Fixed point theorems through rational expression in Altering distance functions

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Abstract

In this paper we proves a generalised results of J.R. Morales , E.M.Rojas , B.K.Dasand, S.Gupta .Also the results given by B.Samet and H.Yazid using altering distance functions and property P for the contraction mappings.

Keywords: Fixed point, Altering distance functions, Complete metric space.

Mathematical Subject Classification: 45H10, 54H25.

Introduction and Preliminaries

The fixed point theorems in metric spaces are playing a major role to solve many problems in a mathematical analysis. So the attraction of metric spaces to a large numbers of mathematicians is understandable. Some generalizations of the notion of a metric space have been proposed by some authors.

Altering distance function for self-mapping on a metric space established by M.S. Khan in 1984 and it can be expanded by M. Swalesh, S. Sessa that they introduced a control function which they called as altering distance function in the research of fixed point theory. The author Mier- Keeler type (ε, δ)- contractive condition to study of fixed point by using a control function with extended contractive conditions.

Definition 1 A function $\psi: \mathbb{R}_+ \to \mathbb{R}_+ := [0, +\infty)$ is called an altering distance function if the following properties are satisfied.

(φ_1) $\psi(t) = 0 \Leftrightarrow t = 0.$

(φ_2) ψ is monotonically non decreasing.

(φ_3) ψ is continuous.

By ψ wedenotes the set of all altering distance function.

Using those control functions the author extend the Banach contraction principle by taking $\psi = \mathrm{Id}$, (the identity mapping), in the inequality contraction (1.1) of the following theorem.

Theorem1.1 Let (M, d)be a complete metric space, let $\psi \in \Psi$ and let $Q : M \to M$

be a mapping which satisfies the following inequality

$$\psi[d(Q_x, Q_y)] \leq a\psi[d(x, y)] \qquad [1.1]$$

for all $x, y \in M$ and for some $0 < a < 1$. Then , shas a unique fixed point $v_0 \in M$

and moreover for each $x \in M$, $\lim_{n\to\infty} Q^n x = v_0$.

Fixed point theorems involving the notion of altering distance functions has been widely studied, On the other hand, in 1975, B.K. Das and S. Gupta [3] proves the following result.

Theorem1.2 Let (M, d)be a metric space and let $Q: M \to M$ be a given mapping such that,

(i) $d(Qx, Qy) \leq \alpha\, d(x, y) + \beta\, m(x, y)$ [1.2]

for all $x, y \in M, \alpha > 0, \beta > 0, \alpha + \beta < 1$ where

$m(x, y) = \left[\frac{d^2(x,Qx)+d(x,Qy)\,d(y,Qx)+d^2(y,Qy)}{1+d(x,Qx)d(y,Qy)}\right]$[1.3]

for all $x, y \in M$.

(ii) for some $x_0 \in M$, the sequence of iterates $(Q^n x_0)$has a subsequence $\left(Q^{nk} x_0\right)$

With$\lim_{k\to\infty} Q^{nk} x_0 = v_0$. Then v_0 is the unique fixed point of Q.

Definition1.2. Let (M, d) be a metric space for a self-mapping Q with a nonempty fixed point set E(Q). Then Q is said to satisfy the property P If E(Q) = E(Q^n) for each n∈ N.

Lemma 1.3. Let (M, d) be a metric space. Let $\{y_n\}$ be a sequence in M such that

$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$ [1.4]

If $\{y_n\}$ is not a Cauchy sequence in M, then there exist an $\varepsilon_0 > 0$ and sequence of integers positive (m(k)) and (n(k)) with

(m(k))> (n(k)) > k, such that,

$d\left(y_{(m(k))}, y_{(n(k))}\right) \geq \varepsilon_0, \; d\left(y_{(m(k))-1}, y_{(n(k))}\right) < \varepsilon_0,$ and

i. $\lim_{k\to\infty} d\left(y_{(m(k))-1}, y_{(n(k))+1}\right) = \varepsilon_0$
ii. $\lim_{k\to\infty} d\left(y_{(m(k))}, y_{(n(k))}\right) = \varepsilon_0$
iii. $\lim_{k\to\infty} d\left(y_{(m(k))-1}, y_{(n(k))}\right) = \varepsilon_0$

Remark 1.4. From Lemma 1.3 is easy to get

$$\lim_{k\to\infty} d\left(y_{(m(k))+1}, y_{(n(k))+1}\right) = \varepsilon_0$$

In this paper we will study the property introduced by G.S. Jeong and B.E. Rhoades in [5] which they called the property P in metric spaces

<u>Main Result</u>

Theorem 2.1 Let a complete metric space (M, d) , we have $\psi \in \Psi. Let\ Q : M \to M$ be a mapping which satisfies the condition:

$\psi[d(Qx, Qy)] \leq \alpha\, \psi[d(x, y)] + \beta\, \psi\left[\frac{d^2(x,Qx)+d(x,Qy)\,d(y,Qx)+d^2(y,Qy)}{1+d(x,Qx)d(y,Qy)}\right]$ [2.1]

for all $x, y \in M, \alpha > 0, \beta > 0, \alpha + 2\beta < 1\ and\ m(x, y)$ is given by [1.2]. Then Q has a unique fixed point $v_0 \in M$, and for each $x \in M$ $lim_{n\to\infty} Q^n x = v_0.$

Proof: Let $x \in M$ be an arbitrary point and let $\{x_n\}$ be a sequence defined as:

$$x_{n+1} = Qx_n = Q^{n+1}x$$

For all n ≥1, Now

$\psi[d(x_n, x_{n+1})] = \psi[d(Qx_{n-1}, Qx_n)]$ [2.2]

$$\leq \alpha\, \psi[d(x_{n-1}, x_n)] + \beta\, \psi\left[\frac{d^2(x_{n-1}, Qx_{n-1}) + d(x_{n-1}, Qx_n)\, d(x_n, Qx_{n-1}) + d^2(x_n, Qx_n)}{1 + d(x_{n-1}, Qx_{n-1})d(x_n, Qx_n)}\right]$$

$$\psi[d(x_n, x_{n+1})] \leq \alpha\, \psi[d(x_{n-1}, x_n)] + \beta\, \psi\left[\frac{d^2(x_{n-1}, Qx_{n-1})}{1 + d(x_{n-1}, Qx_{n-1})d(x_n, Qx_n)}\right]$$

$$+ \beta\, \psi\left[\frac{+d(x_{n-1}, Qx_n)\, d(x_n, Qx_{n-1})}{1+d(x_{n-1}, Qx_{n-1})d(x_n, Qx_n)}\right] + +\beta\, \psi\left[\frac{d^2(x_n, Qx_n)}{1+d(x_{n-1}, Qx_{n-1})d(x_n, Qx_n)}\right]$$

$$\leq \alpha\, \psi[d(x_{n-1}, x_n)] + \beta\, \psi\left[\frac{d^2(x_{n-1}, x_n)}{1 + d(x_{n-1}, x_n)d(x_n, x_{n+1})}\right]$$

$$+ \beta\, \psi\left[\frac{+d(x_{n-1}, x_{n+1})\, d(x_n, x_n)}{1+d(x_{n-1}, x_n)d(x_n, x_{n+1})}\right] + +\beta\, \psi\left[\frac{d^2(x_n, x_{n+1})}{1+d(x_{n-1}, x_n)d(x_n, x_{n+1})}\right]$$

$$\leq \alpha\, \psi[d(x_{n-1}, x_n)] + \beta\, \psi[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$

$$\psi[d(x_n, x_{n+1})] \leq (\alpha + \beta)\psi[d(x_{n-1}, x_n)] + \beta\, \psi d(x_n, x_{n+1})$$

$$\psi[d(x_n, x_{n+1})] \leq (\alpha + \beta)\psi[d(x_{n-1}, x_n)]$$

$$\psi[d(x_n, x_{n+1})] \leq \frac{(\alpha + \beta)}{(1 - \beta)}\psi[d(x_{n-1}, x_n)]$$

$$\leq \left[\frac{(\alpha + \beta)}{(1 - \beta)}\right]^2 \psi[d(x_{n-2}, x_{n-1})] \leq \_\_\_\_$$

$\psi[d(x_n, x_{n+1})] \leq \left[\frac{(\alpha+\beta)}{(1-\beta)}\right]^n \psi[\mathrm{d}(x_0, x_1)]$ [2.3]

since $\frac{\alpha}{1-\beta} \in (0,1)$ from (3), we obtain

$$\lim_{n\to\infty} \psi[d(x_n, x_{n+1})] = 0$$

From the result given that $\psi \in \Psi$, we have

$lim_{n\to\infty}\, d(x_n, x_{n+1}) = 0$ [2.4]

Now, we will show that (x_n) is Cauchy sequence in M. Suppose that (x_n) is not a Cauchy sequence, which means that there is a constant $\in > 0$ such that for each positive integer k, there exist a positive integer m(k)and n(k) with m(k)>n(k)>k such that

$\mathrm{d}(x_{m(k)}, x_{n(k)}) \geq \in_0$, $d(x_{m(k)-1}, x_{n(k)}) < \in_0$

From lemma 1.3 and remark 1.4 we have,

$lim_{k\to\infty}\, d\big(x_{m(k)}, x_{n(k)}\big) = \in_0$ [2.5]

And $lim_{k\to\infty}\, d\big(x_{m(k)+1}, x_{n(k)+1}\big) = \in_0$ [2.6]

For $x = x_{m(k)} and\ y = y_{n(k)}$ from [1] we have

$d\big(x_{m(k)+1}, x_{n(k)+1}\big) = \psi\big[d\big(Qx_{m(k)}, x_{n(k)}\big)\big]$

$$\leq \alpha\, \psi\big[d\big(x_{m(k)}, x_{n(k)}\big)\big] + \beta\, \psi\left[\frac{d^2\big(x_{m(k)}, x_{n(k)}\big) + d\big(x_{m(k)}, x_{n(k)+1}\big)\, d\big(x_{n(k)}, x_{n(k)}\big) + d^2\big(x_{n(k)}, x_{m(k)+1}\big)}{1 + d\big(x_{m(k)}, x_{n(k)}\big)d\big(x_{n(k)}, x_{n(k)+1}\big)}\right]$$

Using [4], [5] and [6] we have

$$\psi(\in) = \lim_{k\to\infty} \beta\, \psi\left[d\left(x_{n(k),}x_{n(k)+1}\right)\right] \le \beta \lim_{k\to\infty} \psi\left[d\left(x_{n(k)-1,}x_{n(k)}\right)\right]$$

$$\le \beta \lim_{k\to\infty} \psi\left[d\left(x_{m(k),}x_{n(k)}\right)\right]$$

$$\le \alpha\, \psi(\in)$$

Since $\alpha \in (0,1), we\ get\ a\ contradiction$. Thus $\{x_n\}$ is a Cauchy sequence in the complete metric space M, Thus there exist $v_0 \in M$such that

$$\lim_{n\to\infty} x_n = v_0$$

Setting $x = x_n\ and\ y = v_0\ in\ [1], we\ have$

$$\psi\left[d\left(x_{n+1,}\, Qv_0\right)\right] = \psi\left[d\left(Qx_{n,}\, Tv_0\right)\right]$$

$$\le \alpha\ \psi[d(x_n, v_0)] + \beta\, \psi\left[\frac{d^2(x_n, Qx_n) + d(x_n, Qv_0)\, d(v_0, Qx_n) + d^2(v_0, Qv_0)}{1 + d(x_n, Qx_n) + d(v_0, Qv_0)}\right]$$

Therefore $\lim_{n\to\infty} \psi\left[d\left(x_{n+1,}\, Qv_0\right)\right] \le\ \beta\, \psi\ d(v_0, Qv_0)$

$$i.e. \qquad \psi\ d(v_0, Qv_0) \le\ \beta\, \psi\ d(v_0, Qv_0)$$

$since\ \beta \in (0,1), then\ \psi\ d(v_0, Qv_0) = 0, which\ implies\ that\ d(v_0, Qv_0) = 0$

$thus\ v_0 = Qv_0.$

Now we are going to establish the uniqueness of the fixed point, Lety_0, v_0 be two fixed point of Q such that $y_0 \ne\ v_0,\ putting\ x = y_0\ and\ y = v_0\ in$ [1], we get

$$\psi\ d(Qv_0, Qy_0)\ \le \alpha\, \psi[d(v_0, y_0)]$$

$$+\beta\, \psi\left[\frac{d^2(v_0, Qv_0) + d(v_0, Qy_0)\, d(y_0, Qv_0) + d^2(y_0, Qy_0)}{1 + d(v_0, Qv_0) + d(y_0, Qy_0)}\right]$$

$i.e. \quad \psi\ d(Qv_0, Qy_0)\ \le \alpha\, \psi[d(v_0, y_0)]$

whichimplies that $\psi[d(v_0, y_0)] = 0, so\ d(v_0, y_0) = 0$

Thus$v_0 = y_0.$

Corollary 2.2 Let (M, d) be a complete metric space and let Q : M → M be a mapping. We assume that for each $x, y \in M,$

$$\int_0^{d(Qx,Qy)} \psi(u)du \le \alpha \int_0^{d(x,y)} \psi(u)du + \beta \int_0^{\psi\left[\frac{d^2(v_0,Qv_0)+d(v_0,Qy_0)\, d(y_0,Qv_0)+d^2(y_0,Qy_0)}{1+d(v_0,Qv_0)+d(y_0,Qy_0)}\right]} \psi(u)du \ [2.7]$$

Where 0< $\alpha + \beta < 1\ and\ \psi : R_+ \to\ R_+$ is a Lebesgue integrable mapping which is summable on each compact subset of [0, +∞) $, non\ negative\ and\ such\ that$

$$\int_0^{\epsilon} \psi(u)du > 0, \quad for\ all\ \epsilon > 0.$$

Then Q has a unique fixed point $v_0 \in M\ such\ that\ for\ each\ x \in M, \lim_{n\to\infty} Q^n x = v_0.$

Proof: Let $\psi : R_+ \to\ R_+$be a mapping as we define $\psi_0(u) = \int_0^u \psi(u)du, u \in\ R_+$. It is clear that $\psi_0(0) = 0. \psi_0$is monotonically non decreasing and by hypothesis $\Psi_0$is absolutely continuous. Hence $\psi_0$is continuous. Therefore, $\psi_0 \in \Psi$, so by (2.1)becomes

$$\psi_0\big(d(Qx,Qy)\big) \le \alpha\psi_0\big(d(x,y)\big) + \beta\,\psi_0\left[\frac{d^2(v_0,Qv_0)+d(v_0,Qy_0)\,d(y_0,Qv_0)+d^2(y_0,Qy_0)}{1+d(v_0,Qv_0)+d(y_0,Qy_0)}\right]$$

Hence from theorem 2.1 there exists a unique fixed point $v_0 \in M$ *such that for each*

$x \in M, \lim_{n\to\infty} Q^n x = v_0$.

Remarks 2.3.

i. If we take $\beta = 0$, then (2.1) reduces to (1.2), thus the Theorem 1.1 is a corollary of theorem 2.1.
ii. If we take $\psi = I\rho$ *in* (2.1), then we obtain (1.2). Therefore the Theorem 2.1is a generalisation of Theorem 1.2.

3 The property P.

In this section we are going to prove that the mappings satisfying the contractive conditions [1.1], [1.2], [2.1] and [2.7] fulfil the property P.

Theorem 3.1 Let (M, d) be a completemetric space, we have $\psi \in \Psi$. *Let* $Q : M \to M$ be a mapping which satisfies the condition:

$$\psi[d(Qx,Qy)] \le \alpha\,\psi[d(x,y)]$$

for all $x, y \in M$, *and for some* $0 < \alpha < 1$. *Then* $E_Q \neq \phi$ and Q has a property P.
Proof: From Theorem [1.1], Q has a fixed point therefore $E_{Q^n} \neq \phi$ for every n ∈ N,
Fix $n > 1$and we assume that$v \in E_{Q^n}$ we have to prove that $v \in E_Q$, Assume that
$v \neq Qv$, *from* [1.1]
$\psi[d(v,Qv)] = \psi[d(Q^n v, Q^{n+1}v)] \le a\psi[d(Q^{n-1}v, Q^n v)] \le \ldots\ldots \le a^n \psi[d(v,Qv)]$.

Since a ∈ (0,1), $lim_{n\to\infty}\,\psi[d(v,Qv)] = 0$. From the fact that, $\psi \in \Psi$ we get $v = Qv$ which is a contradiction. Therefore $v \in E_Q$ i.e. Q has a property P.

Theorem 3.2 Let (M, d) be a complete metric space, and *Let* $Q : M \to M$ be a mapping which satisfies the contractive condition:

$$\psi[d(Qx,Qy)] \le \alpha\,[d(x,y)] + \beta\, m(x,y)$$

for all $x, y \in M, \alpha > 0, \beta > 0, \alpha + \beta < 1$ *where*

$$m(x,y) = \left[\frac{d^2(x,Qx)+d(x,Qy)\,d(y,Qx)+d^2(y,Qy)}{1+d(x,Qx)d(y,Qy)}\right]$$

Then $E_Q \neq \phi$and Q has a property.
Proof: From Theorem [1.2], $E_Q \neq \phi$,therefore $E_{Q^n} \neq \phi$ for every n ∈ N,
Fix $n > 1$and we assume that $v \in E_{Q^n}$ we have to prove that $v \in E_Q$, Assume that
$v \neq Qv$

$$\begin{aligned} d(v,Qv) = {} & d(Q^n v, Q^{n+1}v) \\ & \le ad(Q^{n-1}v, Q^n v) \\ & + b\left[\frac{d^2(Q^{n-1}v,Q^n v)+d(Q^{n-1}v,Q^{n+1}v)d(Q^n v,Q^n v)+d^2(Q^n v,Q^{n+1}v)}{1+d(Q^{n-1}v,Q^n v)+d(Q^n v,Q^{n+1}v)}\right] \\ & = ad(Q^{n-1}v,Q^n v) + bd(Q^n v,Q^{n+1}v) \end{aligned}$$

Therefore $d(v,Qv) = d(Q^n v, Q^{n+1}v) \le \frac{a}{1-b}\, d(Q^{n-1}v, Q^n v) \le \ldots . \le \left(\frac{a}{1-b}\right)^n d(v,Qv)$
Which is a contradiction. Consequently $v \in E_Q$ and Q has the property P.

Theorem 3.3 Let (M, d) be a complete metric space, let $\psi \in \Psi$ and *Let* $Q : M \to M$ be a mapping which satisfies the contractive condition:

$$\psi[d(Qx,Qy)] \leq \alpha\, \psi[d(x,y)] + \beta\psi\left[\frac{d^2(x,Qx) + d(x,Qy)\, d(y,Qx) + d^2(y,Qy)}{1 + d(x,Qx)d(y,Qy)}\right]$$

$Then\ \ E_Q \neq \phi$and Q has a property P.

Proof: From Theorem [1.1], Q has a fixed point therefore $E_{Q^n} \neq \phi$ for every n $\in N$,
$Fix\ n > 1$and we assume that $v \in E_{Q^n}$ we have to prove that $v \in E_Q$, Assume that
$v \neq Qv, from$ [2.1]

$$\psi[d(v,Qv)] = \psi\,[d(Q^n v, Q^{n+1}v)]$$
$$\leq a\, \psi[d(Q^{n-1}v, Q^n v)] + b\,\psi\left[\frac{d^2(Q^{n-1}v,Q^n v) + d(Q^{n-1}v,Q^{n+1}v)d(Q^n v,Q^n v) + d^2(Q^n v,Q^{n+1}v)}{1 + d(Q^{n-1}v,Q^n v) + d(Q^n v,Q^{n+1}v)}\right]$$
$$= a\,\psi d(Q^{n-1}v,Q^n v) + b\,\psi d(Q^n v,Q^{n+1}v)$$

Hence $\psi\, d(v,Qv) = \psi\, d(Q^n v,Q^{n+1}v) \leq \frac{a}{1-b}\psi\, d(Q^{n-1}v,Q^n v) \leq \ldots. \leq \left(\frac{a}{1-b}\right)^n \psi d(v,Qv)$

$$\psi d(v,Qv) \leq \left(\frac{a}{1-b}\right)^n \psi d(v,Qv)$$

Which is a contradiction, therefore$\psi\, d(v,Qv) = 0$, since $\psi \in \Psi$
We conclude that $d(v,Qv) = 0$, thus $v \in E_Q$ and Q has the property P.

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