

The Univalence of Some Integral Operators

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Abstract

In this paper, we obtain the conditions for univalence of the integral operators of the form:

And

$$F(z) = \lambda \int_0^z t^{\lambda-1} \prod_{i=1}^k \left(\frac{g_i(s)}{s} \right)^{1/\lambda} ds$$

$$H(z) = \left\{ \lambda \int_0^z t^{\lambda-1} \prod_{i=1}^k \left(\frac{g_i(s)}{s} \right)^{\alpha-1} ds \right\}^{1/\lambda}$$

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1. Introduction

Let A be the class of the functions in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ such that $f(0) = f'(0) - 1 = 0$. Let S be the class of functions $f \in A$ which are univalent in U . Ozaki and Nunokawa [2] showed the condition for the univalence of the function $f \in A$ as given in the lemma below.

Lemma 1 [3]: Let $f \in A$ satisfy the condition

$$\left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq 1 \quad (1)$$

For all $z \in U$, then f is univalent in U

Lemma 2 [2]: Let α be a complex number, $\alpha > 0$, and $f \in A$. If

$$\frac{1 - |z|^{2\text{Re}\alpha}}{\text{Re}\alpha} \left| \frac{z f''(z)}{f'(z)} \right| \leq 1$$

For all $z \in U$, then the function $F_\alpha(z) = \left[\alpha \int_0^z U^{\alpha-1} f'(u) du \right]^{1/\alpha}$

Is in the class S .

The Schwartz lemma [1] : Let the analytic function f be regular in the unit disk and let $f(0) = 0$. If $|f(z)| \leq 1$, then

$$|f(z)| \leq |z|$$

For all $z \in U$, where equality can hold only if $|f(z)| \equiv Kz$ and $k = 1$

Lemma 3 [4]: Let $g \in A$ satisfies (1) and $a + b_i$ a complex number, a, b satisfies the conditions

$$a \in \left[\frac{3}{4}, \frac{3}{2} \right]; b \in \left[0, \frac{1}{2\sqrt{2}} \right] \quad (2)$$

$$8a^2 + a^9 b^2 - 18a + 9 \leq 0 \quad (3)$$

If $|g(z)| \leq 1$ for all $z \in U$ then the function $G(z) = \left\{ a + b_i \int_0^z \left(\frac{g(u)}{u} \right)^{a+b_i-1} du \right\}^{\frac{1}{a+b_i}}$

Is univalent in U

1. MAIN RESULTS

Theorem 1: Let $g \in A$ satisfies (1) and λ is a complex number, λ satisfies the conditions

$$Re\lambda \in (0, \sqrt{k+2}], k \geq 1 \tag{4}$$

$$(Re\lambda)^4(Re\lambda)^2(Im\lambda)^2 - (k+2)^2 \geq 0 \tag{5}$$

If $|g(z)| \leq 1$ for all $z \in U$ then the function

$$F(z) = \lambda \int_0^z t^{\lambda-1} \prod_{i=1}^k \left(\frac{g_i(s)}{s} \right)^{1/\lambda} ds \text{ Is univalent in } U$$

Proof: Let

$$F(z) = \lambda \int_0^z t^{\lambda-1} \prod_{i=1}^k \left(\frac{g_i(s)}{s} \right)^{1/\lambda} ds$$

And let $f(z) = \int_0^z \prod_{i=1}^k \left(\frac{g_i(s)}{s} \right)^{1/\lambda} ds$ (6)

Then $f'(z) = \prod_{i=1}^k \left(\frac{g_i(z)}{z} \right)^{1/\lambda}$

$$= \left(\frac{g_1(s)}{s} \right)^{1/\lambda} \left(\frac{g_2(s)}{s} \right)^{1/\lambda} \dots \left(\frac{g_k(s)}{s} \right)^{1/\lambda}$$

$$f''(z) = \left(\frac{g_1(s)}{s} \right)^{1/\lambda} \left[\left(\frac{g_2(s)}{s} \right)^{1/\lambda} \dots \left(\frac{g_k(s)}{s} \right)^{1/\lambda} \right]' + \left[\left(\frac{g_1(s)}{s} \right)^{1/\lambda} \right]' \left[\left(\frac{g_2(s)}{s} \right)^{1/\lambda} \dots \left(\frac{g_k(s)}{s} \right)^{1/\lambda} \right]$$

$$\frac{zf''(z)}{f'(z)} = \frac{1}{\lambda} \left(\frac{zg'_1(z)}{g_1(z)} - 1 \right) + \frac{1}{\lambda} \left(\frac{zg'_2(z)}{g_1(z)} - 1 \right) \dots \frac{1}{\lambda} \left(\frac{zg'_k(z)}{g_1(z)} - 1 \right)$$

$$= \frac{1}{\lambda} \sum_{i=1}^k \left(\frac{zg'_i(z)}{g_1(z)} - 1 \right)$$

This implies that

$$\begin{aligned} \frac{1-|z|^{2Re\lambda}}{Re\lambda} \left| \frac{zf''(z)}{f'(z)} \right| &= \frac{1-|z|^{2Re\lambda}}{Re\lambda} \frac{1}{|\lambda|} \left| \sum_{i=1}^k \frac{zg'_i(z)}{g_1(z)} - 1 \right| \\ &\leq \frac{1-|z|^{2Re\lambda}}{Re\lambda} \frac{1}{|\lambda|} \left(\left| \sum_{i=1}^k \frac{zg'_i(z)}{g_1(z)} \right| + 1 \right) \\ &\leq \frac{1-|z|^{2Re\lambda}}{Re\lambda} \frac{1}{|\lambda|} \left(\left| \sum_{i=1}^k \frac{z^2g'_i(z)}{g^2_1(z)} \right| \left| \frac{g_i(z)}{z} \right| + 1 \right) \end{aligned} \tag{7}$$

Using Schwartz-lemma in (7), we have

$$\frac{1-|z|^{2Re\lambda}}{Re\lambda} \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{1-|z|^{2Re\lambda}}{Re\lambda} \frac{1}{|\lambda|} \left(\left| \sum_{i=1}^k \frac{z^2g'_i(z)}{g^2_1(z)} - 1 \right| + 2 \right)$$

But g satisfies (1), thus

$$\left| \sum_{i=1}^k \frac{z^2g'_i(z)}{g^2_1(z)} - 1 \right| \leq k$$

This implies that

$$\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{1-|z|^{2Re\lambda}}{Re\lambda} \frac{1}{|z|} (k+2) \leq \frac{k+2}{Re\lambda|z|} \quad (8)$$

From (4) and (5), we have

$$\frac{k+2}{Re\lambda|z|} \leq 1 \quad (9)$$

Using (9) in (8), we obtain $\frac{1-|z|^{2Re\lambda}}{Re\lambda} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1$

And from (6) $f'(z) = \prod_{i=1}^k \left(\frac{g_i(s)}{s} \right)^{1/\lambda}$

And by lemma 2 $F(z)$ is univalent ■

Theorem 2 Let $g \in A$ satisfies (1) and let λ be a complex number with $Re\lambda, Im\lambda$ satisfying (2) and (3). If $|g(z)| \leq 1$ for all $z \in U$ then the function

$$H(z) = \left\{ \lambda \int_0^z t^{\lambda-1} \prod_{i=1}^k \left(\frac{g_i(s)}{s} \right)^{\lambda-1} \right\}^{1/\lambda}$$

Is univalent in U .

Proof: Let

$$H(z) = \left\{ \lambda \int_0^z t^{\lambda-1} \prod_{i=1}^k \left(\frac{g_i(s)}{s} \right)^{\lambda-1} \right\}^{1/\lambda} \quad (10)$$

Following the procedure of the proof of theorem 1, we obtain

$$\frac{1-|z|^{2Re\lambda}}{Re\lambda} \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{1-|z|^{2Re\lambda}}{Re\lambda} |\lambda-1| \left(\left| \sum_{i=1}^k \frac{z^2 g'_i(z)}{g_i^2(z)} - 1 \right| + 2 \right)$$

But g satisfies (1), thus

$$\frac{1-|z|^{2Re\lambda}}{Re\lambda} \left| \frac{zh''(z)}{h'(z)} \right| \leq \frac{1-|z|^{2Re\lambda}}{Re\lambda} |\lambda-1|(k+2) \leq \frac{|\lambda-1|(k+2)}{Re\lambda}$$

From (2) and (3) we have

$$\frac{|\lambda-1|(k+2)}{Re\lambda} \leq 1$$

This implies that

$$\frac{1-|z|^{2Re\lambda}}{Re\lambda} \left| \frac{zh''(z)}{h'(z)} \right| \leq 1 \text{ for all } z \in U$$

From (10), $h'(z) = \prod_{i=1}^k \left(\frac{g_i(z)}{z} \right)^{\lambda-1}$

And by lemma 2, $H(z)$ is univalent in U ■

Remark: Theorem 1 and theorem 2 gives a generalization of theorem 1[4] and lemma 3 respectively.

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