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Adjoint Operator in Probabilistic Hilbert Space

Assit.Prof. Dr. Radhi I.M Dalia Sami Ali

Department of Mathematic, College of Science for Women, University of Baghdad
radi52@yahoo.com
sm_dl016@yahoo.com

Abstract: The purpose of this paper is to give a definition for the adjoint operator in Probabilistic Hilbert Space and its properties by using Riesz Representation Theorem.

Keywords: Probabilistic Hilbert Space; Adjoint Operator; properties of adjoint operator.

Introduction

The notion of probabilistic inner product spaces can be considered as the generalization of that of inner product spaces. The definition of these spaces has been introduced in [1].Induced norms by these spaces may have very important applications in quantum particle physics particularly in connections with both string and E-infinity theories [2], [3]. The definition of probabilistic Hilbert Space also introduced in [4]. The definition of the adjoint operator in the ordinary Hilbert Space defined in many books and papers by many authors [5], [6].Self-adjoint operators are used in functional analysis and quantum mechanics. In quantum mechanics their importance lies in the Dirac–von Neumann formulation of quantum mechanics, in which physical observables such as position, momentum, angular momentum and spin are represented by self-adjoint operators on a Hilbert space. Our main purpose of this paper is how to give the definition for the adjoint operator in Probabilistic Hilbert Space and some of its properties by using the Riesz representation theorem that has been introduced in [4].We also published paper in the name (Some Results in Modified Probabilistic Hilbert Space) in Journal of the College Science for Women in 2014.

1. Basic definitions

Definition (1-1) [1]

Let $\mathbb{R} = (-\infty, \infty)$, define the set D to be the set of all left continuous distributions such that:

$D = \{F: F \; is \; left \; continous \; distibution \; function\}$

and let $H(t) = \begin{cases} 1 \;, & t > 0 \\ 0 \;, & t \leq 0 \end{cases}$

be distribution function which belongs to D.

Definition (1-2) [4]

A convolution of two functions $f, g \; is \; define \; as \; follows$:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau \, ;$$

$$= \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau \quad (commutative)$$

Definition (1-3) (PIP-space) [4]

Let E be a real linear space and let $F: E \times E \to D \ be\ a\ function$, then the Probabilistic inner product space is the triple $(E, F, *)$ where F is assumed to satisfy the following conditions:-
($F_{x,y}(t)$ will represent the value of $F_{x,y}\ at\ t \in R$)

$(PI-1)$ $$F_{x,x}(0) = 0$$

$(PI-2)$ $$F_{x,y} = F_{y,x}$$

$(PI-3)$ $$F_{x,x}(t) = H(t) \leftrightarrow x = 0$$

$(PI-4)$

$$F_{\lambda x,y}(t) = \begin{cases} F_{x,y}\left(\frac{t}{\lambda}\right) & , \quad \lambda > 0 \\ H(t) & , \quad \lambda = 0 \\ 1 - F_{x,y}\left({}^{t}/_{\lambda} +\right) & , \quad \lambda < 0 \end{cases}$$

where h is real number, $F_{x,y}\left({}^{t}/_{\lambda} +\right)$is the right hand limit of $F_{x,y}$ at $\frac{t}{\lambda}$

$(PI-5)$

$$if\ x, y\ are\ linearly\ independent\ then$$
$$F_{x+y,z}(t) = \left(F_{x,z} * F_{y,z}\right)(t)$$

Where

$$\left(F_{x,z} * F_{y,z}\right)(t) = \int_{-\infty}^{\infty} F_{x,z}(t-u)dF_{z,y}(u)$$

Note: If $x, y\ are\ linearly\ dependent\ then\ let\ y = \alpha x\ , \alpha\ is\ scalar\ (\in R)\ then$

$$x + y = x + \alpha x = (1+\alpha)x \ \ , now\ let\ h = \alpha x\ then$$
$$x + y = hx$$

$$F_{x+y,z}(t) = F_{\lambda x,z}(t) = \begin{cases} F_{x,z}\left(\frac{t}{\lambda}\right) & , \quad \lambda > 0 \\ H(t) & , \quad \lambda = 0 \\ 1 - F_{x,z}\left({}^{t}/_{\lambda} +\right) & , \quad \lambda < 0 \end{cases}$$

$which\ is\ (PI-4).$

Then $(E, F, *)$ is called Probabilistic Inner Product Space.

Definition (1-4) [4]

A PIP-space $(E, F, *) is\ called\ with\ \ Mathematical\ expectation\ if:$

$$\int_{-\infty}^{\infty} t dF_{x,y}(t) < \infty \qquad , \forall x, y \in E$$

Theorem (1-5) [4]

Let $(E, F, *)\ be\ PIP-space\ with\ Mathematical\ expectation\ then$

$$< x, y > = \int_{-\infty}^{\infty} t dF_{x,y}(t) \quad , \forall x, y \in E$$

$(E, < \ >)\ is\ inner\ product\ space\ , (E, \| \ \|) is\ anormed\ space\ where\ \ \| x \| = \sqrt{< x, x >}$

Note $< x, x > \geq 0 \quad \forall x \in E.$

Definition (1-6) [4]

Let $(E, F. *) be\ PIP-space, then$

1. A sequence $\{x_n\}\ in\ E\ is\ said\ to\ be\ \tau - converges\ to\ x \in E, if\ \forall \epsilon > 0\ , \forall \lambda > 0\ \exists n_0(\epsilon, \lambda)\ \ such\ that$
   $$F_{x_{n-x}, x_{n-x}}(\epsilon) > 1 - \lambda \qquad , \forall n > n_0(\epsilon, \lambda)$$
2. A linear functional $f(x) defined\ on\ E\ is\ said\ to\ be\ continuos\ , if\ \{x_n\}\ is\ \ \tau - converges\ to\ x \in E$ Implies that $f(x_n) \to f(x)\ \ , \forall\ \{x_n\} \in E$
3. $we\ say\ that\ f(x) is\ linear\ functional\ if\ :$
   $f: E \to R\ such\ that\ :$
   $$f(x+y) = f(x) + f(y)\ , \forall x, y \in E$$
   $$f(\alpha x) = \alpha f(x)\ \forall \alpha\ scalar\ , \forall x \in E$$

Definition (1-7) [4]

Let$(E, F, *)\ be\ PIP-space\ with\ Mathematical\ expectation.$ If E is complete in$\|.\|$, then E is called probabilistic Hilbert space, where $\| x \| = \sqrt{< x, x >}\ \ \forall x \in E.$

Theorem (1-8) [4]

Let$(E,F,*)$ $be\ PIP-space\ with\ Mathematical\ expectation$.For sequence$\{x_n\}$ $in\ E\ m-convergent\ (in\ norm\ \parallel x \parallel = \sqrt{<x,x>})$ implies $\tau-convergent.$

Theorem (1-9) [4] **(Riesz Representation Theorem)**

Let $(E,F,*)$ be Probabilistic Hilbert Space, for any linear continuous functional $f(x), \exists!\, y \in E\ such\ that$:

$$f(x) = \int_{-\infty}^{\infty} t dF_{x,y}(t) \quad , \forall x \in E$$

2. Main results

In this section we will define the Adjoint operator in Probabilistic Hilbert Space by using Riesz representation theorem.

Theorem (2-1) **(Adjoint Operator in Probabilistic Hilbert Space)**

Let $(E,F,*)$ be Probabilistic Hilbert Space, Let
$T \in E\ be\ continuos\ linear\ functional\ , then\ \exists!\, T^* \in E$ such that:

$$<Tx, y> = <x, T^*\ y> \qquad \forall x, y \in E$$

Proof:

Fix $y \in E\ , let\ \varphi_y(x) = <Tx, y>\ , x \in E$

$\varphi_y\ is\ linear\ functional\ on\ E\ i,e\ \varphi_y : E \to R\ such\ that$

$$\varphi_y(x+z) = \varphi_y(x) + \varphi_y(z) \quad \forall x, z \in E$$

$$\varphi_y(\alpha x) = \alpha \varphi_y(x) \quad \forall \alpha\ scalar$$

Also φ_y is continuous

Then by Riesz Representation theorem $\exists!\, z_y \in E\ such\ that$:

$$\varphi_y(x) = \int_{-\infty}^{\infty} t dF_{x,y}(t) \quad , \forall x \in E$$

Define $T^*: E \to R \text{ such that } T_y^* = z_y \ y \in E$

So

$$< Tx, y > = \ \varphi_y(x) = \int_{-\infty}^{\infty} t dF_{x,T^*y}(t) =< x, T^*y >$$

$T^* \text{ is linear map since}$

$let\ y, z \in E\ , \alpha, \beta \text{ are scalars than } \forall x \in E$

$< x, T^*(\alpha y + \beta z) > = < Tx, \alpha y + \beta z >$

$$= \int_{-\infty}^{\infty} t dF_{Tx,\alpha y+\beta z}(t) = \int_{-\infty}^{\infty} t dF_{Tx,\alpha y}(t) +$$

$$\int_{-\infty}^{\infty} t dF_{Tx,\beta z}(t)$$

$$= \alpha \int_{-\infty}^{\infty} t dF_{Tx,y}(t)$$

$$+ \beta \int_{-\infty}^{\infty} t dF_{Tx,z}(t) = \ \alpha < Tx, y > \ + \beta < Tx, z >$$

$= \alpha < x, T^*y > + \beta < x, T^*z >$

Uniqueness of the adjoint operator

$$let\ T_1^*, T_2^* \text{ be two adjoint operators for } T \in E\ i,e$$

$$< Tx, y > =< x, T_1^*y >$$

$, < Tx, y > =< x, T_2^*y >$

Then

$< x, T_1^*y > = < x, T_2^*y >$

$$\int_{-\infty}^{\infty} t dF_{x,T_1^*y}(t) = \int_{-\infty}^{\infty} t dF_{x,T^*{}_2y}(t) = 0$$

$$\int_{-\infty}^{\infty} t dF_{x,T_1^*y - T_2^*y}(t) = 0$$

$$F_{x,T_1^*y - T_2^*y} = 0\ iff\ , (T_1^* - T_2^*)y = 0\ \forall x \in E$$

$$T_1^* = T_2^*$$

So T^* is unique.

Theorem (2-2)

Let $(E, F, *)$ be Probabilistic Hilbert Space, let T^* be the adjoint operator of $T \in E$, then T^* has the following properties:

1. $(T^*)^* = T$

Proof:

$$< Tx, y > = \int_{-\infty}^{\infty} tdF_{x,T^*y}(t) =< x, T^*y >$$
$$= \int_{-\infty}^{\infty} tdF_{T^*y,x}(t) =< T^*y, x > = \int_{-\infty}^{\infty} tdF_{y,T^{**}x}(t) =< y, T^{**}x >$$
$$= \int_{-\infty}^{\infty} tdF_{T^{**}x,y}(t) =< T^{**}x, y >$$

$$< Tx, y > -< T^{**}x, y > = 0$$

$$< (T - T^{**})(x), y > = 0 \quad \forall x, y \in E$$

$$(T - T^{**})(x) = 0 \ \forall x \in E$$

So

$$(T^*)^* = T$$

2. $(\alpha T)^* = \alpha T^*$

Proof:

$$< \alpha Tx, y >= \int_{-\infty}^{\infty} tdF_{x,(\alpha T)^*y}(t) = < x, (\alpha T)^*y > \qquad (1)$$

$$< \alpha Tx, y >= \int_{-\infty}^{\infty} tdF_{\alpha Tx,y}(t)$$
$$= \alpha \int_{-\infty}^{\infty} tdF_{x,y}(t) = \alpha < Tx, y >= \alpha \int_{-\infty}^{\infty} tdF_{x,T^*y}(t) = \alpha < x, T^*y > \qquad (2)$$

By (1), (2)

$$< x, (\alpha T)^*y > = \alpha < x, T^*y >$$

$$< x, (\alpha T)^*y > -< x, \alpha T^*y >= 0$$

$$< x, (\alpha T)^* y - \alpha T^* y >= 0 \;\; \forall x, y \in E$$

$$< x, ((\alpha T)^* - \alpha T^*)(y) \geq 0$$

$$((\alpha T)^* - \alpha T^*)(y) = 0 \;\; \forall x \in E$$

$$(\alpha T)^* = \alpha T^*$$

3. $(T_1^* + T_2^*) = T_1^* + T_2^*$

Proof:

$$< (T_1 + T_2)x, y >= \int_{-\infty}^{\infty} t dF_{x,(T_1+T_2)^* y}(t) = < x, (T_1 + T_2)^* y > \qquad (1)$$

$$< (T_1 + T_2)x, y > = \int_{-\infty}^{\infty} t dF_{(T_1+T_2)x,y} = \int_{-\infty}^{\infty} t dF_{T_1 x,y} + \int_{-\infty}^{\infty} t dF_{T_2 x,y} = < T_1 x, y > + < T_2 x, y >$$

$$= \int_{-\infty}^{\infty} t dF_{x,T_1^* y} + \int_{-\infty}^{\infty} t dF_{x,T_2^* y} = \int_{-\infty}^{\infty} t dF_{x,(T_1^*+T_2^*)y}(t) =< x, (T_1^* + T_2^*)y > \qquad (2)$$

By (1), (2)

$$< x, (T_1 + T_2)^* y > =< x, (T_1^* + T_2^*)y >$$

$$< x, (T_1 + T_2)^* y > - < x, (T_1^* + T_2^*)y > = 0$$

$$< x, (T_1 + T_2)^* y - (T_1^* + T_2^*)y >= 0$$

$$< x, \left((T_1 + T_2)^* - (T_1^* + T_2^*)\right)(y) > = 0 \quad \forall x, y \in E$$

$$\left((T_1 + T_2)^* - (T_1^* + T_2^*)\right)(y) = 0 \;\; \forall y \in E$$

$$(T_1^* + T_2^*) = T_1^* + T_2^*$$

4. $(T_1 T_2)^* = T_2^* T_1^*$

$$< T_1 T_2 x, y >= \int_{-\infty}^{\infty} t dF_{(T_1 T_2)^* x,y} =< x, (T_1 T_2)^* y > \qquad (1)$$

$$< T_1 T_2 x, y >=< T_2 x, T_1^* y > = \int_{-\infty}^{\infty} t dF_{T_2 x, T_1^* y} = \int_{-\infty}^{\infty} t dF_{x,T_2^* T_1^* y} =< x, T_2^* T_1^* y > \qquad (2)$$

By (1), (2)

$$< x, (T_1T_2)^*y > =< x, T_2^*T_1^*y >$$

$$< x, (T_1T_2)^*y > -< x, T_2^*T_1^*y > = 0$$

$$< x, (T_1T_2)^*y - T_2^*T_1^*y > = 0$$

$$< x, ((T_1T_2)^* - T_2^*T_1^*)(y) > = 0 \ \ \forall x, y \in E$$

$$((T_1T_2)^* - T_2^*T_1^*)(y) = 0 \ \ \forall y \in E$$

$$(T_1T_2)^* = T_2^*T_1^*$$

Conclusion

In this paper we tried to introduce a reasonable definition of the adjoint operator in probabilistic Hilbert Space. This definition can be extend as the extension of probabilistic Hilbert space. Also we proved some properties of the adjoint operator.

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