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# Some Estimation Methods for the Shape Parameter and Reliability Function of Burr Type XII Distribution / Comparison Study

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**Abstract**

Burr type XII distribution plays an important role in reliability modeling, risk analyzing and process capability estimation. The choice of the best estimation method is one of the goals in estimating parameters of the distribution. The main aim of this paper is to obtain and compare the classical "maximum likelihood and uniformly minimum variance unbiased" estimators and the Bayesian estimators of the shape parameter, $\theta$ and reliability function based on a complete sample when the other shape parameter, $\lambda$ known. The Bayes estimators are obtained under non-informative priors "Jeffrey's prior, modified and extension of Jeffrey's prior" as well as under informative gamma prior based on different symmetric and asymmetric loss functions "squared error, quadratic, LINEX, precautionary and entropy". The Monte Carlo experiment was performed under a wide range of cases and sample size. The estimates of the unknown shape parameter were compared by employing the mean square errors and the estimates of reliability function were compared by employing the integrated mean squared error.

**Keywords:** Burr type XII distribution; Maximum likelihood estimator; Uniformly Minimum Variance Unbiased estimator; Bayes estimators; non-informative Prior; informative Prior; Squared error loss function; quadratic loss function; LINEX loss function; Precautionary loss function; Entropy Loss function; Mean squared error; integrated mean squared error.

## 1. Introduction

Burr introduced twelve different forms of cumulative distribution functions for modeling lifetime data or survival data [7]. Out of those twelve distributions, Burr Type XII and Burr Type X have received the maximum attention due to its application in the study of biological, industrial, reliability and life testing, and several industrial and economic experiments [23]. The Burr Type XII has the following distribution function for $t > 0$ :

$$F(t;\theta,\lambda) = 1 - \frac{1}{(1+t^{\lambda})^{\theta}} \quad ; \theta > 0, \lambda > 0 \tag{1}$$

Therefore, the Burr Type XII has the density function for $t > 0$ as :

$$f(t;\theta,\lambda) = \theta\,\lambda\; t^{\lambda-1}\,\frac{1}{(1+t^{\lambda})^{\theta+1}} \quad ; \theta > 0, \lambda > 0 \tag{2}$$

where $\theta$ and $\lambda$ are the shape parameters of the distribution.

The reliability function, $R(t)$, and the hazard function, $h(t)$ are given as follows [3]:

$$R(t) = \left(1+t^{\lambda}\right)^{-\theta} \tag{3}$$

$$h(t) = \frac{f(t)}{R(t)} = \frac{\lambda\,\theta\,t^{\lambda-1}}{1+t^{\lambda}} \tag{4}$$

Inferences on the Burr type XII distribution have been studied by many authors. Rodriguez (1977) [24] gave a comprehensive overview of Burr type XII distribution. A Bayesian approach to inference about the parameters of a Burr distribution was taken by Papadopoulos (1978) [22] where he used the distribution as a failure model. Tadikamalla (1980) [28] gave additional attention to the type XII distribution because it include a variety of distributions with varying degrees of skew and kurtosis. Lewis (1981) [16] stated the fact that many standard theoretical distributions, including the two common failure time distributions, the Weibull and the Exponential are special cases or limiting cases of the Burr type XII distribution. Evans and Ragab (1983) [11] obtained Bayes estimates of the shape parameter $\theta$ and the reliability function based on type-II censored samples. AL-Marzoug and Ahmad (1985) [5] studied some of the properties of the Burr probability model. Cook and Johnson (1986) [8] used the Burr type XII model to obtain better fits to a uranium survey data set. Khan and Khan (1987) [15] studied the moments of order statistics from Burr's distribution and its characterization. AL-Hussaini et al. (1992) [1], based on type-II censored samples, computed and compared the maximum likelihood, uniformly minimum variance unbiased, Bayes "based on squared error loss function with a gamma conjugate prior" and empirical Bayes estimators of one of the two shape parameters, θ, and reliability function R(t) when the other shape parameter, λ, is known. Ali Mousa (1995) [2] obtained empirical Bayes estimation of the shape parameter θ and the reliability function based on accelerated type-II censored data. Wang and Keats (1996) [30] used the maximum likelihood method for obtaining point and interval estimates of the parameters based on censored data as well as complete data. Zimmer et al. (1998) [33] presented statistical and probabilistic properties of the Burr type XII distribution and described its relationship to other distributions used in reliability analyses. Moore and Papadopoulos (2000) [19] studied the Burr-XII distribution as a failure model under various symmetric loss functions "absolute difference loss, squared error loss and logarithmic loss". Ali Mousa and Jaheen (2002)[3] obtained maximum likelihood and Bayes approximate estimates for the shape parameters and the reliability function under squared error loss function based on progressive type-II censored samples. Soliman (2002) [26] obtained the Bayes estimators of the reliability function in generalized life model relative to symmetric loss function (quadratic loss) and asymmetric loss functions (LINEX loss and general entropy loss), Comparisons are made between those estimators and the maximum likelihood estimator applying to the Burr-XII model using the Bayes approximation due to Lindley. Soliman (2005) [27] derived the maximum likelihood and Bayes estimators for some lifetime parameters (reliability and hazard functions), as well as the parameters of the Burr-XII model based on progressively type-II censored samples. Wahed (2006) [29] presented Bayes estimators for the parameters under the symmetric squared error loss function and the asymmetric LINEX loss function based on a simple prior distribution. Asgharzadeh and Valiollahi (2008) [6] derived the uniformly minimum variance unbiased, Bayes and empirical Bayes estimates for the unknown parameter and the reliability function based on progressively type-II censored samples. Jalali and Watkins (2009) [14] considered three related aspects of maximum likelihood estimation of parameters in the two-parameter Burr type XII distribution. Headrick et al. (2010) [15] obtained the classical estimators of the shape parameter, such as, the maximum likelihood estimator, the uniformly minimum variance unbiased estimator, and the minimum mean squared error estimator, then they obtained the minimax estimators of this parameter under the squared log error and precautionary loss functions. Yarmohammadi and Pazira (2010) [31] compared the classical estimators of the shape parameter with the minimax estimators under weighted balanced squared error, squared log error and special case of precautionary loss functions. Yatomma (2011) [32] compared different estimation methods (maximum likelihood, Bayes and empirical Bayes) for reliability function based on (squared error, absolute error, squared logarithmic error and LINEX) loss functions with gamma (1,β) as prior distribution using progressive type II censored data. Also, Makhdoom and Jafari (2011) [18] compared empirically using Monte-Carlo simulation the point and interval Bayesian estimators for the shape parameter with the special form of the distribution when $(\lambda = 1)$ using grouped and un-grouped data. Nasir and AL-Anber (2012) [20] did comparative study for the maximum likelihood estimator, median estimator and Bayesian estimators for estimation the reliability function under Jeffrey, modified Jeffrey and extension of Jeffrey priors information with squared error loss function. Also, Rastogi and Tripathi (2012) [23] obtained several Bayesian estimates under different symmetric and asymmetric loss functions such as squared error, LINEX and general entropy on the basis of a progressively type II censored sample. The Bayesian estimates are evaluated by applying the Lindley approximation method. Saracoglu et al. (2013) [25] obtained the maximum likelihood, ordinary least squares, weighted least squares and best linear unbiased estimators for the shape parameter θ based on progressive type-II right censored samples. The main aim of this paper is to obtain and compare the classical "maximum likelihood and uniformly minimum variance unbiased" estimators and the Bayesian estimators of the shape parameter, $\theta$ and reliability function based on a complete sample when the other shape parameter, λ known. The Bayes estimators are obtained under non-informative priors as well as under informative gamma prior based on different symmetric and asymmetric loss functions. The Monte Carlo experiment was performed under a wide range of cases and sample size. The estimates of the unknown shape parameter were compared by employing the mean square errors and the estimates of reliability function were compared by employing the integrated mean squared error.

## 2. Different Estimators of Shape Parameter θ and Reliability Function R(t)

In this section classical and Bayes estimators of the shape parameter, θ, and reliability function has been determined with the assumption that the other shape parameter, λ , is known.

### 2.1 Maximum Likelihood Estimator (MLE)

Let $t_1, t_2, \dots\dots, t_n$ be a random sample of size n drawn from the Burr type XII distribution defined by (2). Then the Likelihood function for the given random sample is given by:

 

$$L(\theta,\lambda|\underline{t}) = \prod_{i=1}^{n} f(t_i|\theta,\lambda) = \frac{\theta^n \lambda^n \prod_{i=1}^{n} t_i^{\lambda-1}}{\prod_{i=1}^{n}\left(1+t_i^{\lambda}\right)^{\theta+1}} \quad (5)$$

From which we calculate the natural log-likelihood function:

$$l(\theta,\lambda|\underline{t}) = n\ln\theta + n\ln\lambda + (\lambda-1)\sum_{i=1}^{n}\ln(t_i) - (\theta+1)\sum_{i=1}^{n}\ln\left(1+t_i^{\lambda}\right)$$

Finding the maximum with respect to θ by taking the derivative and setting it equal to zero yields the maximum likelihood Estimator of the θ parameter, denoted by $\hat{\theta}_{ML}$:

$$\hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^{n}\ln\left(1+t_i^{\lambda}\right)} \quad (6)$$

Since the maximum likelihood estimator is invariant and one to one mapping, the maximum likelihood estimator of reliability function, denoted by $\hat{R}(t)_{ML}$ is:

$$\hat{R}(t)_{ML} = \left(1+t^{\lambda}\right)^{-\hat{\theta}_{ML}} \quad (7)$$

### 2.2 Uniformly Minimum Variance Unbiased Estimator (UMVUE)

The function of Burr type XII distribution is belongs to exponential family. Therefor $\sum_{i=1}^{n}\ln\left(1+t_i^{\lambda}\right)$ is a complete sufficient statistic for θ [31]. If T has a Burr type XII distribution, then:

$$\ln\left(1+t_i^{\lambda}\right) \sim \text{Exponential } (\theta) \Longrightarrow \sum_{i=1}^{n}\ln\left(1+t_i^{\lambda}\right) \sim \text{Gamma } (n, \theta)$$

Now, depending on the theorem of Lehmann-Scheffe [13], taking the mathematical expected of the complete sufficient statistic yields the Uniform Minimum Variance Unbiased Estimator of the θ parameter, denoted by $\hat{\theta}_{UMVUE}$:

$$\hat{\theta}_{UMVU} = \frac{n-1}{\sum_{i=1}^{n}\ln\left(1+t_i^{\lambda}\right)} \quad (8)$$

The approximate uniformly minimum variance unbiased estimator of R(t), denoted by $\hat{R}(t)_{UMVU}$ is given by:

$$\hat{R}(t)_{UMVU} \cong \left(1+t^{\lambda}\right)^{-\hat{\theta}_{UMVU}} \quad (9)$$

### 2.3 Bayes Estimation

In this subsection we studied Bayes estimators of shape parameter θ and reliability function R(t) based on non-informative and informative priors with different symmetric and asymmetric loss functions.

#### 2.3.1 Prior and Posterior Density Function

For Bayesian estimation we need to specify a prior distribution for the parameter. We consider non-informative (Jeffrey with modified and extension Jeffrey) prior along with informative (Gamma) prior. Then the posterior density function of θ, denoted by $\pi(\theta|\underline{x})$, for the given random sample T with prior information is given by combining the specified prior with the likelihood (5) such as:

$$\pi(\theta|\underline{t}) = \frac{\prod_{i=1}^{n} f(t_i;\theta)\, g(\theta)}{\int_0^{\infty} \prod_{i=1}^{n} f(t_i;\theta)\, g(\theta)\, d\theta} \quad (10)$$

The considered priors and corresponding posterior density functions are summarized in table (1).

#### 2.3.2 Loss Functions

Here we have determined Bayes estimators of shape parameter θ and reliability function R(t) based on different symmetric loss functions "squared error and quadratic" and asymmetric loss functions "LINEX, precautionary and entropy". The symmetric loss function associates equal importance to the losses due to overestimation and underestimation of equal magnitude. However, in real applications, the estimation of the parameters "or function as reliability function" an overestimation is more serious than the underestimate; thus, the use of a symmetrical loss function is inappropriate. In this case, an asymmetric loss functions must be considered. The Bayes estimators of θ and R(t) corresponding to each loss function are given by the formulas which summarized in table (2). The obtained Bayes estimators of $\theta$ and R(t) for Burr type XII distribution are shown in table (3).

**3. Simulation Study and Results**

In this section we introduce the simulation study which has been conducted to assess the statistical performance of the shape parameter and reliability function that we obtain in the previous section. The simulation program is written by using MATLAB (R2011b) program. Simulation experience includes four basic and important stages to estimate the shape parameter and reliability function of the Burr type XII distribution, namely:

**First Stage**: this is the most important stage that depends on it the rest of the stages. The first stage involves determining the default values (true values) for:

- The parameters of the Burr type XII distribution $(\lambda, \theta)$: The default values are varied to observe the effect of parameters on the estimators when $\lambda > \theta, \lambda = \theta, \lambda < \theta$. We consider the value of $\lambda = 0.5$ with $\theta = 0.25, 0.5, 1.5$.
- Sample size (n): the sample size used are (n = 10, 15, 25, 30, 50, and 100) to represent small, medium, and large dataset.
- Values of Jeffrey extension (c): the values used are 0.4, 1 and 2.
- The parameters of gamma prior distribution $(\alpha, \beta)$: the default values of the hyper-parameters chosen are (2, 4), (4, 4) and (4, 2).
- Values of LINEX loss function constant (a): the values used are (0.8) and (- 0.8).
- Number of sample replicated size (L): The process is repeated 1000 times to obtain 1000 independent samples of size n.

**Second Stage**: this stage involves data generating. A random data are generated as uniform distribution (U) for the period (0, 1). Then data generated are converted from uniform distribution to data distribute as Burr type XII distribution with shape parameters $\lambda$ and $\theta$ through the adoption of cumulative distribution function by using the method of inverse transformation as:

$$t_i = F^{-1}(U_i) = \left((1 - U_i)^{-\frac{1}{\theta}} - 1\right)^{\frac{1}{\lambda}} \quad ; i = 1,2, \dots, n \qquad (11)$$

**Third Stage**: this stage involves calculate the estimators of shape parameter through the estimation methods that we have dealt with in the previous section according to the formulas of parameter $\hat{\theta}_{ML}$, $\hat{\theta}_{UMVU}$, $\hat{\theta}_{SJ}$, $\hat{\theta}_{MJ}$, $\hat{\theta}_{SEJ}$, $\hat{\theta}_{SG}$, $\hat{\theta}_{QJ}$, $\hat{\theta}_{QMJ}$, $\hat{\theta}_{QEJ}$, $\hat{\theta}_{QG}$, $\hat{\theta}_{LJ}$, $\hat{\theta}_{LMJ}$, $\hat{\theta}_{LEJ}$, $\hat{\theta}_{LG}$, $\hat{\theta}_{PJ}$, $\hat{\theta}_{PMJ}$, $\hat{\theta}_{PEJ}$, $\hat{\theta}_{PG}$, $\hat{\theta}_{ENJ}$, $\hat{\theta}_{ENMJ}$, $\hat{\theta}_{ENEJ}$, $\hat{\theta}_{ENG}$ and reliability function according to the formulas of $\hat{R}(t)_{ML}$, $\hat{R}(t)_{UMVU}$, $\hat{R}(t)_{SJ}$, $\hat{R}(t)_{SMJ}$, $\hat{R}(t)_{SEJ}$, $\hat{R}(t)_{SG}$, $\hat{R}(t)_{QJ}$, $\hat{R}(t)_{QMJ}$, $\hat{R}(t)_{QEJ}$, $\hat{R}(t)_{QG}$, $\hat{R}(t)_{LJ}$, $\hat{R}(t)_{LMJ}$, $\hat{R}(t)_{LEJ}$, $\hat{R}(t)_{LG}$, $\hat{R}(t)_{PJ}$, $\hat{R}(t)_{PMJ}$, $\hat{R}(t)_{PEJ}$ , $\hat{R}(t)_{PG}$, $\hat{R}(t)_{ENJ}$, $\hat{R}(t)_{ENMJ}$, $\hat{R}(t)_{ENEJ}$, $\hat{R}(t)_{ENG}$. (See table (3))

**Fourth Stage**: this stage involves the comparison between the estimation methods for the shape parameter and the reliability function through statistical measures. After the shape parameter is estimated, mean squared error (MSE) is calculated to compare the estimation methods, where:

$$MSE(\hat{\theta}) = \frac{\sum_{j=1}^{L}\left(\hat{\theta}_j - \theta\right)^2}{L} \qquad (12)$$

Where $\hat{\theta}_j$ is the estimate of $\theta$ at the j th replicate (run).

The MSE gives an error of the estimator at an arbitrary point, but it is worth to study a global risk for the estimator. The integrated mean squared error (IMSE) is an important global measure. So, to reach to the best estimated for estimate the reliability function and depending on the fact that (MSE) is calculated for each $(t_i)$ of the time, it has been a comparison between the estimation methods under studied by (IMSE) which is an integration of the total area for $(t_i)$ and reduced it in one value that is expressive of the total time. The IMSE of an estimator is defined as [20][32]:

$$IMSE\left(\hat{R}(t)\right) = \frac{1}{L}\sum_{j=1}^{L}\left(\frac{1}{n_t}\sum_{i=1}^{n_t}\left(\hat{R}_j(t_i) - R(t_i)\right)^2\right) = \frac{1}{n_t}\sum_{i=1}^{n_t} MSE\left(\hat{R}(t_i)\right) \quad (13)$$

 

$$\mathrm{MSE}\big(\hat{R}(t)\big) = \frac{\sum_{j=1}^{L}\big(\hat{R}_j(t) - R(t)\big)^2}{L} \qquad (14)$$

L: the number of replications.
$n_t$ : the number of times (bounds of time from lower to upper), we consider four values for t (t=0.5,1,1.5,2).

The simulation results of MSE and IMSE are summarized in tables (4)…(7)

**4. Conclusions and Recommendations**

Based on results of estimating the shape parameter of Burr type XII distribution, tables (4) and (5), the following conclusions could be reached:

- Between classical estimators, the performance of UMVUE is better than MLE for all sample sizes and for all different values of $\theta$.
- The performance of Bayes estimator with modified Jeffrey's prior is better comparing to Jeffrey's prior under all different loss functions except quadratic loss function.
- Among non-informative prior distributions, Jeffrey's and modified Jeffrey's priors don't record any appearance as best prior while extension Jeffrey's recorded appearance for seven times. Gamma informative prior distribution recorded appearance for twelve times as best prior distribution.
- The formula of the Bayes estimator of $\theta$ under squared error loss function with Jeffrey's prior information, $\hat{\theta}_{SJ}$, is the same as the formula of the maximum likelihood estimator, $\hat{\theta}_{ML}$ , and the formula of the Bayes estimator of $\theta$ under entropy loss function with Jeffrey's prior information, $\hat{\theta}_{ENJ}$, is the same as the formula of the uniformly minimum variance unbiased estimator, $\hat{\theta}_{UMVU}$ . So, the Bayes estimator may give the classical estimator in some cases.
- For specific values of Jeffrey's extension (c) can get other estimator, such that when c=1 in the formula of the Bayes estimator of $\theta$ under squared error loss function, we get the Bayes estimator under entropy loss function with Jeffrey's prior. Also when c=1 in the formula of the Bayes estimator of $\theta$ under entropy loss function, we get the Bayes estimator under quadratic loss function with Jeffrey's prior. When c=2 in the formula of the Bayes estimator of $\theta$ under squared error loss function, we get the Bayes estimator under quadratic loss function with extension Jeffrey's prior when c=1. So, the Bayes estimator may give other Bayes estimator in some specific values.
- Through the MSE values for the best prior distributions under different loss functions, the performance of the squared error loss function was better than some of the functions, but in spite of that, it doesn't record any appearance as the best loss function. Each of LINEX and precautionary loss functions recorded one time while quadratic loss function record two times as the best loss function.
- When $\theta = 0.5$, quadratic loss function with gamma prior (α=2, β=4), records full appearance "for all sample sizes" as the best loss function. Also, when $\theta = 0.25$ it is recorded the best loss function with same prior for $n \leq 25$ and with extension Jeffrey's prior (c=0.4) for $n \geq 30$.
- For gamma prior (α=4, β=4) and $\theta = 1.5$, precautionary loss function records as the best loss function for $n \leq 15$ while LINEX loss function with (a=0.8) records as the best loss function for $n \geq 25$.
- For all cases, as the sample size increase the values of MSE decrease and this conforms to the statistical theory. Also the results showed a convergence between most of the estimators to increase the sample size.

Now, based on results of estimating the reliability function of Burr type XII distribution, tables (6) and (7), the following conclusions could be reached:

- Between classical estimators, the performance of UMVUE is better than MLE for all sample sizes when $\lambda > \theta$ and $\lambda = \theta$ while MLE is better than UMVUE for all sample sizes when $\lambda < \theta$ .
- The performance of Bayes estimator under squared error loss function with modified Jeffrey's prior when $\theta = 0.25, 0.5$ is better comparing to Jeffrey's prior for all sample sizes while when $\theta = 1.5$ the Jeffrey's prior is outperformed. The performance of Bayes estimator under LINEX loss function with Jeffrey's prior is better comparing to modified Jeffrey's prior for $\lambda < \theta$ while the opposite is true for the rest of the cases. The performance of Bayes estimator under precautionary loss function with modified Jeffrey's prior is better comparing to Jeffrey's prior when $\theta = 0.25, 0.5$ while the situation is vice versa when $\theta = 1.5$. The performance of Bayes estimator under quadratic and entropy loss functions with modified Jeffrey's prior is better comparing to Jeffrey's prior for all sample sizes and all different values of $\theta$.
- The squared error and LINEX loss functions don't record any appearance as the best loss function. Precautionary loss function records two times while each of quadratic and entropy loss functions recorded one time as the best loss function.

- Precautionary loss function with gamma prior (α=2, β=4), records full appearance "for all sample sizes" as the best loss function when $\theta = 0.5$. Moreover, when $\theta = 0.25$, precautionary loss function with extension Jeffrey's prior when (c=1) for $n \leq 25$ and with (c=0.4) for $n \geq 30$ recorded appearance as the best loss function.
- When $\theta = 1.5$, quadratic loss function with gamma prior (α=4, β=4), records appearance as the best loss function for $n = 10$ while entropy loss function with the same prior records appearance as the best loss function for the rest of sample sizes $(n \geq 15)$ .
- For all cases, as the sample size increase the values of IMSE decrease and this conforms to the statistical theory. Also the results show a convergence between most of the estimators to real values with increasing the sample size.

Among non-informative prior distributions, Jeffrey's and modified Jeffrey's priors don't record any appearance as best prior while extension Jeffrey's with extension value (c=0.4,1) recorded appearance as best prior with all different loss functions in the first case, $\lambda = 0.5, \theta = 0.25$ . For all other cases, gamma informative prior distribution recorded appearance as best prior distribution with all different loss functions.

Now, through the conclusions that have been obtained for the shape parameter, we recommend:

- For classical estimators, using UMVUE to estimate the shape parameter.
- Dealing with modified Jeffrey's prior instead of Jeffrey's prior for all different loss functions under study except quadratic loss function. And dealing with extension Jeffrey's prior as non-informative prior.
- Using quadratic loss function as symmetric loss function with gamma prior (α=2, β=4) when $\theta = 0.5$ and using precautionary loss function as asymmetric loss function with gamma prior (α=4, β=4) when $\theta = 1.5$.

For the reliability function, we recommend:

- For classical estimators, using UMVUE when $\lambda > \theta$ , $\lambda = \theta$ and using MLE when $\lambda < \theta$ .
- Using precautionary loss function with gamma prior (α=2, β=4) when $\theta = 0.5$ .
- Dealing with extension Jeffrey's prior as non-informative prior when $\theta = 0.25$, and dealing with informative gamma prior when $\theta = 0.5, 1, 1.5$.
- Conduct future research to estimate the reliability function of Burr type XII distribution taking into account changing: loss function, prior distribution or type of data.

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**Table (1): The Prior and Posterior Density Functions**

| Prior Distribution | Posterior Distribution |
|---|---|
| Jeffrey's Prior [20]: $g(\theta) \propto \frac{1}{\theta}$ | $\pi(\theta\|\underline{t})_J = \frac{\left(\sum_{i=1}^{n}\ln\left(1+t_i^{\lambda}\right)\right)^{n}}{\Gamma(n)}\theta^{n-1}e^{-\theta\sum_{i=1}^{n}\ln\left(1+t_i^{\lambda}\right)}$ <br> $\Rightarrow (\theta\|\underline{t})_J \sim \text{Gamma}\left(n\,,\ \sum_{i=1}^{n}\ln\left(1+t_i^{\lambda}\right)\right)$ |
| Modified Jeffrey's Prior [20]: $g(\theta) \propto \frac{1}{\sqrt{\theta^3}}$ | $\pi(\theta\|\underline{t})_{MJ} = \frac{\left(\sum_{i=1}^{n}\ln\left(1+t_i^{\lambda}\right)\right)^{n-\frac{1}{2}}}{\Gamma\left(n-\frac{1}{2}\right)}\theta^{n-\frac{3}{2}}\ e^{-\theta\sum_{i=1}^{n}\ln\left(1+t_i^{\lambda}\right)}$ <br> $\Rightarrow (\theta\|\underline{t})_{MJ} \sim \text{Gamma}\left(n-\frac{1}{2}\ ,\ \sum_{i=1}^{n}\ln\left(1+t_i^{\lambda}\right)\right)$ |
| Extension Jeffrey's Prior [20]: $g(\theta) \propto \frac{1}{\theta^{2c}}$ | $\pi(\theta\|\underline{t})_{EJ} = \frac{\left(\sum_{i=1}^{n}\ln\left(1+t_i^{\lambda}\right)\right)^{n-2c+1}}{\Gamma(n-2c+1)}\theta^{n-2c}\ e^{-\theta\sum_{i=1}^{n}\ln\left(1+t_i^{\lambda}\right)}$ <br> $\Rightarrow (\theta\|\underline{t})_{EJ} \sim \text{Gamma}\left(n-2c+1\ ,\ \sum_{i=1}^{n}\ln\left(1+t_i^{\lambda}\right)\right)$ |

The Jeffrey's prior considered as [4]: $g(\theta) \propto \sqrt{I(\theta)}$

The Modified Jeffrey's prior considered as: $g(\theta) \propto \left[\sqrt{I(\theta)}\right]^{\frac{3}{2}}$

The Extension Jeffrey's prior considered as [12]: $g(\theta) \propto [I(\theta)]^c$ ; $c \in R^+$

Where $I(\theta) = -n\,E\left[\frac{\partial^2 \ln f(t;\lambda,\theta)}{\partial\theta^2}\right]$ is the Fisher's information matrix.

| | |
|---|---|
| Gamma Prior: $g(\theta) = \frac{\beta^\alpha \theta^{\alpha-1} e^{-\beta\theta}}{\Gamma(\alpha)}; \alpha, \beta > 0$ | $\pi(\theta\|\underline{t})_G = \frac{\left(\sum_{i=1}^n \ln(1+t_i^\lambda)+\beta\right)^{n+\alpha}}{\Gamma(n+\alpha)} \theta^{n+\alpha-1} e^{-\theta\left(\sum_{i=1}^n \ln(1+t_i^\lambda)+\beta\right)}$ <br> $\Rightarrow (\theta\|\underline{t})_G \sim \text{Gamma}\left(n+\alpha\ ,\ \sum_{i=1}^n \ln(1+t_i^\lambda)+\beta\right)$ |

**Table (2): Bayes Estimator under Different Loss Function**

| Loss Function | Shape Parameter | Reliability Function |
|---|---|---|
| Squared Error Loss [23] $L(\hat\theta,\theta) = (\hat\theta-\theta)^2$ | $\hat\theta_S = E_\pi(\theta\|\underline{t})$ | $\hat R(t)_S = E_\pi(R(t)\|\underline{t})$ |
| Quadratic Loss [10] $L(\hat\theta,\theta) = \left(\frac{\hat\theta-\theta}{\theta}\right)^2$ | $\hat\theta_Q = \frac{E_\pi\left(\frac{1}{\theta}\|\underline{t}\right)}{E_\pi\left(\frac{1}{\theta^2}\|\underline{t}\right)}$ | $\hat R(t)_Q = \frac{E_\pi\left(\frac{1}{R(t)}\|\underline{t}\right)}{E_\pi\left(\frac{1}{(R(t))^2}\|\underline{t}\right)}$ |
| LINEX Loss [17] $L(\hat\theta,\theta) =$ $b\left[e^{a(\hat\theta-\theta)} - a(\hat\theta-\theta) - 1\right]; a \neq 0, b > 0$ | $\hat\theta_L = -\frac{1}{a}\ln\left(E_\pi(e^{-a\theta}\mid\underline{t})\right)$ | $\hat R(t)_L = -\frac{1}{a}\ln\left(E_\pi(\mathbf{e}^{-aR(t)}\mid\underline{t})\right)$ |
| Precautionary Loss [21] $L(\hat\theta,\theta) = \frac{(\hat\theta-\theta)^2}{\hat\theta}$ | $\hat\theta_P = \sqrt{E_\pi(\theta^2\|\underline{t})}$ | $\hat R(t)_P = \sqrt{E_\pi((R(t))^2\|\underline{t})}$ |
| Entropy Loss [9] $L(\hat\theta,\theta) = b\left[\frac{\hat\theta}{\theta} - \ln\left(\frac{\hat\theta}{\theta}\right) - 1\right]; b > 0$ | $\hat\theta_{EN} = \left[E_\pi\left(\frac{1}{\theta}\|\underline{t}\right)\right]^{-1}$ | $\hat R(t)_{ENJ} = \left[E_\pi\left(\frac{1}{R(t)}\|\underline{t}\right)\right]^{-1}$ |

**Table (3): Bayes Estimators for Burr Type XII Distribution**

| **Prior Distribution** | **Bayes Estimator** | |
|---|---|---|
| | **Shape Parameter** | **Reliability Function** |
| | **Squared Error Loss** | |
| **Jeffrey** | $\hat\theta_{SJ} = \frac{n}{\sum_{i=1}^n \ln(1+t_i^\lambda)}$ | $\hat R(t)_{SJ} = \left(1 + \frac{\ln(1+t^\lambda)}{\sum_{i=1}^n \ln(1+t_i^\lambda)}\right)^{-n}$ |
| **Modified Jeffrey** | $\hat\theta_{SMJ} = \frac{n-\frac{1}{2}}{\sum_{i=1}^n \ln(1+t_i^\lambda)}$ | $\hat R(t)_{SMJ} = \left(1 + \frac{\ln(1+t^\lambda)}{\sum_{i=1}^n \ln(1+t_i^\lambda)}\right)^{-(n-\frac{1}{2})}$ |
| **Extension Jeffrey** | $\hat\theta_{SEJ} = \frac{n-2c+1}{\sum_{i=1}^n \ln(1+t_i^\lambda)}$ | $\hat R(t)_{SEJ} = \left(1 + \frac{\ln(1+t^\lambda)}{\sum_{i=1}^n \ln(1+t_i^\lambda)}\right)^{-(n-2c+1)}$ |
| **Gamma** | $\hat\theta_{SG} = \frac{n+\alpha}{\sum_{i=1}^n \ln(1+t_i^\lambda)+\beta}$ | $\hat R(t)_{SG} = \left(1 + \frac{\ln(1+t^\lambda)}{\sum_{i=1}^n \ln(1+t_i^\lambda)+\beta}\right)^{-(n+\alpha)}$ |
| | **Quadratic Loss Function** | |
| **Jeffrey** | $\hat\theta_{QJ} = \frac{n-2}{\sum_{i=1}^n \ln(1+t_i^\lambda)}$ | $\hat R(t)_{QJ} = \left(\frac{\sum_{i=1}^n \ln(1+t_i^\lambda) - 2\ln(1+t^\lambda)}{\sum_{i=1}^n \ln(1+t_i^\lambda) - \ln(1+t^\lambda)}\right)^{n}$ |
| **Modified Jeffrey** | $\hat\theta_{QMJ} = \frac{n-\frac{5}{2}}{\sum_{i=1}^n \ln(1+t_i^\lambda)}$ | $\hat R(t)_{QMJ} = \left(\frac{\sum_{i=1}^n \ln(1+t_i^\lambda) - 2\ln(1+t^\lambda)}{\sum_{i=1}^n \ln(1+t_i^\lambda) - \ln(1+t^\lambda)}\right)^{n-\frac{1}{2}}$ |
| **Extension Jeffrey** | $\hat\theta_{QEJ} = \frac{n-2c-1}{\sum_{i=1}^n \ln(1+t_i^\lambda)}$ | $\hat R(t)_{QEJ} = \left(\frac{\sum_{i=1}^n \ln(1+t_i^\lambda) - 2\ln(1+t^\lambda)}{\sum_{i=1}^n \ln(1+t_i^\lambda) - \ln(1+t^\lambda)}\right)^{n-2c+1}$ |
| **Gamma** | $\hat\theta_{QG} = \frac{n+\alpha-2}{\sum_{i=1}^n \ln(1+t_i^\lambda)+\beta}$ | $\hat R(t)_{QG} = \left(\frac{\sum_{i=1}^n \ln(1+t_i^\lambda) - 2\ln(1+t^\lambda)+\beta}{\sum_{i=1}^n \ln(1+t_i^\lambda) - \ln(1+t^\lambda)+\beta}\right)^{n+\alpha}$ |
| | **LINEX Loss Function** | |

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| | | |
|---|---|---|
| **Jeffrey** | $\hat{\theta}_{LJ} = \frac{n}{a}\ \ln\left(1+\frac{a}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})}\right)$ | $\hat{R}(t)_{LJ} = -\frac{1}{a}\ln\sum_{m=0}^{\infty}\frac{(-a)^m}{m!}\left(1+\frac{m\ln(1+t^{\lambda})}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})}\right)^{-n}$ |
| **Modified Jeffrey** | $\hat{\theta}_{LMJ} = \frac{n-\frac{1}{2}}{a}\ \ln\left(1+\frac{a}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})}\right)$ | $\hat{R}(t)_{LMJ} = -\frac{1}{a}\ln\sum_{m=0}^{\infty}\frac{(-a)^m}{m!}\left(1+\frac{m\ln(1+t^{\lambda})}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})}\right)^{-\left(n-\frac{1}{2}\right)}$ |
| **Extension Jeffrey** | $\hat{\theta}_{LEJ} = \frac{n-2c+1}{a}\ \ln\left(1+\frac{a}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})}\right)$ | $\hat{R}(t)_{LEJ} = -\frac{1}{a}\ln\sum_{m=0}^{\infty}\frac{(-a)^m}{m!}\left(1+\frac{m\ln(1+t^{\lambda})}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})}\right)^{-(n-2c+1)}$ |
| **Gamma** | $\hat{\theta}_{LG} = \frac{n+\alpha}{a}\ \ln\left(1+\frac{a}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})+\beta}\right)$ | $\hat{R}(t)_{LG} = -\frac{1}{a}\ln\sum_{m=0}^{\infty}\frac{(-a)^m}{m!}\left(1+\frac{m\ln(1+t^{\lambda})}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})+\beta}\right)^{-(n+\alpha)}$ |
| | **Precautionary Loss Function** | |
| **Jeffrey** | $\hat{\theta}_{PJ} = \frac{\sqrt{n(n+1)}}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})}$ | $\hat{R}(t)_{PJ} = \sqrt{\left(1+\frac{2\ln(1+t^{\lambda})}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})}\right)^{-n}}$ |
| **Modified Jeffrey** | $\hat{\theta}_{PMJ} = \frac{\sqrt{n^2-\frac{1}{4}}}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})}$ | $\hat{R}(t)_{PMJ} = \sqrt{\left(1+\frac{2\ln(1+t^{\lambda})}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})}\right)^{-\left(n-\frac{1}{2}\right)}}$ |
| **Extension Jeffrey** | $\hat{\theta}_{PEJ} = \frac{\sqrt{(n-2c+1)(n-2c+2)}}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})}$ | $\hat{R}(t)_{PEJ} = \sqrt{\left(1+\frac{2\ln(1+t^{\lambda})}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})}\right)^{-(n-2c+1)}}$ |
| **Gamma** | $\hat{\theta}_{PG} = \frac{\sqrt{(n+\alpha)(n+\alpha+1)}}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})+\beta}$ | $\hat{R}(t)_{PG} = \sqrt{\left(1+\frac{2\ln(1+t^{\lambda})}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})+\beta}\right)^{-(n+\alpha)}}$ |
| | **Entropy Loss Function** | |
| **Jeffrey** | $\hat{\theta}_{ENJ} = \frac{n-1}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})}$ | $\hat{R}(t)_{ENJ} = \left(1-\frac{\ln(1+t^{\lambda})}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})}\right)^{n}$ |
| **Modified Jeffrey** | $\hat{\theta}_{ENMJ} = \frac{n-\frac{3}{2}}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})}$ | $\hat{R}(t)_{ENMJ} = \left(1-\frac{\ln(1+t^{\lambda})}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})}\right)^{n-\frac{1}{2}}$ |
| **Extension Jeffrey** | $\hat{\theta}_{ENEJ} = \frac{n-2c}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})}$ | $\hat{R}(t)_{ENEJ} = \left(1-\frac{\ln(1+t^{\lambda})}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})}\right)^{n-2c+1}$ |
| **Gamma** | $\hat{\theta}_{ENG} = \frac{n+\alpha-1}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})+\beta}$ | $\hat{R}(t)_{ENG} = \left(1-\frac{\ln(1+t^{\lambda})}{\sum_{i=1}^{n}\ln(1+t_i^{\lambda})+\beta}\right)^{n+\alpha}$ |

**Table (4): MSE Values for Classical Estimators of $\theta$**

| | | | $\lambda = 0.5$ , $\theta = 0.25$ | | | |
|---|---|---|---|---|---|---|
| **Estimator** | **n = 10** | **n = 15** | **n = 25** | **n = 30** | **n = 50** | **n = 100** |
| MLE | 0.0105486 | 0.0063409 | 0.0029351 | 0.0026966 | 0.0014629 | 0.0006334 |
| UMVUE | 0.0079447 | 0.0052335 | 0.0026791 | 0.0024366 | 0.0013846 | 0.0006256 |
| Best Estimator | UMVUE | UMVUE | UMVUE | UMVUE | UMVUE | UMVUE |
| | | | $\lambda = 0.5$ , $\theta = 0.5$ | | | |
| MLE | 0.0416255 | 0.0245246 | 0.0121591 | 0.0099986 | 0.0058778 | 0.0023355 |
| UMVUE | 0.0311350 | 0.0201690 | 0.0107211 | 0.0089680 | 0.0055140 | 0.0022706 |
| Best Estimator | UMVUE | UMVUE | UMVUE | UMVUE | UMVUE | UMVUE |
| | | | $\lambda = 0.5$ , $\theta = 1.5$ | | | |
| MLE | 0.4038231 | 0.1929136 | 0.1167631 | 0.0815982 | 0.0474192 | 0.0229250 |
| UMVUE | 0.3039927 | 0.1590291 | 0.1022410 | 0.0753817 | 0.0447033 | 0.0222478 |
| Best Estimator | UMVUE | UMVUE | UMVUE | UMVUE | UMVUE | UMVUE |

**Table (5): MSE Values for Bayesian Estimators of $\theta$ under Different Loss Function**

| under Squared Error Loss Function | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\lambda = 0.5$ , $\theta = 0.25$ | | | | | | | |
| **Prior Distribution** | | **n = 10** | **n = 15** | **n = 25** | **n = 30** | **n = 50** | **n = 100** |
| Jeffrey | | 0.0105486 | 0.0063409 | 0.0029351 | 0.0026966 | 0.0014629 | 0.0006334 |
| Modified Jeffrey | | 0.0090300 | 0.0057005 | 0.0027796 | 0.0025472 | 0.0014171 | 0.0006279 |
| Extension Jeffrey | c = 0.4 | 0.0115195 | 0.0061656 | 0.0033299 | 0.0024854 | 0.0013884 | 0.0006292 |
| | c = 1 | 0.0079447 | 0.0052335 | 0.0026791 | 0.0024366 | 0.0013846 | 0.0006256 |
| | c = 2 | 0.0079363 | 0.0050978 | 0.0028268 | 0.0023827 | 0.0013871 | 0.0006479 |
| Gamma | α = 2 , β = 4 | 0.0105351 | 0.0066093 | 0.0030774 | 0.0028491 | 0.0015203 | 0.0006403 |
| | α = 4 , β = 4 | 0.0198293 | 0.0113678 | 0.0043945 | 0.0036520 | 0.0018428 | 0.0008077 |
| | α = 4 , β = 2 | 0.0266249 | 0.0133805 | 0.0054829 | 0.0047784 | 0.0022029 | 0.0008452 |
| Best Prior Distribution | | E. (c=2) | E. (c=2) | E. (c=1) | E. (c=2) | E. (c=1) | E. (c=1) |
| $\lambda = 0.5$ , $\theta = 0.5$ | | | | | | | |
| Jeffrey | | 0.0416255 | 0.0245246 | 0.0121591 | 0.0099986 | 0.0058778 | 0.0023355 |
| Modified Jeffrey | | 0.0355100 | 0.0220006 | 0.0113260 | 0.0094055 | 0.0056691 | 0.0022966 |
| Extension Jeffrey | c = 0.4 | 0.0477099 | 0.0229297 | 0.0125713 | 0.0099823 | 0.0058628 | 0.0025735 |
| | c = 1 | 0.0311350 | 0.0201690 | 0.0107211 | 0.0089680 | 0.0055140 | 0.0022706 |
| | c = 2 | 0.0310393 | 0.0197667 | 0.0105828 | 0.0087753 | 0.0054289 | 0.0022949 |
| Gamma | α = 2 , β = 4 | 0.0228872 | 0.0169165 | 0.0100166 | 0.0085102 | 0.0053699 | 0.0022407 |
| | α = 4 , β = 4 | 0.0454469 | 0.0278333 | 0.0154226 | 0.0114173 | 0.0068435 | 0.0028946 |
| | α = 4 , β = 2 | 0.0788497 | 0.0441759 | 0.0184412 | 0.0154459 | 0.0070591 | 0.0032563 |
| Best Prior Distribution | | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) |
| $\lambda = 0.5$ , $\theta = 1.5$ | | | | | | | |
| Jeffrey | | 0.4038231 | 0.1929136 | 0.1167631 | 0.0815982 | 0.0474192 | 0.0229250 |
| Modified Jeffrey | | 0.3460065 | 0.1729173 | 0.1084620 | 0.0778133 | 0.0458226 | 0.0225284 |
| Extension Jeffrey | c = 0.4 | 0.3736989 | 0.1979973 | 0.1122144 | 0.0947591 | 0.0545204 | 0.0246776 |
| | c = 1 | 0.3039927 | 0.1590291 | 0.1022410 | 0.0753817 | 0.0447033 | 0.0222478 |
| | c = 2 | 0.2939639 | 0.1645559 | 0.0981597 | 0.0791893 | 0.0449979 | 0.0222840 |
| Gamma | α = 2 , β = 4 | 0.1665473 | 0.1084101 | 0.0719006 | 0.0638248 | 0.0385004 | 0.0205010 |
| | α = 4 , β = 4 | 0.0934324 | 0.0731802 | 0.0617842 | 0.0486490 | 0.0362256 | 0.0197030 |
| | α = 4 , β = 2 | 0.2205180 | 0.1581932 | 0.1015950 | 0.0762435 | 0.0500098 | 0.0245190 |
| Best Prior Distribution | | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) |
| **under Quadratic Loss Function** | | | | | | | |
| $\lambda = 0.5$ , $\theta = 0.25$ | | | | | | | |
| **Prior Distribution** | | **n = 10** | **n = 15** | **n = 25** | **n = 30** | **n = 50** | **n = 100** |
| Jeffrey | | 0.0070739 | 0.0048191 | 0.0026430 | 0.0023319 | 0.0013594 | 0.0006304 |
| Modified Jeffrey | | 0.0072884 | 0.0048718 | 0.0027074 | 0.0023379 | 0.0013666 | 0.0006376 |
| Extension Jeffrey | c = 0.4 | 0.0070299 | 0.0046500 | 0.0026345 | 0.0021395 | 0.0011679 | 0.0005915 |
| | c = 1 | 0.0079363 | 0.0050978 | 0.0028268 | 0.0023827 | 0.0013871 | 0.0006479 |
| | c = 2 | 0.0148827 | 0.0076816 | 0.0036564 | 0.0028149 | 0.0014905 | 0.0006763 |
| Gamma | α = 2 , β = 4 | 0.0058443 | 0.0043349 | 0.0024424 | 0.0022446 | 0.0013240 | 0.0006131 |
| | α = 4 , β = 4 | 0.0105351 | 0.0066093 | 0.0030774 | 0.0028491 | 0.0015203 | 0.0006403 |
| | α = 4 , β = 2 | 0.0138857 | 0.0079524 | 0.0036510 | 0.0033947 | 0.0016714 | 0.0007238 |
| Best Prior Distribution | | G (2,4) | G (2,4) | G (2,4) | E. (c=0.4) | E. (c=0.4) | E. (c=0.4) |
| $\lambda = 0.5$ , $\theta = 0.5$ | | | | | | | |
| Jeffrey | | 0.0276062 | 0.0185831 | 0.0101956 | 0.0085603 | 0.0053644 | 0.0022571 |
| Modified Jeffrey | | 0.0284525 | 0.0188287 | 0.0102751 | 0.0085900 | 0.0053699 | 0.0022696 |
| Extension Jeffrey | c = 0.4 | 0.0266261 | 0.0171335 | 0.0101708 | 0.0085420 | 0.0052591 | 0.0022076 |
| | c = 1 | 0.0310393 | 0.0197667 | 0.0105828 | 0.0087753 | 0.0054289 | 0.0022949 |
| | c = 2 | 0.0593883 | 0.0313428 | 0.0147946 | 0.0118050 | 0.0060181 | 0.0028110 |
| Gamma | α = 2 , β = 4 | 0.0177248 | 0.0137330 | 0.0086081 | 0.0074107 | 0.0049315 | 0.0021689 |
| | α = 4 , β = 4 | 0.0228872 | 0.0169165 | 0.0100166 | 0.0085102 | 0.0053699 | 0.0022407 |
| | α = 4 , β = 2 | 0.0400359 | 0.0257064 | 0.0122548 | 0.0108365 | 0.0054175 | 0.0028035 |
| Best Prior Distribution | | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) |
| $\lambda = 0.5$ , $\theta = 1.5$ | | | | | | | |
| Jeffrey | | 0.2673729 | 0.1495765 | 0.0960399 | 0.0745787 | 0.0438961 | 0.0220341 |
| Modified Jeffrey | | 0.2727671 | 0.1540122 | 0.0960597 | 0.0762073 | 0.0442084 | 0.0221011 |
| Extension Jeffrey | c = 0.4 | 0.2459124 | 0.1469057 | 0.0960167 | 0.0797387 | 0.0488650 | 0.0232952 |
| | c = 1 | 0.2939639 | 0.1645559 | 0.0981597 | 0.0791893 | 0.0449979 | 0.0222840 |
| | c = 2 | 0.5220861 | 0.2743745 | 0.1347209 | 0.1020044 | 0.0531225 | 0.0247942 |
| Gamma | α = 2 , β = 4 | 0.3168923 | 0.1919237 | 0.1046214 | 0.0907944 | 0.0483992 | 0.0230931 |
| | α = 4 , β = 4 | 0.1665473 | 0.1084101 | 0.0719006 | 0.0638248 | 0.0385004 | 0.0205010 |
| | α = 4 , β = 2 | 0.1731731 | 0.1101353 | 0.0787151 | 0.0647713 | 0.0425115 | 0.0223706 |
| Best Prior Distribution | | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) |
| **under LINEX Loss Function** | | | | | | | |
| $\lambda = 0.5$ , $\theta = 0.25$ , a = 0.8 | | | | | | | |
| **Prior Distribution** | | **n = 10** | **n = 15** | **n = 25** | **n = 30** | **n = 50** | **n = 100** |
| Jeffrey | | 0.0098346 | 0.0060678 | 0.0028706 | 0.0026404 | 0.0014459 | 0.0006308 |
| Modified Jeffrey | | 0.0084664 | 0.0054784 | 0.0027284 | 0.0025004 | 0.0014031 | 0.0006259 |
| Extension Jeffrey | c = 0.4 | 0.0107243 | 0.0059054 | 0.0032417 | 0.0024326 | 0.0012716 | 0.0006059 |
| | c = 1 | 0.0075194 | 0.0050595 | 0.0026406 | 0.0023990 | 0.0013735 | 0.0006243 |
| | c = 2 | 0.0080384 | 0.0051263 | 0.0027509 | 0.0024450 | 0.0013989 | 0.0006318 |
| Gamma | α = 2 , β = 4 | 0.0098726 | 0.0063173 | 0.0029968 | 0.0027813 | 0.0014987 | 0.0006365 |
| | α = 4 , β = 4 | 0.0186747 | 0.0108655 | 0.0042566 | 0.0035533 | 0.0018094 | 0.0007995 |
| | α = 4 , β = 2 | 0.0250415 | 0.0127732 | 0.0053163 | 0.0046534 | 0.0021608 | 0.0008361 |
| Best Prior Distribution | | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=1) |
| $\lambda = 0.5$ , $\theta = 0.5$ , a = 0.8 | | | | | | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Jeffrey | | 0.0363565 | 0.0225036 | 0.0115643 | 0.0095826 | 0.0057351 | 0.0023088 |
| Modified Jeffrey | | 0.0313922 | 0.0203740 | 0.0108431 | 0.0090639 | 0.0055504 | 0.0022753 |
| Extension Jeffrey | c = 0.4 | 0.0414853 | 0.0210288 | 0.0119306 | 0.0095681 | 0.0057114 | 0.0025471 |
| | c = 1 | 0.0280749 | 0.0189142 | 0.0103460 | 0.0086986 | 0.0054188 | 0.0022546 |
| | c = 2 | 0.0330178 | 0.0197088 | 0.0109820 | 0.0095180 | 0.0055430 | 0.0025362 |
| Gamma | α = 2 , β = 4 | 0.0206083 | 0.0157234 | 0.0095654 | 0.0081790 | 0.0052448 | 0.0022156 |
| | α = 4 , β = 4 | 0.0403185 | 0.0255145 | 0.0145294 | 0.0108293 | 0.0066245 | 0.0028450 |
| | α = 4 , β = 2 | 0.0698699 | 0.0404075 | 0.0173235 | 0.0146413 | 0.0067955 | 0.0031877 |
| Best Prior Distribution | | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) |
| **λ = 0.5 , θ = 1.5 , a = 0.8** | | | | | | | |
| Jeffrey | | 0.2788938 | 0.1513085 | 0.1009357 | 0.0738708 | 0.0443762 | 0.0221828 |
| Modified Jeffrey | | 0.2494503 | 0.1406915 | 0.0956214 | 0.0718932 | 0.0433977 | 0.0219309 |
| Extension Jeffrey | c = 0.4 | 0.2580313 | 0.1579581 | 0.0982021 | 0.0837746 | 0.0506297 | 0.0237593 |
| | c = 1 | 0.2334804 | 0.1356312 | 0.0922783 | 0.0712121 | 0.0428845 | 0.0217934 |
| | c = 2 | 0.2858601 | 0.1739716 | 0.1034709 | 0.0825876 | 0.0450933 | 0.0230630 |
| Gamma | α = 2 , β = 4 | 0.1902665 | 0.1222128 | 0.0767672 | 0.0682939 | 0.0400136 | 0.0208745 |
| | α = 4 , β = 4 | 0.1015162 | 0.0752267 | 0.0605684 | 0.0484623 | 0.0360704 | 0.0194967 |
| | α = 4 , β = 2 | 0.1613369 | 0.1258405 | 0.0875308 | 0.0672150 | 0.0458318 | 0.0233926 |
| Best Prior Distribution | | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) |
| **λ = 0.5 , θ = 0.25 , a = - 0.8** | | | | | | | |
| **Prior Distribution** | | **n = 10** | **n = 15** | **n = 25** | **n = 30** | **n = 50** | **n = 100** |
| Jeffrey | | 0.0115491 | 0.0059581 | 0.0032761 | 0.0023446 | 0.0013482 | 0.0006607 |
| Modified Jeffrey | | 0.0097986 | 0.0053153 | 0.0030762 | 0.0022490 | 0.0013064 | 0.0006475 |
| Extension Jeffrey | c = 0.4 | 0.0143774 | 0.0065755 | 0.0031110 | 0.0027100 | 0.0014054 | 0.0006808 |
| | c = 1 | 0.0085006 | 0.0048468 | 0.0029329 | 0.0022943 | 0.0012778 | 0.0006376 |
| | c = 2 | 0.0084813 | 0.0045583 | 0.0027766 | 0.0022368 | 0.0012520 | 0.0005924 |
| Gamma | α = 2 , β = 4 | 0.0115771 | 0.0063640 | 0.0034452 | 0.0025287 | 0.0014051 | 0.0006819 |
| | α = 4 , β = 4 | 0.0230306 | 0.0105690 | 0.0051468 | 0.0037603 | 0.0018980 | 0.0007905 |
| | α = 4 , β = 2 | 0.0272360 | 0.0139097 | 0.0060076 | 0.0043837 | 0.0021431 | 0.0008474 |
| Best Prior Distribution | | E. (c=2) | E. (c=2) | E. (c=2) | E. (c=2) | E. (c=2) | E. (c=2) |
| **λ = 0.5 , θ = 0.5 , a = - 0.8** | | | | | | | |
| Jeffrey | | 0.0520678 | 0.0273196 | 0.0120660 | 0.0103745 | 0.0052029 | 0.0026599 |
| Modified Jeffrey | | 0.0438533 | 0.0239493 | 0.0111522 | 0.0097404 | 0.0050577 | 0.0026062 |
| Extension Jeffrey | c = 0.4 | 0.0480671 | 0.0251333 | 0.0138426 | 0.0102909 | 0.0055523 | 0.0028519 |
| | c = 1 | 0.0375453 | 0.0213234 | 0.0104699 | 0.0092631 | 0.0049648 | 0.0025654 |
| | c = 2 | 0.0327331 | 0.0201256 | 0.0108829 | 0.0093826 | 0.0052536 | 0.0026585 |
| Gamma | α = 2 , β = 4 | 0.0272537 | 0.0187579 | 0.0099145 | 0.0088045 | 0.0049376 | 0.0025476 |
| | α = 4 , β = 4 | 0.0509088 | 0.0275402 | 0.0148617 | 0.0110297 | 0.0067024 | 0.0029888 |
| | α = 4 , β = 2 | 0.0877320 | 0.0467107 | 0.0208725 | 0.0156646 | 0.0074607 | 0.0031868 |
| Best Prior Distribution | | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) |
| **λ = 0.5 , θ = 1.5 , a = - 0.8** | | | | | | | |
| Jeffrey | | 0.6607761 | 0.2767379 | 0.1385072 | 0.1052432 | 0.0530475 | 0.0248300 |
| Modified Jeffrey | | 0.5538507 | 0.2443821 | 0.1269125 | 0.0984595 | 0.0508675 | 0.0243036 |
| Extension Jeffrey | c = 0.4 | 0.6416512 | 0.2998398 | 0.1462821 | 0.1097231 | 0.0582096 | 0.0255518 |
| | c = 1 | 0.4665450 | 0.2187939 | 0.1175147 | 0.0931175 | 0.0491754 | 0.0238944 |
| | c = 2 | 0.3076578 | 0.1816334 | 0.1066507 | 0.0799987 | 0.0472314 | 0.0231521 |
| Gamma | α = 2 , β = 4 | 0.1363087 | 0.1068382 | 0.0700777 | 0.0628908 | 0.0392241 | 0.0212284 |
| | α = 4 , β = 4 | 0.0931786 | 0.0783920 | 0.0620424 | 0.0543641 | 0.0364972 | 0.0210648 |
| | α = 4 , β = 2 | 0.3326011 | 0.1808511 | 0.1121683 | 0.0915993 | 0.0444049 | 0.0247020 |
| Best Prior Distribution | | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) |
| **under Precautionary Loss Function** | | | | | | | |
| **λ = 0.5 , θ = 0.25** | | | | | | | |
| **Prior Distribution** | | **n = 10** | **n = 15** | **n = 25** | **n = 30** | **n = 50** | **n = 100** |
| Jeffrey | | 0.0124489 | 0.0071400 | 0.0031432 | 0.0028832 | 0.0015216 | 0.0006421 |
| Modified Jeffrey | | 0.0105053 | 0.0063288 | 0.0029332 | 0.0026952 | 0.0014626 | 0.0006334 |
| Extension Jeffrey | c = 0.4 | 0.0136889 | 0.0069531 | 0.0036182 | 0.0026660 | 0.0013513 | 0.0006216 |
| | c = 1 | 0.0089959 | 0.0056910 | 0.0027783 | 0.0025461 | 0.0014169 | 0.0006279 |
| | c = 2 | 0.0074425 | 0.0046657 | 0.0026659 | 0.0022033 | 0.0013125 | 0.0006212 |
| Gamma | α = 2 , β = 4 | 0.0124989 | 0.0075322 | 0.0033593 | 0.0030884 | 0.0016007 | 0.0006549 |
| | α = 4 , β = 4 | 0.0230817 | 0.0129192 | 0.0048943 | 0.0040175 | 0.0019735 | 0.0008421 |
| | α = 4 , β = 2 | 0.0306661 | 0.0151037 | 0.0060690 | 0.0052144 | 0.0023682 | 0.0008835 |
| Best Prior Distribution | | E. (c=2) | E. (c=2) | E. (c=2) | E. (c=2) | E. (c=2) | E. (c=2) |
| **λ = 0.5 , θ = 0.5** | | | | | | | |
| Jeffrey | | 0.0492740 | 0.0276836 | 0.0132089 | 0.0107407 | 0.0061385 | 0.0023870 |
| Modified Jeffrey | | 0.0414513 | 0.0244769 | 0.0121497 | 0.0099930 | 0.0058766 | 0.0023354 |
| Extension Jeffrey | c = 0.4 | 0.0564037 | 0.0260126 | 0.0137038 | 0.0107200 | 0.0061461 | 0.0026217 |
| | c = 1 | 0.0353725 | 0.0219632 | 0.0113187 | 0.0094011 | 0.0056682 | 0.0022965 |
| | c = 2 | 0.0312614 | 0.0182280 | 0.0105193 | 0.0091164 | 0.0050456 | 0.0022877 |
| Gamma | α = 2 , β = 4 | 0.0317760 | 0.0189544 | 0.0108399 | 0.0091171 | 0.0056013 | 0.0022893 |
| | α = 4 , β = 4 | 0.0537389 | 0.0317985 | 0.0170594 | 0.0125431 | 0.0072728 | 0.0029961 |
| | α = 4 , β = 2 | 0.0916128 | 0.0501870 | 0.0204824 | 0.0169463 | 0.0075948 | 0.0034010 |
| Best Prior Distribution | | E. (c=2) | E. (c=2) | E. (c=2) | E. (c=2) | E. (c=2) | E. (c=2) |
| **λ = 0.5 , θ = 1.5** | | | | | | | |
| Jeffrey | | 0.4755047 | 0.2185482 | 0.1270327 | 0.0866889 | 0.0494816 | 0.0234361 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Modified Jeffrey | | 0.4021840 | 0.1925301 | 0.1166698 | 0.0815611 | 0.0474101 | 0.0229239 |
| Extension Jeffrey | c = 0.4 | 0.4416242 | 0.2225556 | 0.1211732 | 0.1019457 | 0.0571276 | 0.0253130 |
| | c = 1 | 0.3446975 | 0.1726260 | 0.1083880 | 0.0777870 | 0.0458158 | 0.0225276 |
| | c = 2 | 0.2453156 | 0.1511510 | 0.0991848 | 0.0811083 | 0.0432693 | 0.0225195 |
| Gamma | α = 2 , β = 4 | 0.1417247 | 0.0947282 | 0.0669752 | 0.0593979 | 0.0369716 | 0.0201108 |
| | α = 4 , β = 4 | 0.0863115 | 0.0711766 | 0.0622413 | 0.0484852 | 0.0361196 | 0.0198009 |
| | α = 4 , β = 2 | 0.2612217 | 0.1797643 | 0.1111856 | 0.0825912 | 0.0529393 | 0.0253294 |
| Best Prior Distribution | | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) |
| **under Entropy Loss Function** | | | | | | | |
| **λ = 0.5 , θ = 0.25** | | | | | | | |
| **Prior Distribution** | | **n = 10** | **n = 15** | **n = 25** | **n = 30** | **n = 50** | **n = 100** |
| Jeffrey | | 0.0079447 | 0.0052335 | 0.0026791 | 0.0024366 | 0.0013846 | 0.0006256 |
| Modified Jeffrey | | 0.0072927 | 0.0049397 | 0.0026436 | 0.0023648 | 0.0013654 | 0.0006254 |
| Extension Jeffrey | c = 0.4 | 0.0083914 | 0.0050696 | 0.0029181 | 0.0022358 | 0.0012016 | 0.0005940 |
| | c = 1 | 0.0070739 | 0.0048191 | 0.0026430 | 0.0023319 | 0.0013594 | 0.0006304 |
| | c = 2 | 0.0105928 | 0.0059610 | 0.0030902 | 0.0024533 | 0.0013787 | 0.0006477 |
| Gamma | α = 2 , β = 4 | 0.0075222 | 0.0051771 | 0.0026592 | 0.0024746 | 0.0013968 | 0.0006205 |
| | α = 4 , β = 4 | 0.0142051 | 0.0086717 | 0.0035393 | 0.0030228 | 0.0016183 | 0.0007482 |
| | α = 4 , β = 2 | 0.0195110 | 0.0103547 | 0.0044602 | 0.0040119 | 0.0019106 | 0.0007781 |
| Best Prior Distribution | | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=0.4) | E. (c=0.4) | E. (c=0.4) |
| **λ = 0.5 , θ = 0.5** | | | | | | | |
| Jeffrey | | 0.0311350 | 0.0201690 | 0.0107211 | 0.0089680 | 0.0055140 | 0.0022706 |
| Modified Jeffrey | | 0.0285004 | 0.0190299 | 0.0103443 | 0.0086863 | 0.0054124 | 0.0022574 |
| Extension Jeffrey | c = 0.4 | 0.0351467 | 0.0186803 | 0.0109654 | 0.0089548 | 0.0054538 | 0.0025151 |
| | c = 1 | 0.0276062 | 0.0185831 | 0.0101956 | 0.0085603 | 0.0053644 | 0.0022571 |
| | c = 2 | 0.0427505 | 0.0240956 | 0.0124132 | 0.0103589 | 0.0054700 | 0.0026436 |
| Gamma | α = 2 , β = 4 | 0.0181553 | 0.0143046 | 0.0089286 | 0.0076906 | 0.0050522 | 0.0021802 |
| | α = 4 , β = 4 | 0.0318124 | 0.0213114 | 0.0126987 | 0.0095521 | 0.0061283 | 0.0027282 |
| | α = 4 , β = 2 | 0.0568068 | 0.0337607 | 0.0149377 | 0.0128536 | 0.0061360 | 0.0030044 |
| Best Prior Distribution | | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) |
| **λ = 0.5 , θ = 1.5** | | | | | | | |
| Jeffrey | | 0.3039927 | 0.1590291 | 0.1022410 | 0.0753817 | 0.0447033 | 0.0222478 |
| Modified Jeffrey | | 0.2777815 | 0.1512488 | 0.0981003 | 0.0746035 | 0.0440611 | 0.0220830 |
| Extension Jeffrey | c = 0.4 | 0.2795808 | 0.1656677 | 0.1001099 | 0.0844519 | 0.0507266 | 0.0237532 |
| | c = 1 | 0.2673729 | 0.1495765 | 0.0960399 | 0.0745787 | 0.0438961 | 0.0220341 |
| | c = 2 | 0.3642255 | 0.2072224 | 0.1143947 | 0.0894130 | 0.0477624 | 0.0236600 |
| Gamma | α = 2 , β = 4 | 0.2318750 | 0.1445523 | 0.0856886 | 0.0754818 | 0.0426982 | 0.0215914 |
| | α = 4 , β = 4 | 0.1222857 | 0.0857069 | 0.0647353 | 0.0517732 | 0.0375592 | 0.0198171 |
| | α = 4 , β = 2 | 0.1611259 | 0.1262746 | 0.0869875 | 0.0667833 | 0.0454051 | 0.0232245 |
| Best Prior Distribution | | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) |

### Table (6): IMSE Values for Classical Estimators of $R(t)$

| **λ = 0.5 , θ = 0.25** | | | | | | |
|---|---|---|---|---|---|---|
| **Estimator** | **n = 10** | **n = 15** | **n = 25** | **n = 30** | **n = 50** | **n = 100** |
| MLE | 0.0033417 | 0.0021085 | 0.0010396 | 0.0009461 | 0.0005273 | 0.0002340 |
| UMVUE | 0.0026804 | 0.0018120 | 0.0009724 | 0.0008716 | 0.0005050 | 0.0002324 |
| Best Estimator | UMVUE | UMVUE | UMVUE | UMVUE | UMVUE | UMVUE |
| **λ = 0.5 , θ = 0.5** | | | | | | |
| MLE | 0.0080412 | 0.0051510 | 0.0028071 | 0.0023198 | 0.0014159 | 0.0005846 |
| UMVUE | 0.0067971 | 0.0045852 | 0.0025913 | 0.0021610 | 0.0013588 | 0.0005749 |
| Best Estimator | UMVUE | UMVUE | UMVUE | UMVUE | UMVUE | UMVUE |
| **λ = 0.5 , θ = 1.5** | | | | | | |
| MLE | 0.0128517 | 0.0073236 | 0.0051460 | 0.0041457 | 0.0024452 | 0.0012536 |
| UMVUE | 0.0135983 | 0.0075420 | 0.0052326 | 0.0042773 | 0.0024621 | 0.0012571 |
| Best Estimator | MLE | MLE | MLE | MLE | MLE | MLE |

### Table (7): IMSE Values for Bayesian Estimators of $R(t)$ under Different Loss Function

| **under Squared Error Loss Function** | | | | | | | |
|---|---|---|---|---|---|---|---|
| **λ = 0.5 , θ = 0.25** | | | | | | | |
| **Prior Distribution** | | **n = 10** | **n = 15** | **n = 25** | **n = 30** | **n = 50** | **n = 100** |
| Jeffrey | | 0.0031611 | 0.0020338 | 0.0010207 | 0.0009294 | 0.0005221 | 0.0002332 |
| Modified Jeffrey | | 0.0028016 | 0.0018715 | 0.0009816 | 0.0008884 | 0.0005096 | 0.0002321 |
| Extension Jeffrey | c = 0.4 | 0.0034261 | 0.0020011 | 0.0011224 | 0.0008677 | 0.0005030 | 0.0002333 |
| | c = 1 | 0.0025756 | 0.0017662 | 0.0009619 | 0.0008610 | 0.0005018 | 0.0002321 |
| | c = 2 | 0.0031298 | 0.0018916 | 0.0010406 | 0.0008623 | 0.0005044 | 0.0002357 |
| Gamma | α = 2 , β = 4 | 0.0031868 | 0.0021014 | 0.0010502 | 0.0009688 | 0.0005365 | 0.0002340 |
| | α = 4 , β = 4 | 0.0058291 | 0.0035054 | 0.0014528 | 0.0012269 | 0.0006380 | 0.0002898 |
| | α = 4 , β = 2 | 0.0074994 | 0.0039694 | 0.0018078 | 0.0015744 | 0.0007585 | 0.0003019 |
| Best Prior Distribution | | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=1) |
| **λ = 0.5 , θ = 0.5** | | | | | | | |
| Jeffrey | | 0.0073350 | 0.0048476 | 0.0027044 | 0.0022469 | 0.0013900 | 0.0005796 |
| Modified Jeffrey | | 0.0067114 | 0.0045588 | 0.0025928 | 0.0021647 | 0.0013602 | 0.0005744 |
| Extension Jeffrey | c = 0.4 | 0.0080780 | 0.0045724 | 0.0027540 | 0.0022387 | 0.0013814 | 0.0006390 |
| | c = 1 | 0.0064163 | 0.0044148 | 0.0025331 | 0.0021185 | 0.0013433 | 0.0005724 |
| | c = 2 | 0.0092963 | 0.0055515 | 0.0029877 | 0.0025081 | 0.0013458 | 0.0006588 |
| Gamma | α = 2 , β = 4 | 0.0045977 | 0.0035669 | 0.0022731 | 0.0019420 | 0.0012772 | 0.0005566 |

 

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | α = 4 , β = 4 | 0.0079513 | 0.0053355 | 0.0031872 | 0.0024407 | 0.0015568 | 0.0006923 |
| | α = 4 , β = 2 | 0.0125908 | 0.0079700 | 0.0037436 | 0.0032448 | 0.0015852 | 0.0007679 |
| Best Prior Distribution | | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) |
| **λ = 0.5 , θ = 1.5** | | | | | | | |
| Jeffrey | | 0.0117945 | 0.0068440 | 0.0049852 | 0.0040465 | 0.0023928 | 0.0012396 |
| Modified Jeffrey | | 0.0124407 | 0.0070966 | 0.0050344 | 0.0041505 | 0.0024158 | 0.0012451 |
| Extension Jeffrey | c = 0.4 | 0.0118717 | 0.0074525 | 0.0049900 | 0.0041431 | 0.0026630 | 0.0012885 |
| | c = 1 | 0.0137457 | 0.0076438 | 0.0051862 | 0.0043277 | 0.0024648 | 0.0012571 |
| | c = 2 | 0.0265920 | 0.0144239 | 0.0073187 | 0.0056234 | 0.0029774 | 0.0014280 |
| Gamma | α = 2 , β = 4 | 0.0164669 | 0.0097394 | 0.0056768 | 0.0050310 | 0.0026845 | 0.0013113 |
| | α = 4 , β = 4 | 0.0078772 | 0.0054006 | 0.0039632 | 0.0031388 | 0.0022478 | 0.0011502 |
| | α = 4 , β = 2 | 0.0079341 | 0.0055823 | 0.0042189 | 0.0033968 | 0.0022792 | 0.0012446 |
| Best Prior Distribution | | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) |
| **under Quadratic Loss Function** | | | | | | | |
| **λ = 0.5 , θ = 0.25** | | | | | | | |
| **Prior Distribution** | | **n = 10** | **n = 15** | **n = 25** | **n = 30** | **n = 50** | **n = 100** |
| Jeffrey | | 0.0039915 | 0.0023650 | 0.0011036 | 0.0010013 | 0.0005443 | 0.0002367 |
| Modified Jeffrey | | 0.0034660 | 0.0021413 | 0.0010470 | 0.0009478 | 0.0005277 | 0.0002346 |
| Extension Jeffrey | c = 0.4 | 0.0043653 | 0.0023241 | 0.0012330 | 0.0009261 | 0.0005255 | 0.0002379 |
| | c = 1 | 0.0030814 | 0.0019768 | 0.0010102 | 0.0009081 | 0.0005159 | 0.0002338 |
| | c = 2 | 0.0031080 | 0.0019947 | 0.0010350 | 0.0009125 | 0.0005163 | 0.0002359 |
| Gamma | α = 2 , β = 4 | 0.0039927 | 0.0024634 | 0.0011550 | 0.0010564 | 0.0005651 | 0.0002392 |
| | α = 4 , β = 4 | 0.0071807 | 0.0041136 | 0.0016317 | 0.0013560 | 0.0006824 | 0.0003011 |
| | α = 4 , β = 2 | 0.0092333 | 0.0046574 | 0.0020214 | 0.0017331 | 0.0008145 | 0.0003143 |
| Best Prior Distribution | | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=1) |
| **λ = 0.5 , θ = 0.5** | | | | | | | |
| Jeffrey | | 0.0109906 | 0.0063345 | 0.0031884 | 0.0025859 | 0.0015083 | 0.0006028 |
| Modified Jeffrey | | 0.0096508 | 0.0057608 | 0.0029839 | 0.0024408 | 0.0014573 | 0.0005926 |
| Extension Jeffrey | c = 0.4 | 0.0121515 | 0.0060176 | 0.0032721 | 0.0025754 | 0.0015085 | 0.0006613 |
| | c = 1 | 0.0086464 | 0.0053356 | 0.0028325 | 0.0023323 | 0.0014193 | 0.0005856 |
| | c = 2 | 0.0090614 | 0.0052571 | 0.0028954 | 0.0024873 | 0.0013264 | 0.0006472 |
| Gamma | α = 2 , β = 4 | 0.0064574 | 0.0045226 | 0.0026506 | 0.0022180 | 0.0013820 | 0.0005785 |
| | α = 4 , β = 4 | 0.0117231 | 0.0071188 | 0.0039074 | 0.0029308 | 0.0017424 | 0.0007356 |
| | α = 4 , β = 2 | 0.0183187 | 0.0106641 | 0.0046380 | 0.0038997 | 0.0018130 | 0.0008285 |
| Best Prior Distribution | | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) |
| **λ = 0.5 , θ = 1.5** | | | | | | | |
| Jeffrey | | 0.0210577 | 0.0111393 | 0.0068845 | 0.0050080 | 0.0028043 | 0.0013447 |
| Modified Jeffrey | | 0.0195684 | 0.0102689 | 0.0065098 | 0.0048133 | 0.0027169 | 0.0013222 |
| Extension Jeffrey | c = 0.4 | 0.0199461 | 0.0114465 | 0.0065895 | 0.0055023 | 0.0031809 | 0.0014204 |
| | c = 1 | 0.0184791 | 0.0096180 | 0.0062221 | 0.0046835 | 0.0026538 | 0.0013060 |
| | c = 2 | 0.0190009 | 0.0105152 | 0.0061671 | 0.0050249 | 0.0026479 | 0.0013671 |
| Gamma | α = 2 , β = 4 | 0.0095261 | 0.0059589 | 0.0042806 | 0.0037934 | 0.0022470 | 0.0011970 |
| | α = 4 , β = 4 | 0.0060195 | 0.0047989 | 0.0039406 | 0.0030155 | 0.0021689 | 0.0011585 |
| | α = 4 , β = 2 | 0.0146621 | 0.0096807 | 0.0061261 | 0.0047100 | 0.0028942 | 0.0014209 |
| Best Prior Distribution | | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) |
| **under LINEX Loss Function** | | | | | | | |
| **λ = 0.5 , θ = 0.25 , a = 0.8** | | | | | | | |
| **Prior Distribution** | | **n = 10** | **n = 15** | **n = 25** | **n = 30** | **n = 50** | **n = 100** |
| Jeffrey | | 0.0032538 | 0.0020742 | 0.0010314 | 0.0009390 | 0.0005252 | 0.0002337 |
| Modified Jeffrey | | 0.0028733 | 0.0019034 | 0.0009897 | 0.0008961 | 0.0005120 | 0.0002324 |
| Extension Jeffrey | c = 0.4 | 0.0035328 | 0.0020411 | 0.0011370 | 0.0008669 | 0.0005061 | 0.0002340 |
| | c = 1 | 0.0026269 | 0.0017898 | 0.0009674 | 0.0008668 | 0.0005036 | 0.0002323 |
| | c = 2 | 0.0031091 | 0.0018827 | 0.0010377 | 0.0008409 | 0.0004943 | 0.0002356 |
| Gamma | α = 2 , β = 4 | 0.0032846 | 0.0021480 | 0.0010644 | 0.0009810 | 0.0005406 | 0.0002348 |
| | α = 4 , β = 4 | 0.0059964 | 0.0035855 | 0.0014781 | 0.0012454 | 0.0006445 | 0.0002915 |
| | α = 4 , β = 2 | 0.0077060 | 0.0040576 | 0.0018379 | 0.0015968 | 0.0007668 | 0.0003038 |
| Best Prior Distribution | | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=1) |
| **λ = 0.5 , θ = 0.5 , a = 0.8** | | | | | | | |
| Jeffrey | | 0.0075772 | 0.0049603 | 0.0027465 | 0.0022772 | 0.0014010 | 0.0005818 |
| Modified Jeffrey | | 0.0068863 | 0.0046414 | 0.0026239 | 0.0021874 | 0.0013686 | 0.0005759 |
| Extension Jeffrey | c = 0.4 | 0.0083533 | 0.0046848 | 0.0027999 | 0.0022688 | 0.0013937 | 0.0006410 |
| | c = 1 | 0.0065222 | 0.0044669 | 0.0025533 | 0.0021336 | 0.0013490 | 0.0005732 |
| | c = 2 | 0.0091316 | 0.0054673 | 0.0029597 | 0.0024934 | 0.0013993 | 0.0006563 |
| Gamma | α = 2 , β = 4 | 0.0047322 | 0.0036421 | 0.0023062 | 0.0019668 | 0.0012870 | 0.0005587 |
| | α = 4 , β = 4 | 0.0082732 | 0.0054988 | 0.0032589 | 0.0024910 | 0.0015766 | 0.0006971 |
| | α = 4 , β = 2 | 0.0130573 | 0.0082145 | 0.0038341 | 0.0033127 | 0.0016103 | 0.0007749 |
| Best Prior Distribution | | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) |
| **λ = 0.5 , θ = 1.5 , a = 0.8** | | | | | | | |
| Jeffrey | | 0.0116926 | 0.0068102 | 0.0049533 | 0.0040269 | 0.0023896 | 0.0012389 |
| Modified Jeffrey | | 0.0121889 | 0.0069901 | 0.0050059 | 0.0041116 | 0.0024055 | 0.0012426 |
| Extension Jeffrey | c = 0.4 | 0.0106002 | 0.0074000 | 0.0049541 | 0.0041451 | 0.0026650 | 0.0012893 |
| | c = 1 | 0.0133187 | 0.0074561 | 0.0051292 | 0.0042683 | 0.0024471 | 0.0012527 |
| | c = 2 | 0.0251788 | 0.0137916 | 0.0071088 | 0.0054915 | 0.0029245 | 0.0014161 |
| Gamma | α = 2 , β = 4 | 0.0156584 | 0.0093027 | 0.0054999 | 0.0048868 | 0.0026307 | 0.0012968 |
| | α = 4 , β = 4 | 0.0074968 | 0.0052085 | 0.0038858 | 0.0030769 | 0.0022192 | 0.0011441 |
| | α = 4 , β = 2 | 0.0076314 | 0.0056332 | 0.0042463 | 0.0034149 | 0.0022919 | 0.0012488 |
| Best Prior Distribution | | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) |

 

| **λ = 0.5 , θ = 0.25 , a = - 0.8** | | | | | | | |
|---|---|---|---|---|---|---|---|
| **Prior Distribution** | | **n = 10** | **n = 15** | **n = 25** | **n = 30** | **n = 50** | **n = 100** |
| Jeffrey | | 0.0031426 | 0.0018263 | 0.0010807 | 0.0007927 | 0.0004768 | 0.0002390 |
| Modified Jeffrey | | 0.0027808 | 0.0016810 | 0.0010354 | 0.0007565 | 0.0004670 | 0.0002355 |
| Extension Jeffrey | c = 0.4 | 0.0036322 | 0.0020145 | 0.0010419 | 0.0009005 | 0.0004911 | 0.0002447 |
| | c = 1 | 0.0025533 | 0.0015925 | 0.0010095 | 0.0007340 | 0.0004619 | 0.0002331 |
| | c = 2 | 0.0033050 | 0.0017958 | 0.0010576 | 0.0008436 | 0.0005076 | 0.0002216 |
| Gamma | α = 2 , β = 4 | 0.0031860 | 0.0019130 | 0.0011124 | 0.0008365 | 0.0004885 | 0.0002444 |
| | α = 4 , β = 4 | 0.0061216 | 0.0030690 | 0.0016163 | 0.0012150 | 0.0006411 | 0.0002783 |
| | α = 4 , β = 2 | 0.0070106 | 0.0040003 | 0.0018673 | 0.0013931 | 0.0007196 | 0.0002972 |
| Best Prior Distribution | | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=1) |
| **λ = 0.5 , θ = 0.5 , a = - 0.8** | | | | | | | |
| Jeffrey | | 0.0074730 | 0.0047932 | 0.0025113 | 0.0022088 | 0.0012269 | 0.0006396 |
| Modified Jeffrey | | 0.0068192 | 0.0044377 | 0.0024143 | 0.0021424 | 0.0012187 | 0.0006332 |
| Extension Jeffrey | c = 0.4 | 0.0069377 | 0.0044523 | 0.0027876 | 0.0021762 | 0.0012714 | 0.0006139 |
| | c = 1 | 0.0064976 | 0.0042303 | 0.0023693 | 0.0021120 | 0.0012132 | 0.0006300 |
| | c = 2 | 0.0091200 | 0.0056088 | 0.0028800 | 0.0024577 | 0.0014012 | 0.0006325 |
| Gamma | α = 2 , β = 4 | 0.0046740 | 0.0035289 | 0.0021138 | 0.0019115 | 0.0011309 | 0.0006137 |
| | α = 4 , β = 4 | 0.0075875 | 0.0047015 | 0.0028659 | 0.0022048 | 0.0014580 | 0.0006985 |
| | α = 4 , β = 2 | 0.0121649 | 0.0074996 | 0.0038913 | 0.0030215 | 0.0015960 | 0.0007363 |
| Best Prior Distribution | | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) |
| **λ = 0.5 , θ = 1.5 , a = - 0.8** | | | | | | | |
| Jeffrey | | 0.0111951 | 0.0079848 | 0.0052317 | 0.0042302 | 0.0025047 | 0.0012843 |
| Modified Jeffrey | | 0.0117974 | 0.0083856 | 0.0053138 | 0.0043339 | 0.0025419 | 0.0012936 |
| Extension Jeffrey | c = 0.4 | 0.0110619 | 0.0077439 | 0.0051430 | 0.0041751 | 0.0026043 | 0.0012874 |
| | c = 1 | 0.0130804 | 0.0090874 | 0.0055002 | 0.0045111 | 0.0026054 | 0.0013094 |
| | c = 2 | 0.0293804 | 0.0152812 | 0.0077423 | 0.0056691 | 0.0032237 | 0.0014509 |
| Gamma | α = 2 , β = 4 | 0.0162717 | 0.0111366 | 0.0060746 | 0.0051471 | 0.0028744 | 0.0013793 |
| | α = 4 , β = 4 | 0.0080139 | 0.0059620 | 0.0040516 | 0.0034397 | 0.0022603 | 0.0012352 |
| | α = 4 , β = 2 | 0.0072555 | 0.0051920 | 0.0038616 | 0.0033572 | 0.0021777 | 0.0012067 |
| Best Prior Distribution | | G (4,2) | G (4,2) | G (4,2) | G (4,2) | G (4,2) | G (4,2) |
| **under Precautionary Loss Function** | | | | | | | |
| **λ = 0.5 , θ = 0.25** | | | | | | | |
| **Prior Distribution** | | **n = 10** | **n = 15** | **n = 25** | **n = 30** | **n = 50** | **n = 100** |
| Jeffrey | | 0.0029977 | 0.0019644 | 0.0010029 | 0.0009136 | 0.0005172 | 0.0002324 |
| Modified Jeffrey | | 0.0026751 | 0.0018164 | 0.0009680 | 0.0008755 | 0.0005056 | 0.0002315 |
| Extension Jeffrey | c = 0.4 | 0.0032387 | 0.0019332 | 0.0010981 | 0.0007326 | 0.0004579 | 0.0002223 |
| | c = 1 | 0.0024843 | 0.0015248 | 0.0009524 | 0.0008511 | 0.0004988 | 0.0002318 |
| | c = 2 | 0.0031546 | 0.0019026 | 0.0010441 | 0.0008438 | 0.0004945 | 0.0002358 |
| Gamma | α = 2 , β = 4 | 0.0030193 | 0.0020229 | 0.0010269 | 0.0009490 | 0.0005299 | 0.0002329 |
| | α = 4 , β = 4 | 0.0055362 | 0.0033690 | 0.0014117 | 0.0011970 | 0.0006275 | 0.0002871 |
| | α = 4 , β = 2 | 0.0071244 | 0.0038153 | 0.0017584 | 0.0015374 | 0.0007452 | 0.0002989 |
| Best Prior Distribution | | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=0.4) | E. (c=0.4) | E. (c=0.4) |
| **λ = 0.5 , θ = 0.5** | | | | | | | |
| Jeffrey | | 0.0067510 | 0.0045866 | 0.0026135 | 0.0021818 | 0.0013665 | 0.0005751 |
| Modified Jeffrey | | 0.0062805 | 0.0043618 | 0.0025236 | 0.0021145 | 0.0013419 | 0.0005711 |
| Extension Jeffrey | c = 0.4 | 0.0074072 | 0.0043179 | 0.0026551 | 0.0021740 | 0.0013558 | 0.0006348 |
| | c = 1 | 0.0061354 | 0.0042806 | 0.0024854 | 0.0020831 | 0.0013301 | 0.0005703 |
| | c = 2 | 0.0095299 | 0.0056826 | 0.0030310 | 0.0025282 | 0.0013555 | 0.0006629 |
| Gamma | α = 2 , β = 4 | 0.0043017 | 0.0034010 | 0.0021029 | 0.0018895 | 0.0010965 | 0.0005524 |
| | α = 4 , β = 4 | 0.0072331 | 0.0049798 | 0.0030364 | 0.0023371 | 0.0015165 | 0.0006828 |
| | α = 4 , β = 2 | 0.0114765 | 0.0074184 | 0.0035531 | 0.0031031 | 0.0015349 | 0.0007542 |
| Best Prior Distribution | | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) |
| **λ = 0.5 , θ = 1.5** | | | | | | | |
| Jeffrey | | 0.0114887 | 0.0067120 | 0.0048560 | 0.0040340 | 0.0023723 | 0.0012336 |
| Modified Jeffrey | | 0.0126973 | 0.0072396 | 0.0050088 | 0.0042106 | 0.0024224 | 0.0012460 |
| Extension Jeffrey | c = 0.4 | 0.0123323 | 0.0073551 | 0.0048965 | 0.0040394 | 0.0026172 | 0.0012759 |
| | c = 1 | 0.0146089 | 0.0080752 | 0.0052672 | 0.0044620 | 0.0024988 | 0.0012649 |
| | c = 2 | 0.0303182 | 0.0161980 | 0.0079134 | 0.0059945 | 0.0031371 | 0.0014630 |
| Gamma | α = 2 , β = 4 | 0.0187853 | 0.0110632 | 0.0062209 | 0.0054798 | 0.0028548 | 0.0013572 |
| | α = 4 , β = 4 | 0.0090621 | 0.0059985 | 0.0041815 | 0.0033240 | 0.0023305 | 0.0011660 |
| | α = 4 , β = 2 | 0.0063681 | 0.0052054 | 0.0040126 | 0.0032609 | 0.0022005 | 0.0012200 |
| Best Prior Distribution | | G (4,2) | G (4,2) | G (4,2) | G (4,2) | G (4,2) | G (4,2) |
| **under Entropy Loss Function** | | | | | | | |
| **λ = 0.5 , θ = 0.25** | | | | | | | |
| **Prior Distribution** | | **n = 10** | **n = 15** | **n = 25** | **n = 30** | **n = 50** | **n = 100** |
| Jeffrey | | 0.0035413 | 0.0021889 | 0.0010598 | 0.0009637 | 0.0005328 | 0.0002348 |
| Modified Jeffrey | | 0.0031021 | 0.0019967 | 0.0010121 | 0.0009165 | 0.0005182 | 0.0002333 |
| Extension Jeffrey | c = 0.4 | 0.0038582 | 0.0021525 | 0.0011750 | 0.0008903 | 0.0004738 | 0.0002255 |
| | c = 1 | 0.0028001 | 0.0018625 | 0.0009839 | 0.0008831 | 0.0005084 | 0.0002328 |
| | c = 2 | 0.0031023 | 0.0018776 | 0.0010361 | 0.0008912 | 0.0004950 | 0.0002357 |
| Gamma | α = 2 , β = 4 | 0.0035631 | 0.0022731 | 0.0011003 | 0.0010110 | 0.0005503 | 0.0002365 |
| | α = 4 , β = 4 | 0.0064702 | 0.0037975 | 0.0015395 | 0.0012897 | 0.0006597 | 0.0002953 |
| | α = 4 , β = 2 | 0.0083215 | 0.0042998 | 0.0019115 | 0.0016516 | 0.0007859 | 0.0003080 |
| Best Prior Distribution | | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=1) | E. (c=0.4) | E. (c=0.4) |
| **λ = 0.5 , θ = 0.5** | | | | | | | |
| Jeffrey | | 0.0088902 | 0.0055014 | 0.0029223 | 0.0024007 | 0.0014443 | 0.0005902 |

 

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Modified Jeffrey | | 0.0079267 | 0.0050753 | 0.0027653 | 0.0022876 | 0.0014040 | 0.0005824 |
| Extension Jeffrey | c = 0.4 | 0.0098303 | 0.0052086 | 0.0029886 | 0.0023916 | 0.0014401 | 0.0006491 |
| | c = 1 | 0.0072963 | 0.0047960 | 0.0026607 | 0.0022108 | 0.0013767 | 0.0005779 |
| | c = 2 | 0.0090184 | 0.0053537 | 0.0029246 | 0.0024854 | 0.0013321 | 0.0006521 |
| Gamma | $\alpha = 2$ , $\beta = 4$ | 0.0053878 | 0.0039854 | 0.0024425 | 0.0020668 | 0.0013252 | 0.0005666 |
| | $\alpha = 4$ , $\beta = 4$ | 0.0096579 | 0.0061558 | 0.0035244 | 0.0026708 | 0.0016448 | 0.0007129 |
| | $\alpha = 4$ , $\beta = 2$ | 0.0152056 | 0.0092219 | 0.0041649 | 0.0035550 | 0.0016939 | 0.0007971 |
| Best Prior Distribution | | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) | G (2,4) |
| **$\lambda = 0.5$ , $\theta = 1.5$** | | | | | | | |
| Jeffrey | | 0.0148260 | 0.0082079 | 0.0056517 | 0.0043392 | 0.0025313 | 0.0012758 |
| Modified Jeffrey | | 0.0143244 | 0.0078891 | 0.0054881 | 0.0042942 | 0.0024993 | 0.0012673 |
| Extension Jeffrey | c = 0.4 | 0.0147216 | 0.0087020 | 0.0055072 | 0.0046300 | 0.0028542 | 0.0013380 |
| | c = 1 | 0.0143231 | 0.0078310 | 0.0054200 | 0.0042883 | 0.0024925 | 0.0012652 |
| | c = 2 | 0.0208842 | 0.0117054 | 0.0064733 | 0.0051337 | 0.0027477 | 0.0013814 |
| Gamma | $\alpha = 2$ , $\beta = 4$ | 0.0123916 | 0.0074879 | 0.0048064 | 0.0042901 | 0.0024143 | 0.0012399 |
| | $\alpha = 4$ , $\beta = 4$ | 0.0062792 | 0.0047038 | 0.0037716 | 0.0029468 | 0.0021563 | 0.0011398 |
| | $\alpha = 4$ , $\beta = 2$ | 0.0098387 | 0.0071026 | 0.0049530 | 0.0038970 | 0.0025262 | 0.0013171 |
| Best Prior Distribution | | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) | G (4,4) |