) = \delta(x_1, x_2, \dots, x_{n-1}, q_n - k) = \begin{cases} 1 & k = q_n \\ 0 & k \neq q_n \end{cases}$

Note that  $c$  is a constant and  $n$  is a nonnegative integer.

**2. Applications:**

Consider a general nonlinear non-homogenous partial differential equation with initial conditions of the form:

$$Du(x, t) + Ru(x, t) + Nu(x, t) = g(x, t) \tag{3}$$

$$u(x, 0) = h(x) \quad , \quad u_t(x, 0) = f(x)$$

Where  $D$  is linear differential operator of order two,  $R$  is linear differential operator of less order than  $D$ ,  $N$  is the general nonlinear differential operator and  $g(x, t)$  is the source term.

Taking Elzaki transform on both sides of equation (3), to get:

$$E[Du(x, t)] + E[Ru(x, t)] + E[Nu(x, t)] = E[g(x, t)] \quad (4)$$

Using the differentiation property of Elzaki transforms and above initial conditions, we have:

$$E[u(x, t)] = v^2 E[g(x, t)] + v^2 h(x) + v^3 f(x) - v^2 E[Ru(x, t) + Nu(x, t)] \quad (5)$$

Applying the inverse Elzaki transform on both sides of equation (5), to find:

$$u(x, t) = G(x, t) - E^{-1} \left\{ v^2 E [Ru(x, t) + Nu(x, t)] \right\} \quad (6)$$

Where  $G(x, t)$  represents the term arising from the source term and the prescribed initial conditions.

Now, we apply the projected differential transform method.

$$u(x, m+1) = -E^{-1} \left\{ v^2 E [A_m + B_m] \right\}, \quad u(x, 0) = G(x, t) \quad (7)$$

Where  $A_m, B_m$  are the projected differential transform of  $Ru(x, t), Nu(x, t)$ .

From equation (7), we have:

$$\begin{aligned} u(x, 0) &= G(x, t), \quad u(x, 1) = -E^{-1} \left\{ v^2 E [A_0 + B_0] \right\} \\ u(x, 2) &= -E^{-1} \left\{ v^2 E [A_1 + B_1] \right\}, \quad u(x, 3) = -E^{-1} \left\{ v^2 E [A_2 + B_2] \right\} \\ &\dots \end{aligned}$$

Then the solution of equation (3) is

$$u(x, t) = u(x, 0) + u(x, 1) + u(x, 2) + \dots$$

To illustrate the capability and simplicity of the method, some examples for different linear and nonlinear partial differential equations will be discussed.

*Example 2.1:*

Consider the simple first order partial differential equation

$$\frac{\partial y}{\partial x} = 2 \frac{\partial y}{\partial t} + y, \quad y(x, 0) = 6e^{-3x} \quad (8)$$

Taking Elzaki transform of (8), leads to

$$E[y(x, t)] = 6v^2 e^{-3x} + \frac{v}{2} E[A_m - B_m]$$

Take the inverse Elzaki transform to find,

$$y(x, m+1) = E^{-1} \left\{ \frac{v}{2} E [A_m - B_m] \right\}, \quad y(x, 0) = 6e^{-3x} \quad (9)$$

Where  $A_m = \frac{\partial y(x, m)}{\partial x}$ ,  $B_m = y(x, m)$  are the projected differential transform of  $\frac{\partial y(x, t)}{\partial x}$ ,  $y(x, t)$ .

The standard Elzaki transform defines the solution  $y(x, t)$  by the series

$$y(x, t) = \sum_{m=0}^{\infty} y(x, m)$$

From equation (9) we find that:  $y(x, 0) = 6e^{-3x}$

$$\begin{aligned} A_0 &= -18e^{-3x}, B_0 = 6e^{-3x}, y(x, 1) = E^{-1}[-12v^3 e^{-3x}] = -12te^{-3x} \\ A_1 &= 36te^{-3x}, B_1 = -12e^{-3x}, y(x, 2) = E^{-1}[24v^4 e^{-3x}] = 12t^2 e^{-3x} \\ &\dots\dots\dots y(x, 3) = -8t^3 e^{-3x} \end{aligned}$$

The solution in a series form is given by

$$y(x, t) = 6e^{-3x} - 12te^{-3x} + 12t^2 e^{-3x} + \dots\dots = 6e^{-3x} e^{-2t} = 6e^{-(3x+2t)}$$

*Example 2.2:*

Consider the following linear second order partial differential equation

$$u_{xx} + u_{tt} = 0, \quad u(x, 0) = 0, \quad u_t(x, 0) = \cos x \tag{10}$$

To find the solution we take Elzaki transform of equation (10) and making use of the conditions to find,

$$E[u(x, t)] = v^3 \cos x - v^2 E[A_m], \quad A_m = \frac{\partial^2 u(x, m)}{\partial x^2}$$

Take the inverse Elzaki transform we get:

$$u(x, m+1) = -E^{-1}\{v^2 E[A_m]\}, \quad u(x, 0) = 0, \quad u(x, 1) = t \cos x \tag{11}$$

By using equation (11), we find that:

$$\begin{aligned} A_1 &= -t \cos x \Rightarrow u(x, 2) = E^{-1}\{v^2 E[t \cos x]\} = \frac{t^3}{3!} \cos x \\ A_2 &= -\frac{t^3}{3!} \cos x \Rightarrow u(x, 3) = E^{-1}\left\{v^2 E\left[\frac{t^3}{3!} \cos x\right]\right\} = \frac{t^5}{5!} \cos x \\ &\dots \\ &\dots \\ &\dots \end{aligned}$$

Then the solution is 
$$u(x, t) = \cos x \left[ t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots \right] = \cos x \sinh t$$

*Example 2.3:*

Consider the following second order nonlinear partial differential equation

$$\frac{\partial u}{\partial t} = \left( \frac{\partial u}{\partial x} \right)^2 + u \frac{\partial^2 u}{\partial x^2}, \quad u(x, 0) = x^2 \quad (12)$$

To find the solution take Elzaki transform of (12) and using the condition we get:

$$E[u(x, t)] = v^2 x^2 + vE[A_m + B_m]$$

Where  $A_m = \sum_{m=0}^h \frac{\partial u(x, m)}{\partial x} \frac{\partial u(x, h-m)}{\partial x}$ ,  $B_m = \sum_{m=0}^h u(x, m) \frac{\partial^2 u(x, h-m)}{\partial x^2}$ , are projected

differential transform of  $\left( \frac{\partial u}{\partial x} \right)^2$ ,  $u \frac{\partial^2 u}{\partial x^2}$

Take the inverse Elzaki transform to get:

$$u(x, m+1) = E^{-1} \{ vE[A_m + B_m] \}, \quad u(x, 0) = x^2 \quad (13)$$

From equation (13) we find that:

$$\begin{aligned} A_0 &= 4x^2, B_0 = 2x^2 \Rightarrow u(x, 1) = E^{-1} [6x^2 v^3] = 6x^2 t \\ A_1 &= 48x^2 t, B_1 = 24x^2 t \Rightarrow u(x, 2) = E^{-1} [72x^2 v^4] = 36x^2 t^2 \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned}$$

Then the solution of equation (12) is

$$u(x, t) = x^2 + 6x^2 t + 36x^2 t^2 + \dots = x^2 (1 - 6t)^{-1} = \frac{x^2}{1 - 6t}$$

Which is an exactly the same solution obtained by the Adomian decomposition method.

*Example 2.4:*

Let us consider the nonlinear partial differential equation

$$\frac{\partial u}{\partial t} = 2u \left( \frac{\partial u}{\partial x} \right)^2 + u^2 \frac{\partial^2 u}{\partial x^2}, \quad u(x, 0) = \frac{x+1}{2} \quad (14)$$

Taking Elzaki transform of equation (14) and using the condition leads to:

$$E[u(x,t)] = v^2 \left( \frac{x+1}{2} \right) + vE[2A_m + B_m]$$

Taking the inverse Elzaki transform to find:

$$u(x, m+1) = E^{-1} \{ vE[2A_m + B_m] \}, \quad u(x, 0) = \frac{x+1}{2} \quad (15)$$

Where

$$A_m = \sum_{k=0}^h \sum_{m=0}^k u(x, m) \frac{\partial u(x, h-m)}{\partial x} \frac{\partial u(x, h-k)}{\partial x}$$

$$B_m = \sum_{k=0}^h \sum_{m=0}^k u(x, m) u(x, h-m) \frac{\partial^2 u(x, h-k)}{\partial x^2}$$

From equation (15) we have:

$$u(x, 0) = \frac{x+1}{2}, \quad u(x, 1) = \frac{x+1}{2}t, \quad u(x, 2) = \frac{3(x+1)}{8}t^2, \dots$$

Then the solution of equation (14) is

$$u(x, t) = \frac{x+1}{2} (1-t)^{-\frac{1}{2}} = \frac{x+1}{2\sqrt{1-t}}$$

### 3. Conclusion

The solution of linear and nonlinear partial differential equations can be obtained using Elzaki transform and projected differential transform method without any discretization of the variables. The results for all examples can be obtained in series form, and all calculations in the method are very easy. This technique is useful to solve linear and nonlinear partial differential equations.

### References

- [1] S. Islam, Yasir Khan, Naeem Faraz and Francis Austin (2010), Numerical Solution of Logistic Differential Equations by using the Laplace Decomposition Method, World Applied Sciences Journal 8 (9):1100-1105.
- [2] Nuran Guzel and Muhammet Nurulay (2008), Solution of Shiff Systems By using Differential Transform Method, Dunlupinar universities Fen Bilimleri Enstitusu Dergisi, ISSN 1302-3055, PP. 49-59.
- [3] Shin- Hsiang Chang , I. Ling Chang (2008), A new algorithm for calculating one – dimensional differential transform of nonlinear functions, Applied Mathematics and Computation 195, 799-808.
- [4] Tarig M. Elzaki (2011), The New Integral Transform “Elzaki Transform” Global Journal of Pure and Applied Mathematics, ISSN 0973-1768, Number 1, pp. 57-64.
- [5] Tarig M. Elzaki & Salih M. Elzaki (2011), Application of New Transform “Elzaki Transform” to Partial Differential Equations, Global Journal of Pure and Applied Mathematics, ISSN 0973-1768, Number 1, pp. 65-70.
- [6] Tarig M. Elzaki & Salih M. Elzaki (2011), On the Connections Between Laplace and Elzaki transforms, Advances in Theoretical and Applied Mathematics, ISSN 0973-4554 Volume 6, Number 1, pp. 1-11.
- [7] Tarig M. Elzaki & Salih M. Elzaki (2011), On the Elzaki Transform and Ordinary Differential Equation With Variable Coefficients, Advances in Theoretical and Applied Mathematics. ISSN 0973-4554 Volume 6,



- [8] Lokenath Debnath and D. Bhatta (2006). Integral transform and their Application second Edition, Chapman & Hall /CRC.
- [9] A.Kilicman and H.E.Gadain (2009). An application of double Laplace transform and Sumudu transform, Lobachevskii J. Math.30 (3), pp.214-223.
- [10] J. Zhang (2007), A Sumudu based algorithm m for solving differential equations, Comp. Sci. J. Moldova 15(3), pp – 303-313.
- [11] Hassan Eltayeb and Adem kilicman (2010), A Note on the Sumudu Transforms and differential Equations, Applied Mathematical Sciences, VOL, 4, no.22,1089-1098
- [12] Kilicman A. & H. ELtayeb (2010). A note on Integral transform and Partial Differential Equation, Applied Mathematical Sciences, 4(3), PP.109-118.
- [13] Hassan ELtayeh and Adem kilicman (2010), on Some Applications of a new Integral Transform, Int. Journal of Math. Analysis, Vol, 4, no.3, 123-132.
- [14] N.H. Sweilam, M.M. Khader (2009). Exact Solutions of some capled nonlinear partial differential equations using the homotopy perturbation method. Computers and Mathematics with Applications 58 2134-2141.
- [15] P.R. Sharma and Giriraj Methi (2011). Applications of Homotopy Perturbation method to Partial differential equations. Asian Journal of Mathematics and Statistics 4 (3): 140-150.
- [16] M.A. Jafari, A. Aminataei (2010). Improved Homotopy Perturbation Method. International Mathematical Forum, 5, no, 32, 1567-1579.
- [17] Jagdev Singh, Devendra, Sushila (2011). Homotopy Perturbation Sumudu Transform Method for Nonlinear Equations. Adv. Theor. Appl. Mech., Vol. 4, no. 4, 165-175.

Table 1. Elzaki transform of some Functions

$f(t)$	$E[f(t)] = T(u)$
1	$v^2$
$t$	$v^3$
$t^n$	$n! v^{n+2}$
$e^{at}$	$\frac{v^2}{1-av}$
$\sin at$	$\frac{av^3}{1+a^2v^2}$
$\cos at$	$\frac{v^2}{1+a^2v^2}$

This academic article was published by The International Institute for Science, Technology and Education (IISTE). The IISTE is a pioneer in the Open Access Publishing service based in the U.S. and Europe. The aim of the institute is Accelerating Global Knowledge Sharing.

More information about the publisher can be found in the IISTE's homepage:

<http://www.iiste.org>

The IISTE is currently hosting more than 30 peer-reviewed academic journals and collaborating with academic institutions around the world. **Prospective authors of IISTE journals can find the submission instruction on the following page:**

<http://www.iiste.org/Journals/>

The IISTE editorial team promises to review and publish all the qualified submissions in a fast manner. All the journals articles are available online to the readers all over the world without financial, legal, or technical barriers other than those inseparable from gaining access to the internet itself. Printed version of the journals is also available upon request of readers and authors.

### **IISTE Knowledge Sharing Partners**

EBSCO, Index Copernicus, Ulrich's Periodicals Directory, JournalTOCS, PKP Open Archives Harvester, Bielefeld Academic Search Engine, Elektronische Zeitschriftenbibliothek EZB, Open J-Gate, OCLC WorldCat, Universe Digital Library, NewJour, Google Scholar

