An Extension of Peano Axioms for the Natural Number System

Stephen Sebastine Akpan[1*] John Usen[2] Thomas Adidaumbe Ugbe[3]

Department of Maths/Statistics & Computer Science, University of Calabar, Cross-River State, Nigeria

E-mail of the corresponding author:ugbe_thomas@yahoo.com

Abstract

As a fundamental requirement, a natural number system must necessarily satisfy Peano axioms. These axioms altogether justify whether or not a number system is a natural number system. In this paper, a set of axioms have been proposed and established to augment and extend the existing Peano axioms for the natural number system.

Key Words: Natural numbers, Rational numbers, Peano axioms, Pearson correlation coefficient, Even and Odd partitions.

Introduction

Natural numbers are the heart of mathematics and related sciences. Besides from their obvious practical use as a measure for counting, natural numbers are of great theoretical importance. They the form the basis for higher-level number constructs, such as rational numbers, Jaeger (2011). Natural numbers also play important role in theoretical computer science Turing [(1936), (1937)]. In particular, the concept of computable enumerable sets, such as languages accepted by Turing machines, is intimately connected to natural numbers.

In mathematics there are two conventions for the set of natural numbers. Natural numbers may either be described as the set of positive integers $\{1, 2, 3, \cdots\}$ according to the traditional definition; or the set of non-negative integers $\{0, 1, 2, 3, \cdots\}$ according to a definition first appearing in the nineteenth century (Wikipedia, 2013). Mathematicians use ℕ or N to refer to the set of all natural numbers. This set is countable-infinite. This is also expressed by saying that the cardinality of the set is aleph null, "ℵ" To be unambiguous about whether zero is included or not, sometimes an under " 0 " is added in the former case and a superscript " $*$ " or subscript " 1" is added in the latter case. That is, $\mathrm{N}_0 = \{0, 1, 2, 3, \cdots\}$ or $\mathrm{N}^* = \mathrm{N}_1 = \{1, 2, 3, \cdots\}$ (Wikipedia, 2013; Aitken, 2009).

Peano axioms state necessary and sufficient conditions that any definition of a natural number system must satisfy (Joyce, 2005; Aitken, 2009). However, granted that these axioms may be necessary and sufficient are there more properties or axioms for the natural number system not explored? If there are, are they sufficient and necessary? These questions have given a prompting which establishes the basis for this paper. It is in this regard that we have proposed two axioms somewhat sufficient and necessary as the Peano axioms.

This paper considers the set N^* in relation to the partitions X and Y of Even and Odd numbers respectively with a view to analysing the correlation between these two partitions. In more general terms, we have defined the set of elementary events for N^* with respect to these partitions as the random variable Z such that

$$Z = \begin{cases} \{2i\}, & \forall i \in \mathrm{N}^* \\ \{2i-1\}, & \forall i \in \mathrm{N}^* \end{cases}$$

This implies that Z could be modelled to have a binomial distribution with probability distribution function given as;

$$b(Z;m,p) = \binom{m}{z} p^k q^{m-k}$$

Where m represents a countable infinite number of trials

Historically the mathematical definition of the natural numbers developed with some difficulty (CC-BY-SA, 2013). Peano axioms, also known as Peano's postulates (in Number Theory), comprises five axioms introduced in 1889 by Italian mathematician "Giuseppe Peano" (CC-BY-SA, 2013; William, 2013). Like the axioms for Geometry devised by Greek mathematician "Euclid" (c. 300 BCE), the Peano axioms were meant to provide a rigorous foundation for the natural numbers $(0,1,2,3,\ldots)$ used in arithmetic, number theory and set theory. In particular, the Peano axioms enable an infinite set to be generated by a finite set of symbols and rules. Certain constructs show that if there is a given set theory, then models of Peano's postulates must exist (Wikipedia, 2013; Aitken, 2009).

Peano axioms give a formal theory of natural numbers (Alozano, 2012; Carl, 1998; Groisser, 2001). The axioms are:

i. There is a natural number "0"
ii. Every natural number "a" has a natural number successor, denoted by - $S(a)$. Intuitively, $S(a)$ is $a+1$
iii. There is no natural number whose successor is zero.
iv. S is injective. That is, distinct natural numbers have distinct successors
v. If a property is possessed by "0" and also by the successor of every natural number which possesses it, then it is possessed by all natural numbers. This postulate ensures that the proof technique of mathematical induction is valid

Furthermore, it should be noted that the "0" in the above definition need not correspond to what we normally consider to be the number zero. "0" simply means some object that when combined with an appropriate succession function satisfies the Peano axioms (Wikipedia, 2013).

Methodology

For our first axiom, the Natural number system is first partitioned into two sets of numbers. These sets are: the Even and the Odd number sets. Next, Pearson's correlation coefficient is used to verify the degree of correlation between these two partitions as well as draw-up other useful features. In our second axiom, we have established the nature of the regression lines of each partition on one another and the implication thereof.

Axiom 1: The correlation of the Even and Odd partitions of N^* (of equal cardinality) gives unity.

Proof 2: Let X and Y be the Even and Odd partitions of N^* respectively, with equal cardinalities - $n = \frac{\mathrm{N}^*}{2}$. Then $X = \{2i\}$ and $Y = \{2i - 1\}$ $\forall\, i \in \mathrm{N}^*$

Let r be the Pearson's correlation coefficient. Then

$$r = \frac{n\sum_{i=1}^{n} X_i Y_i - \sum_{i=1}^{n} X_i \sum_{i=1}^{n} Y_i}{\sqrt{n\sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i\right)^2}\sqrt{n\sum_{i=1}^{n} Y_i^2 - \left(\sum_{i=1}^{n} Y_i\right)^2}}$$

Such that $r = \sqrt{aa'}$

Where

$$a = \frac{n\sum_{i=1}^{n} X_i Y_i - \sum_{i=1}^{n} X_i \sum_{i=1}^{n} Y_i}{n\sum_{i=1}^{n} X_i^2 - \left(\sum_{i=1}^{n} X_i\right)^2}$$

$$a' = \frac{n\sum_{i=1}^{n} X_i Y_i - \sum_{i=1}^{n} X_i \sum_{i=1}^{n} Y_i}{n\sum_{i=1}^{n} Y_i^2 - \left(\sum_{i=1}^{n} Y_i\right)^2}$$

Now we substitute for $X = \{2i\}$ and $Y = \{2i - 1\}$ in a

$$a = \frac{n\sum_{i=1}^{n} (2i)(2i-1) - \sum_{i=1}^{n} (2i) \sum_{i=1}^{n} (2i-1)}{n\sum_{i=1}^{n} (2i)^2 - \left(\sum_{i=1}^{n} (2i)\right)^2}$$

$$\Rightarrow a = \frac{n\sum_{i=1}^{n} \left(4i^2 - 2i\right) - 2\sum_{i=1}^{n} i \sum_{i=1}^{n} (2i-1)}{n\sum_{i=1}^{n} 4i^2 - \left(2\sum_{i=1}^{n} i\right)^2}$$

$$\Rightarrow a = \frac{n\sum_{i=1}^{n}\left(4i^2 - 2i\right) - 2\left(\sum_{i=1}^{n} i\right)\left(2\sum_{i=1}^{n} i - n\right)}{n\sum_{i=1}^{n} 4i^2 - \left(2\sum_{i=1}^{n} i\right)^2}$$

$$\Rightarrow a = \frac{4n\sum_{i=1}^{n} i^2 - 2n\sum_{i=1}^{n} i - 4\left(\sum_{i=1}^{n} i\right)^2 + 2n\sum_{i=1}^{n} i}{4n\sum_{i=1}^{n} i^2 - \left(2\sum_{i=1}^{n} i\right)^2}$$

$$\Rightarrow a = \frac{4n\sum_{i=1}^{n} i^2 - 2n\sum_{i=1}^{n} i - 4\left(\sum_{i=1}^{n} i\right)^2 + 2n\sum_{i=1}^{n} i}{4n\sum_{i=1}^{n} i^2 - 4\left(\sum_{i=1}^{n} i\right)^2}$$

$$\Rightarrow a = \frac{4n\sum_{i=1}^{n} i^2 - 4\left(\sum_{i=1}^{n} i\right)^2}{4n\sum_{i=1}^{n} i^2 - \left(4\sum_{i=1}^{n} i\right)^2}$$

$$\Rightarrow a = 1$$

Next, we substitute for $X = \{2i\}$ and $Y = \{2i - 1\}$ in a'

$$a' = \frac{n\sum_{i=1}^{n}(2i)(2i-1) - \sum_{i=1}^{n}(2i)\sum_{i=1}^{n}(2i-1)}{n\sum_{i=1}^{n}(2i-1)^2 - \left(\sum_{i=1}^{n}(2i-1)\right)^2}$$

$$\Rightarrow a' = \frac{4n\sum_{i=1}^{n} i^2 - 4\left(\sum_{i=1}^{n} i\right)^2}{n\sum_{i=1}^{n}(2i-1)^2 - \left(\sum_{i=1}^{n}(2i-1)\right)^2}$$

$$\Rightarrow a' = \frac{4n\sum_{i=1}^{n} i^2 - 4\left(\sum_{i=1}^{n} i\right)^2}{n\sum_{i=1}^{n}\left(4i^2 - 4i + 1\right) - \left(2\sum_{i=1}^{n} i - n\right)^2}$$

$$\Rightarrow a' = \frac{4n\sum_{i=1}^{n} i^2 - 4\left(\sum_{i=1}^{n} i\right)^2}{4n\sum_{i=1}^{n}\left(4i^2 - 4i + 1\right) - 4\left(\sum_{i=1}^{n} i\right)^2 + 2\left(2n\sum_{i=1}^{n} i\right) - n^2}$$

$$\Rightarrow a' = \frac{4n\sum_{i=1}^{n} i^2 - 4\left(\sum_{i=1}^{n} i\right)^2}{4n\sum_{i=1}^{n} i^2 - 4n\sum_{i=1}^{n} i + n^2 - 4\left(\sum_{i=1}^{n} i\right)^2 + 4n\sum_{i=1}^{n} i - n^2}$$

$$\Rightarrow a' = \frac{4n\sum_{i=1}^{n} i^2 - 4\left(\sum_{i=1}^{n} i\right)^2}{4n\sum_{i=1}^{n} i^2 - \left(4\sum_{i=1}^{n} i\right)^2}$$

$$\Rightarrow a' = 1$$

Next, we evaluate the Pearson's correlation coefficient - r . This gives

$$r = \sqrt{aa'} = 1$$

□

The proof above implies that there is perfect correlation between - X & - Y . It also establishes the following corollary.

Corollary 1: Let X and Y be the Even and Odd partitions of N^* respectively, with equal cardinalities - $n = \frac{\mathrm{N}^*}{2}$. Then,

i. $Cov(X,Y) = Var(X) = Var(Y)$
ii. $Cov(X,Y) = \sqrt{Var(X)Var(Y)}$

Axiom 2: Let X and Y be the Even and Odd partitions of N^* with equal cardinalities. The linear regression lines of X on Y and Y on X are equal, and are respectively $X = Y + 1$ and $Y = X + 1$

Proof 2: From Axiom 1 $a = 1$ and - $a' = 1$. We recall that the regression lines of a set of observations X on another set of observations Y and vice-versa are respectively given as $X = aY + b$ and $Y = a'X + b'$

Also, $b = \overline{X} - a\overline{Y}$

$$\Rightarrow b = \frac{\sum_{i=1}^{n} X_i}{n} - \frac{\sum_{i=1}^{n} Y_i}{n}$$

$$\Rightarrow b = \frac{\sum_{i=1}^{n} (2i)}{n} - \frac{\sum_{i=1}^{n} (2i-1)}{n}$$

$$\Rightarrow b = \frac{2\sum_{i=1}^{n} i}{n} - \frac{2\sum_{i=1}^{n} i - n}{n}$$

$$\Rightarrow b = \frac{2\sum_{i=1}^{n} i}{n} - \frac{2\sum_{i=1}^{n} i}{n} + \frac{n}{n}$$

$$\Rightarrow b = \frac{2\sum_{i=1}^{n} i}{n} - \frac{2\sum_{i=1}^{n} i}{n} + 1$$

$$\Rightarrow b = 1$$

$$\Rightarrow X = (1)Y + 1$$

$$\Rightarrow X = Y + 1$$

Similarly,

Also, $b' = \overline{X} - a'\overline{Y}$

$$\Rightarrow b' = \frac{\sum_{i=1}^{n} X_i}{n} - \frac{\sum_{i=1}^{n} Y_i}{n}$$

$$\Rightarrow b' = \frac{\sum_{i=1}^{n} (2i)}{n} - \frac{\sum_{i=1}^{n} (2i-1)}{n}$$

$$\Rightarrow b' = \frac{2\sum_{i=1}^{n} i}{n} - \frac{2\sum_{i=1}^{n} i - n}{n}$$

$$\Rightarrow b' = \frac{2\sum_{i=1}^{n} i}{n} - \frac{2\sum_{i=1}^{n} i}{n} + \frac{n}{n}$$

$$\Rightarrow b' = \frac{2\sum_{i=1}^{n} i}{n} - \frac{2\sum_{i=1}^{n} i}{n} + 1$$

$$\Rightarrow b' = 1$$

$$\Rightarrow Y = (1)X + 1$$

$$\Rightarrow X = Y + 1$$

□

It is evident from this second axiom that

i. The explanatory power of the model is very high. That is, 100%.
ii. The error term is zero. Hence, there is perfect stability.

Conclusion

The concept of correlation has been used in the analysis of two partitions of the natural number system. Significant results have been established, and summarised as axioms. These established axioms extend the existing axioms necessarily satisfied by a number system which is a natural number system. This implies that if N^* is natural number system as defined above, then, in addition to the Peano axioms above,

i. The correlation of the Even and Odd partitions of N^* (of equal cardinality) must give unity.
ii. The linear regression lines of the Even partition (X) on the Odd partition (Y) and vice-versa are respectively $X = Y + 1$ & $Y = X + 1$

It is expected that further research may unravel deeper theoretical and applied results.

References

Aitken, (2009), Chapter 1: The Peano Axioms. MATH 378, CSUSM. SPRING. (*http://publics.csusm.edu/aitken_html/m378/ch1PeanoAxioms.pdf*)

Alozano, (2012), Peano Arithmetic. (*http://planetmath.org/peanoarithmetic*).

Carl, L. (1998), Peano's Axioms. (*http://ms.uky.edu/~lee/ma502/notes2/node7.html*)

CCSA,(2013),PeanoAxioms.(*http://www.priceton.edu/~achaney/tmvc/wiki100k/docs/ Peano_axioms.html*)

Groisser, D. (2001), Peano Axioms for the Natural Numbers. (*http://www.math.ufl.edu/~groisser/classes/4211_f01/misc_handouts/peano_axioms.pdf*)

Jaeger, S. (2011), Computational Complexity on Signed Numbers.

(http://www.jaeger.info)

Joyce, D. (2005), The Dedekind/Peano Axioms. Clark University. (*http://www.alephi.clarku.edu./~djoyce/numbers/peano.pdf*)

Turing, A. (1936), On Computable numbers, with an Application to the Entscheidungsproblem. *Proc. Of the London Mathematical Society*. **42**(2), 230-265.

Turing, A. (1937), On Computable Numbers, with an Application to the Entscheidungs problem: A Correction. *Proc. Of the London Mathematical Society*. **43**(2), 544-546.

Wikipedia, (2013), Natural Numbers. (*http://en.wikipedia.org/wiki/Natural-numbers*)

William,L.H.(2013),PeanoAxioms.EncyclopaediaBritannica. (*http://www.britannica.com/EBchecked/topic/447921/peano.axioms*)