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# The Univalence of Some Integral Operators

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**Abstract**

In this paper, we obtain the conditions for univalence of the integral operators of the form:

$$F(z) = \lambda \int_0^z t^{\lambda-1} \prod_{i=1}^{k} \left( \frac{g_i(s)}{s} \right)^{1/\lambda} ds$$

And

$$H(z) = \left\{ \lambda \int_0^z t^{\lambda-1} \prod_{i=1}^{k} \left( \frac{g_i(s)}{s} \right)^{\lambda-1} ds \right\}^{1/\lambda}$$

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## 1. Introduction

Let $A$ be the class of the functions in the unit disk $U = \{Z \in C: |z| < 1$ such that $f(0) = f'(0) - 1 = 0$. Let $S$ be the class of functions $f \in A$ which are univalent in $U$. Ozaki and Nunokawa [2] showed the condition for the univalence of the function $f \in A$ as given in the lemma below.

**Lemma 1 [3]:** Let $f \in A$ satisfy the condition

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq 1 \tag{1}$$

For all $z \in U$, then $f$ is univalent in $U$

**Lemma 2 [2]:** Let $\alpha$ be a complex number, $\alpha > 0$ , and $f \in A$. If

$$\frac{1 - |z|^{2Re\alpha}}{Re\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1$$

For all $z \in U$, then the function $F_\alpha(z) = \left[ \alpha \int_0^z U^{\alpha-1} f'(u) du \right]^{1/\alpha}$

Is in the class $S$.

The Schwartz lemma [1] : Let the analytic function $f$ be regular in the unit disk and let $f(0) = 0$. If $|f(z)| \leq 1$, then

$|f(z)| \leq |z|$

For all $z \in U$, where equality can hold only if $|f(z)| \equiv Kz$ and $k = 1$

**Lemma 3 [4]:** Let $g \in A$ satisfies (1) and $a + b_i$ a complex number, $a, b$ satisfies the conditions

$$a \in \left[ \frac{3}{4}, \frac{3}{2} \right]; b \in [0, \frac{1}{2\sqrt{2}}] \tag{2}$$

$$8a^2 + a^9 b^2 - 18a + 9 \leq 0 \tag{3}$$

If $|g(z) \leq 1|$ for all $z \in U$ then the function $G(z) = \left\{ a + b_i \int_0^z \left( \frac{g(u)}{u} \right)^{a+b_i-1} du \right\}^{\frac{1}{a+b_i}}$

Is univalent in $U$

## 1. MAIN RESULTS

**Theorem 1:** Let $g \in A$ satisfies (1) and $\lambda$ is a complex number, $\lambda$ satisfies the conditions

$$Re\lambda \in \left(0, \sqrt{k+2}\right], k \geq 1 \tag{4}$$

$$(Re\lambda)^4 (Re\lambda)^2 (Im\lambda)^2 - (k+2)^2 \geq 0 \tag{5}$$

If $|g(z)| \leq 1$ for all $z \in U$ then the function

$$F(z) = \lambda \int_0^z t^{\lambda-1} \prod_{i=1}^{k} \left(\frac{g_i(s)}{s}\right)^{1/\lambda} ds$$ Is univalent in $U$

**Proof:** Let

$$F(z) = \lambda \int_0^z t^{\lambda-1} \prod_{i=1}^{k} \left(\frac{g_i(s)}{s}\right)^{1/\lambda} ds$$

And let $f(z) = \int_0^z \prod_{i=1}^{k} \left(\frac{g_i(s)}{s}\right)^{1/\lambda} ds$ (6)

Then $f'(z) = \prod_{i=1}^{k} \left(\frac{g_i(z)}{z}\right)^{1/\lambda}$

$$= \left(\frac{g_1(s)}{s}\right)^{1/\lambda} \left(\frac{g_2(s)}{s}\right)^{1/\lambda} \dots \left(\frac{g_k(s)}{s}\right)^{1/\lambda}$$

$$f''(z) = \left(\frac{g_1(s)}{s}\right)^{1/\lambda} \left[\left(\frac{g_2(s)}{s}\right)^{\frac{1}{\lambda}} \dots \left(\frac{g_k(s)}{s}\right)^{\frac{1}{\lambda}}\right]' + \left[\left(\frac{g_1(s)}{s}\right)^{1/\lambda}\right]' \left[\left(\frac{g_2(s)}{s}\right)^{\frac{1}{\lambda}} \dots \left(\frac{g_k(s)}{s}\right)^{\frac{1}{\lambda}}\right]$$

$$\frac{zf''(z)}{f'(z)} = \frac{1}{\lambda}\left(\frac{zg'_1(z)}{g_1(z)} - 1\right) + \frac{1}{\lambda}\left(\frac{zg'_2(z)}{g_1(z)} - 1\right) \dots \frac{1}{\lambda}\left(\frac{zg'_k(z)}{g_1(z)} - 1\right)$$

$$= \frac{1}{\lambda} \sum_{i=1}^{k} \left(\frac{zg'_i(z)}{g_1(z)} - 1\right)$$

This implies that

$$\frac{1-|z|^{2Re\lambda}}{Re\lambda} \left|\frac{zf''(z)}{f'(z)}\right| = \frac{1-|z|^{2Re\lambda}}{Re\lambda} \frac{1}{|\lambda|} \left|\sum_{i=1}^{k} \frac{zg'_i(z)}{g_1(z)} - 1\right|$$

$$\leq \frac{1-|z|^{2Re\lambda}}{Re\lambda} \frac{1}{|\lambda|} \left(\left|\sum_{i=1}^{k} \frac{zg'_i(z)}{g_1(z)}\right| + 1\right)$$

$$\leq \frac{1-|z|^{2Re\lambda}}{Re\lambda} \frac{1}{|\lambda|} \left(\left|\sum_{i=1}^{k} \frac{z^2 g'_i(z)}{g^2{}_1(z)}\right| \left|\frac{g_i(z)}{z}\right| + 1\right) \tag{7}$$

Using Schwartz-lemma in (7), we have

$$\frac{1-|z|^{2Re\lambda}}{Re\lambda} \left|\frac{zf''(z)}{f'(z)}\right| \leq \frac{1-|z|^{2Re\lambda}}{Re\lambda} \frac{1}{|\lambda|} \left(\left|\sum_{i=1}^{k} \frac{z^2 g'_i(z)}{g^2{}_1(z)} - 1\right| + 2\right)$$

But $g$ satisfies (1), thus

$$\left|\sum_{i=1}^{k} \frac{z^2 g'_i(z)}{g^2{}_1(z)} - 1\right| \leq k$$

This implies that

$$\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{1-|z|^{2Re\lambda}}{Re\lambda}\frac{1}{|\lambda|}(k+2) \le \frac{k+2}{Re\lambda|\lambda|} \tag{8}$$

From (4) and (5), we have

$$\frac{k+2}{Re\lambda|\lambda|} \le 1 \tag{9}$$

Using (9) in (8), we obtain $\frac{1-|z|^{2Re\lambda}}{Re\lambda}\left|\frac{zf''(z)}{f'(z)}\right| \le 1$

And from (6) $f'(z) = \prod_{i=1}^{k}\left(\frac{g_k(s)}{s}\right)^{1/\lambda}$

And by lemma 2 $F(z)$ is univalent ■

**Theorem 2** Let $g \in A$ satisfies (1) and let $\lambda$ be a complex number with $Re\lambda, Im\lambda$ satisfying (2) and (3). If $|g(z)| \le 1$ for all $z \in U$ then the function

$$H(z) = \left\{\lambda\int_0^z t^{\lambda-1}\prod_{i=1}^{k}\left(\frac{g_i(s)}{s}\right)^{\lambda-1}\right\}^{1/\lambda}$$

Is univalent in $U$.

**Proof:** Let

$$H(z) = \left\{\lambda\int_0^z t^{\lambda-1}\prod_{i=1}^{k}\left(\frac{g_i(s)}{s}\right)^{\lambda-1}\right\}^{1/\lambda} \tag{10}$$

Following the procedure of the proof of theorem 1, we obtain

$$\frac{1-|z|^{2Re\lambda}}{Re\lambda}\left|\frac{zh''(z)}{h'(z)}\right| \le \frac{1-|z|^{2Re\lambda}}{Re\lambda}|\lambda-1|\left(\left|\sum_{i=1}^{k}\frac{z^2g'_i(z)}{g^2{}_1(z)}-1\right|+2\right)$$

But $g$ satisfies (1), thus

$$\frac{1-|z|^{2Re\lambda}}{Re\lambda}\left|\frac{zh''(z)}{h'(z)}\right| \le \frac{1-|z|^{2Re\lambda}}{Re\lambda}|\lambda-1|(k+2) \le \frac{|\lambda-1|(k+2)}{Re\lambda}$$

From (2) and (3) we have

$$\frac{|\lambda-1|(k+2)}{Re\lambda} \le 1$$

This implies that

$\frac{1-|z|^{2Re\lambda}}{Re\lambda}\left|\frac{zh''(z)}{h'(z)}\right| \le 1$ for all $z \in U$

From (10), $h'(z) = \prod_{i=1}^{k}\left(\frac{g(z)}{z}\right)^{\lambda-1}$

And by lemma 2, $H(z)$ is univalent in $U$ ■

**Remark**: Theorem 1and theorem 2 gives a generalization of theorem 1[4] and lemma 3 respectively,.

 

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