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Conditional Expectation on Extended Positive Part of Crossed Product of a Von Neumann Algebra

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Abstract

We extend the notion of conditional expectation onto the set of generalised positive operators (the extended positive part) of crossed product of a von Neumann algebra.

Keywords: conditional expectation, generalised positive operators, crossed product

1.0 Introduction

The concept of conditional expectation in non-commutative theory of operator algebra was first studied by Umegaki [12]. Takesaki [10] gave the necessary and sufficient conditions for the existence of such expectation.The motivation for this paper is as a result of Lance [7] where the possibility of extending conditional expectation to crossed product of von Neumann algebras was shown. In this paper, we extend the notion of conditional expectation to the extended positive part of crossed product of von Neumann algebras defined in Haagerup [3].

2.0 Preliminaries:

We recall the notion and results on generalised positive operators as discussed by Haagerup [3]. We also recall the construction of crossed product of a von Neumann algebra as done by Vandaele [13].

Definition

A weight φ on a von Neumann algebra M is a function $\varphi : M_+ \to [0,\infty]$, such that

i. $\varphi(x+y) = \varphi(x) + \varphi(y)$, $x, y \in M_+$
ii. $\varphi(\lambda x) = \lambda\varphi(x)$, $x \in M_+$, $\lambda \geq 0$.

We say that φ is faithful if $\varphi(x^* x) = 0 \Rightarrow x = 0$. φ is normal if $\varphi(x) = \sup \varphi(x_i)$, with x as the limit of a bounded increasing net of operators $\{x_i\}_{i\in I}$ in M_+ and φ is semifinite if n_φ is σ - weakly dense in M . To any weight φ is associated a σ - weakly continuous one parameter group of $*$ automorphisms $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ on the von Neumann algebra M , called the modular automorphisms group defined by $\sigma_t(x) = \pi^{-1}(\Delta_\varphi^{it}\pi(x)\Delta_\varphi^{-it})$.

Definition

A generalised positive operator affiliated with a von Neumann algebra M is the set of maps $\hat{x} : M_*^+ \to [0,\infty]$ which is positively homogeneous, additive and lower semi continuous, where M_*^+ is the positive part of the predual $M_{\bullet}$ of M. The set of all such maps is called the extended positive part of M and is denoted by $M_{\bullet}^{\wedge}$. They are "weights " on the predual of the von Neumann algebra M .Each element $x \in M_{+}$ defines an element in $M_{+}^{\wedge}$ by $\varphi \to \varphi(x)$, $\varphi \in M_{\bullet}^{+}$, hence we can regard $M_{+} \subset M_{+}^{\wedge}$ [3].

Definition

Let $x, y \in M_{+}, a \in M$ and $\lambda \geq 0$ we define $x+y, \lambda x$ and $a^{*}xa$ by

(1) $(\lambda x)\phi = \lambda x(\phi)$, $\phi \in M_{+}^{*}$

(2) $(x+y)\phi = x(\phi) + y(\phi)$, $\phi \in M_{+}^{*}$

(3) $(a^{*}xa)(\phi) = x(a\phi a^{*})$, $\phi \in M_{+}^{*}$

Remark: $a\phi a^{*}(x) = \phi(a^{*}xa)$, $x \in M$,

hence we have $a\phi a^{*}(1) = \phi(a^{*}1a) = \phi(aa^{*})$, $1 \in M$ (**)

Definition

If $\hat{x} \in M_{+}^{\wedge}$, a weight on $M_{+}^{\wedge}$ is given by ;

$$\varphi(\hat{a}) = \lim_{m} \varphi(a_m) = \lim_{n,m} \tau(x_n \bullet a_m) = \lim_{n,m} \tau(x_n^{\frac{1}{2}} a_m x_n^{\frac{1}{2}}) , \quad a \in M_{+} \text{ [3]}$$

We have a theorem from Haagerup [3] which states the form of the spectral resolution for the generalised positive operator.

Theorem

L et M be a von Neumann algebra. Each element $x \in M_{+}^{\wedge}$ has a spectral resolution of the form

$\hat{x}(\varphi) = \int_{0}^{\infty} \lambda d\varphi(e_{\lambda}) + \infty(p)$, $\varphi \in M_{\bullet}^{+}$. Where $(e_{\lambda})_{\lambda \in [0,\infty]}$ is an increasing family of projections in M such that $\lambda \to e_{\lambda}$ is strongly continuous from right and $\lim_{\lambda \to \infty} = 1 - p$. [3]

Here we recall the construction of crossed product of von Neumann algebra by Vandaele[13]

Definition

Let M be a von Neumann algebra acting on Hilbert space H with a faithful normal state φ. We denote by $\overline{M} = R(M, \sigma_{t}^{\varphi})$, the crossed product of M by the modular automorphism group $\{\sigma_{\varphi}^{t} : t \in \mathbb{R}\}$ associated with φ. $\overline{M}$ is a semifinite von Neumann algebra acting on a Hilbert space $\overline{H} = L_2(\mathbb{R}, H)$

generated by the operators $(\pi(x)\xi)(s)=\sigma_{-s}(x)\xi(s)$ and $(\lambda(t)\xi)(s)=\xi(s-t)$, $x\in M$, $\xi\in C_c(\mathbb{R},H)$, $s,t\in\mathbb{R}$ such that $\tau\circ\theta_s(x)=e^{-s}\tau(x)$, where $C_c(\mathbb{R},H)$ is a complex vector space of H valued functions on $\mathbb{R}$ with compact support, τ is a faithful normal semifinite trace (f.n.s.) on M and θ_s is the dual action of $\mathbb{R}$ on $\bar{M}$. [13]

3.0 Conditional Expectation on Extended Positive Part of Crossed Product of a von Neumann Algebra.

Let M be a semfinite von Neumann algebra and N its von Neumann subalgebra. Then there exists a conditional expectation E from M onto N which is a projection of norm one [11]. The conditional expectation exists only when N is globally invariant under the modular automorphism group $(\sigma_t^{\varphi})_{t\in\mathbb{R}}$ associated with a faithful normal weight φ [9]. Goldstein [2] extends conditional expectation E to the extended positive part of von Neumann Algebras (denoted by E) and also to the crossed product of von Neumann algebra (denoted by $\overline{E}$). Here we define our conditional expectation on the extended positive part of the crossed product von Neumann algebra $\overline{\hat{M}}_+$. We denote the extended conditional expectation by $\overline{\hat{E}}$ and show that $\overline{\hat{E}}$ is invariant with respect to a given normal weight on $\overline{\hat{M}}_+$. To show the possibility of extending the conditional expectation onto the generalised positive operators of crossed product of a von Neumann algebra, we follow the same idea of [2].

We define our generalised positive operators affiliated with crossed product of a von Neumann algebra M

M using the same argument of [3].

Definition 3.1

A generalised positive operator $\overline{x}^{\wedge}$ affiliated with a crossed product of a von Neumann algebra M is the set of maps $\hat{x}:\overline{M}_*^+\to[0,\infty]$ which is positively homogeneous, additive and lower semi continuous, where $\overline{M}_*^+$ is the positive part of the predual $\overline{M}_*$ of $\overline{M}$. The set of all such maps is called the the extended positive of $\overline{M}$ and is denoted by $\overline{\hat{M}}_+$. They are "weights" on the predual of the crossed product $\overline{M}$. Each element $\overline{x}\in\overline{M}_+$ defines an element in $\overline{\hat{M}}_+$ by $\overline{\varphi}\to\overline{\varphi}(\overline{x})$, $\overline{\varphi}\in\overline{M}_+^{\wedge}$ hence we can regard $\overline{M}_+\subset\overline{\hat{M}}_+$.

Definition 3.2

Let $\hat{x}\in\overline{\hat{M}}_+$ we define a weight on $\overline{\hat{M}}_+$ by

$$\overline{\varphi}_{\hat{x}}(\overline{y})=\lim_{n,m}\overline{\tau}^{\wedge}(\overline{x}_m\bullet\overline{y}_n)=\lim_{n,m}\overline{\tau}^{\wedge}(\overline{x}_m^{\frac{1}{2}}\overline{y}_n\overline{x}_m^{\frac{1}{2}}),\quad \overline{y}\in\overline{M}_+$$

Remark: The generalised positive operators are added multiplied by scalars in a natural way.

We now define the crossed product operators of the extended positive part $\overline{M}^{\wedge}_{+}$ and the Hilbert space it act on using [13].

Definition 3.3

Let H be a Hilbert space. Denote by $C_c(R)$ the space of continuous functions with compact support on R. The tensor product $\hat{L}_2(R.ds)\otimes\hat{H}$ is spanned by elements of the form $f\otimes\hat{\xi}$ where $C_c(R)$, $\hat{\xi}\in\hat{H}$. The Hilbert space completion of the linear combination of such function is denoted by $\overline{\hat{H}}$. Let $\hat{M}$ be a von Neumann algebra acting in $\hat{H}$ and $\hat{\varphi}_0$ a fixed faithful normal state on $\hat{M}$. Denote by $\hat{\sigma}$ the modular automorphism group associated with $\hat{M}$ and $\hat{\varphi}_0$. The crossed product of $\hat{M}$ by $\hat{\sigma}$ is the von Neumann algebra $\overline{\hat{M}}$ acting on $\overline{\hat{H}}$ generated by $\pi(x)$, $x\in\hat{M}$ and $\hat{\lambda}(s), s\in R$ where

$$(\pi(x)\hat{\xi})(t)=\sigma_{-t}(x)\hat{\xi}(t)$$

$$(\hat{\lambda}(s)\hat{\xi})(t)=\hat{\xi}(t-s).$$

Let $\overline{\hat{M}_{\circ}}$ be the * algebra generated algebraically by operators $\pi(x)$, $x\in\hat{M}$ and $\hat{\lambda}(s), s\in R$. Then $\overline{\hat{M}}$ is the $\hat{\sigma}$ weak closure of $\overline{\hat{M}_{\circ}}$. Every element $\overline{x}\in\overline{\hat{M}}$ may be represented as $\hat{x}=\sum_k\hat{\lambda}(s_k)\hat{\pi}(\hat{x}_k)$ for some $s_1,.,.,s_k\in R,\hat{x}_1,.,.,\hat{x}_k\in\hat{M}$.

We show the possibility of extending $\overline{E}$ onto $\overline{M}^{\wedge}$ using the same argument of the proof in [2].

Theorem 1

The $\overline{E}$ restricted to M_+ extends uniquely to a map of $\overline{\hat{M}}$ onto $\overline{\hat{N}}$ which is positive, additive, order preserving , normal and satisfies $(\hat{E}\hat{x})(\phi)=\hat{x}(\phi\circ\overline{E})$.

Proof : Using Goldstein [2]

Let $\overline{x}\in\overline{\hat{M}}$, $\overline{x_n}\in\overline{M}_+$ and $\overline{x_n}\nearrow\overline{x}$ since $\overline{E}$ is positive, $\overline{E}\overline{x_n}\nearrow\overline{y}$ for some $\overline{y}\in\overline{N}_+$

Put $\hat{E}\hat{x}=\hat{y}$, if $\overline{z_n}\in\overline{M}_+$, $\overline{z_n}\nearrow\hat{x}$, then for each $\overline{\phi}\in\overline{M}_*$

$\lim_n\overline{\phi}(\overline{x_n})=\lim_n\overline{\phi}(\overline{z_n})$ i.e. $\overline{x_n}-\overline{z_n}\to 0$, σ - weakly

Where $\overline{E}\overline{x_n}-\overline{E}\overline{z_n}\to 0$, σ - weakly, and thus $\lim_n\overline{\phi}(\overline{x_n})-\lim_n\overline{\phi}(\overline{E}\overline{z_n}-\overline{E}\overline{z_n})\to 0$, σ - weakly

Implies $\overline{E}\overline{z}_n \nearrow \hat{\overline{E}}\hat{x}$

We have $(\hat{E}\hat{x})(\overline{\phi}) = \lim_n \overline{\phi}(\overline{E}\overline{x}_n) = \hat{x}(\overline{\phi} \circ \overline{E})$

Hence, $(\hat{E}\hat{x})(\overline{\phi}) = \hat{x}(\overline{\phi} \circ \overline{E})$. It is obvious that $\hat{E}$ is positive, additive and also normal.

To show that $\hat{E}$ is invariant with respect to a faithful normal weight we refer to the theorem

Theorem 2:

Let $\overline{M}$ be semifinite and $\overline{N}$ its subalgebra and $\overline{M}_+$ and $\overline{N}_+$ be their respective extended positive part such that $\overline{N}_+ \subset \overline{M}_+$, then $\overline{\varphi}_{\hat{x}} = \overline{\varphi}_{\hat{x}} \circ \hat{E}$

Proof:

Let $\hat{x} \in \hat{\overline{M}}_+$, we define a weight on $\hat{\overline{M}}_+$ by

$$\overline{\varphi}_{\hat{x}}(\hat{a}) = \lim_n \overline{\varphi}_{\hat{x}}(\overline{a}_n) = \lim_{m,n} \overline{\tau}(\overline{x}_m \cdot \overline{a}_n) = \lim_{m,n} \overline{\tau}(\overline{x}_m^{-1/2}\overline{a}_n \overline{x}_m^{-1/2}) \ , \ \hat{a} \in \hat{\overline{M}}_+$$

Let $\hat{E} : \hat{\overline{M}}_+ \to \hat{N}_+$ be our extended generalised conditional expectation on $\hat{\overline{M}}_+$ onto $\hat{\overline{N}}_+$,

If $\hat{a} \in \hat{\overline{M}}_+$, $\hat{x} \in \hat{N}_+$ then, $\overline{x} \nearrow \hat{x}$ and $\overline{a}_n \nearrow \hat{a}$ with $\overline{x}_m \in \overline{N}_+$ and $\overline{a}_n \in \overline{M}_+$

$$\overline{\varphi}_{\hat{x}}(\hat{a}) = \hat{\tau}(\hat{x} \cdot \hat{a}) = \lim_{m,n} \overline{\tau}(\overline{x}_m \cdot \overline{E}(\overline{a}_n)) = \lim_{m,n} \overline{\tau}(\overline{x}_m^{-1/2}\overline{E}(\overline{a}_n)\overline{x}_m^{-1/2})$$

$$= \lim_m \overline{\tau}(\overline{x}_m^{-1/2} \lim_n \overline{\phi}(\overline{E}\overline{a}_n)\overline{x}_m^{-1/2}) = \lim_m \overline{\tau}(\overline{x}_m^{-1/2}\hat{E}(\hat{a})\overline{x}_m^{-1/2})$$

$$= \lim_m \overline{\tau}(\hat{E}(\hat{a})(\overline{x}_m^{-1/2}\overline{\phi}\overline{x}_m^{-1/2}) = \lim_m \overline{\tau}(\hat{E}(\hat{a})(\overline{\phi}(\overline{x}_m^{-1/2} 1 \overline{x}_m^{-1/2})))$$

$$= \lim_m \overline{\tau}(\hat{E}(\hat{a})\overline{\phi}(\overline{x}_m)) = \overline{\tau}(\hat{E}(\hat{a}) \lim_m \overline{\phi}(\overline{x}_m)) = \hat{\tau}(\hat{E}(\hat{a})(\hat{x})$$

$$\overline{\varphi}_{\hat{x}}(\hat{a}) = \hat{\tau}(\hat{E}(\hat{a})(\hat{x}) = \overline{\varphi}_{\hat{x}}(\hat{E}\hat{a})$$

Hence $\overline{\varphi}_{\hat{x}} = \overline{\varphi}_{\hat{x}} \circ \hat{E}$

We used the relation in remark (**) and the assumption that the increasing sequences are densely defined on $\overline{M}_+$ and $\overline{N}_+$ to prove the above theorem.

Theorem:

If $\overline{\varphi}_{\hat{x}}$ is a weight on $\hat{\overline{M}}_{+}$ and $\hat{\overline{E}}$, $\overline{E}$ are the conditional expectations on $\hat{\overline{M}}_{+}$ and $\overline{\overline{M}}_{+}$ respectively then

$$\overline{\varphi}_{\hat{x}}(\hat{\overline{E}}\,\hat{\overline{a}}) = \overline{\varphi}_{\hat{x}}(\hat{a}(\overline{\phi}\circ\overline{E}))$$

Proof

$$\overline{\varphi}_{\hat{x}}(\hat{\overline{E}}\,\hat{\overline{a}}) = \overline{\varphi}_{\hat{x}}(\lim_n \overline{\phi}(\overline{\overline{E a_n}})) \quad \text{where } \overline{a}_n \in \overline{\overline{M}}_{+} \text{ and } \overline{a}_n \nearrow \hat{a}$$

$$= \overline{\tau}(\overline{x}_m^{-1/2} \lim_n \overline{\phi}(\overline{\overline{E a_n}})\overline{x}_m^{-1/2}) = \lim_n \overline{\tau}((\overline{x}_m^{-1/2}\overline{\phi}(\overline{\overline{E a_n}})\overline{x}_m^{-1/2})$$

$$= \overline{\tau}(\overline{x}_m^{-1/2} \lim_n \overline{\phi}(\overline{\overline{E a_n}})\overline{x}_m^{-1/2}) = \lim_n \overline{\tau}((\overline{x}_m^{-1/2}(\overline{\phi}\circ\overline{E})\overline{a}_n)\overline{x}_m^{-1/2})$$

$$= \overline{\tau}(\overline{x}_m^{-1/2}\,\hat{a}(\overline{\phi}\circ\overline{E})\overline{x}_m^{-1/2}) = \hat{\tau}(\hat{a}(\overline{x}_m^{-1/2}(\overline{\phi}\circ\overline{E})\overline{x}_m^{-1/2}))$$

$$= \hat{\tau}(\hat{a}(\overline{\phi}\circ\overline{E})\overline{x}_m) = \overline{\varphi}_{\hat{x}}(\hat{a}(\overline{\phi}\circ\overline{E}))$$

Hence $\overline{\varphi}_{\hat{x}}(\hat{\overline{E}}\,\hat{\overline{a}}) = \overline{\varphi}_{\hat{x}}(\hat{a}(\overline{\phi}\circ\overline{E}))$.

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