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Strongly Rickart Rings

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Abstract.

Let R be a ring with identity. In this paper we introduce a *strongly Rickart* ring as a stronger concept of a Rickart ring. A ring R is said to be strongly Rickart ring if the right annihilators of each single element in R is generated by a left semicentral idempotent in R. This class of rings is proper class in right Rickart rings, p.q.-Baer rings, reduced rings and semiprime rings. The relation between strongly Rickart and strongly regular are studied. We discuss some types of extension of strongly Rickart ring such as the Dorroh extension and the idealization ring.

Key words: strongly Rickart ring, Rickart ring, right annihilator element, semicentral idempotent element, Dorroh extension, idealization of a module.

0. Introduction

Throughout this paper all rings are associative with identity. Kaplansky in (1955) introduced a Baer ring R as the right annihilator of every non empty subset of a ring R is generated by an idempotent element[11]. In 1967, Clark introduced a quasi-Baer ring as the right annihilator of every two sided ideal of a ring R is generated by an idempotent element in R[20]. A. Hattori in (1960)[1] introduce p.p. (Rickart) ring as every principle ideal is projective which equivalent to the right annihilator of any single element in R is generated by an idempotent element[4, proposition 1.4]. G.F. Birkenmeier, J. Y. Kim, J.K. Park, in (2001) introduced a p.q.-Baer ring R as a generalization of a quasi-Baer ring[6]. A ring R is said to be p.q.-Baer if the right annihilator of every principle right ideal of a ring R is generated as an R-module by an idempotent element. Recently more than one author investigates in these types of rings and the relation of each other. Recall that a ring R is said to be Von Neumann regular if for each a∈R there is b∈ R such that aba=a[16]. Also a ring R is said to be strongly regular if $a^2 b=a$ for each a∈R and some b∈ R[16].

In this paper we introduce and study the concept of a strongly Rickart rings which are properly stronger than of Rickart ring and p.q-Baer rings.

Notations: For a ring R and a∈R the set $r_R(a) = \{r \in R: ar = 0\}$ (resp. $l_R(a) = \{ r \in R : ra = 0\}$) is said to be the right (resp. left) annihilator of an element a in R. An idempotent e∈R is called left (resp. right) semicentral if xe=exe (resp. ex=exe), for all x∈R. An idempotent e∈R is called central if it commute with each x ∈R. $S_\ell(R)$, $S_r(R)$ and B(R)($B(R) = S_\ell(R) \cap S_r(R)$) is the set of all left semicentral, right semicentral and central idempotents of R respectively. We will refer to the idealization of a module M by R(+)M.

1.Basic structure of strongly Rickart rings.

Its known that a right ideal I of a ring R is a direct summand if it's generated by an idempotent element e ∈ R[16]. The idempotent elements play an important role in the structure theory of rings. In this section we introduce the following concept as stronger than Rickart concept.

Definition 1.1. A ring R is said to be right *strongly Rickart* if the right annihilator of each single element in R is generated by left semicentral idempotent of R.

Examples and Remarks 1.2

1. Follows from [6, lemma 1.1] an idempotent e∈ $S_\ell(R)$ if and only if eR is an ideal of R. Hence a ring R is right strongly Rickart if and only if the right annihilator of each element a∈R is a direct summand ideal in R.

2. Every right strongly Rickart ring is right Rickart, but the converse is not true in general, see section (3).

3. From (1) a commutative ring R is strongly Rickart if and only if R is Rickart ring.

4. Every semisimple ring is strongly Rickart ring.

Proof Straightforward from the fact: a ring R is semisimple if and only if every ideal is generated by a central idempotent [16, 3.5, p.17]. ◘

5. Every right ideal of right strongly Rickart ring is right strongly Rickart.

Proof: Let I be a right ideal in R and a∈ I. Since R is right right strongly Rickart then $r_R(a)$= eR for some e^2= e∈ $S_\ell(R)$. Claim that $r_I(a)$= I∩ $r_R(a)$ =eI. Let y∈ eI. Then y= ei ∈ $r_R(a)$.Thus ay =0. Hence y∈$r_I(a)$ .Thus eI ⊆$r_I(a)$. Now let x ∈$r_I(a)$,then x ∈ $r_R(a)$ and so x= ex ∈ eI .Thus $r_I(a)$ ⊆eI. Hence I is right strongly Rickart. ◘

6. Any ring with no right (left) zero divisors is a right strongly Rickart ring.

H.E. Bell in [10] introduced the insertion factor property (simply IFP) as for each a∈R, ab=0 implies ba=0 and arb=0 for all r∈R. Its well known that if a ring R has IFP then $r_R(a)$ = $r_R(aR)$ for all a∈ R and R is an abelian ring (in sense every idempotent is central)[6]. An idempotent e∈ $S_\ell(R)$ if and only if eR is an ideal of R[6]. So we have the following results.

Proposition 1.3. Every indecomposable right Rickart ring R satisfies IFP.

Proof Let a∈ R. Since R is right Rickart ring then $r_R(a)$ = eR for some e^2=e ∈ R. But R is indecomposable, then either $r_R(a)$ = 0 or $r_R(a)$ = R. In the two cases $r_R(a)$ is two sided ideal in R. ◘

Proposition 1.4. Every right strongly Rickart ring satisfy IFP (and hence is an abelian ring).

It is well known that Baer rings [11] and quasi-Baer rings [6] are left –right symmetric while right Rickart rings is not. In fact, Chase shown in his example the ring A=$\begin{pmatrix} T & T \\ 0 & R \end{pmatrix}$ is left semihereditary ring and hence left Rickart where R is Von Neumann regular ring and T=$\frac{R}{I}$ for a not direct summand ideal I of R(as a submodule of R_R). But A is not right semihereditary, where there is a principle right ideal L =$\begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}$ =$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$A ⊆A not projective as A-module and hence is not right Rickart [19].

The following theorem shows that a right strongly Rickart is left–right symmetry.

Theorem 1.5. A ring R is right strongly Rickart if and only if R is an abelian right Rickart ring.

Proof .⇒) Clear that R is right Rickart ring(remark and example 1.2(2)) and from proposition (1.4) R has IFP and hence R is an abelian.

⇐) Since R is abelian then every idempotent in R is central. Hence if R is right Rickart ring then clear that R is right strongly Rickart. ◘

Corollary1.6. A strongly Rickart ring is left–right symmetry.

Now from the previous corollary we are not needed to write left or right strongly Rickart ring.

Recall that a ring R is said to be directly finite if when ab=1 then ba=1. Hence we have the following result for strongly Rickart.

Corollary 1.7. Every strongly Rickart is directly finite.

Proof. Suppose that ab=1. It's clear that $(ab)^2$ =1=ab. Thus give us ab is an idempotent element in R and hence is central. Then $(ab)^2$=abab =1. So, abab=a(ab)b=a^2b^2=1. Again (aabb)=1 gives (aabb)a= a and so ab(ab)a = aba(ab) = a^2b(ba)a. Hence a(ab)(ba) =a(ba) =a. Thus ba(ba)=ba . Hence $(ba)^2$= ba. Thus ba is a central idempotent in R. Since a^2b(ba) =(ba)ab=a. Finely, (ba)a^2b=a gives baa^2 b^2=ab. Hence ba=1. ◘

Recall that from [15] a ring R is said to be reduced if R has no nonzero nilpotent elements. It's known that every reduced ring is an abelian ring, but the converse is not true in general. In the ring $R=Z_3\oplus Z_3$ where (a, b)$_*$(c, d) =(ac, ad +bc) and addition is compointwise. R is commutative and hence abelian ring. I has identity (1, 0). But (0, 1) is a nonzero nilpotent element in R. So R is not reduced. In fact these conditions are equivalent under right Rickart ring [15].Clear that the Z_4 ring (the ring of integers modulo 4) is abelian but not reduced ring, where it has a nonzero nilpotent element $a=\overline{2}$. A ring R is said to be semiprime if has no nonzero nilpotent ideal [4], equivalently if aRa = 0 then a=0. It's well known that every reduced ring is semiprime and they are equivalently in a commutative ring. The 2x2 matrices over a ring Z of integers is semiprime which is not reduced

One can conclude that reduced rings is a generalization to a strongly Rickart rings.

Proposition1.8. Every strongly Rickart ring is reduced (and hence is a semiprime) ring.

Corollary 1.9. A ring R is strongly Rickart if and only if R is reduced (semiprime) right Rickart ring.

Proof. ⇒) Follows from proposition (1.8) and examples and remarks (1.2).

⇐ Since every reduced and semiprime is an abelian, then from proposition (1.5) the proof holds. ◘

Note that right Rickart and reduced (abelian) are different concepts as the following examples.

Examples 1.10.

1.The Z_4 ring is an abelian which is not Rickart.

2. The 2x2 upper triangular matrix $S=\begin{pmatrix} Z/2Z & Z/2Z \\ 0 & Z/2Z \end{pmatrix}$ is left Rickart ring not abelian (and hence not reduced) ring where there is an idempotent $a=\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ in R and $y=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in R then $ay \neq ya$. So a is not central idempotent in S.

Recall that a ring R is symmetric if and only if when abc =0 then acb=0 for all a, b, c in R) while Rickart ring are not. It's well known that every reduced is symmetric [8] hence we have the following proposition.

Proposition 1.11. Every Strongly Rickart ring is symmetric.

Recall that a left ideal A of R is reflexive if $xRy \subseteq A$, then $yRx \subseteq A$ for all x, y∈R. A ring R is reflexive if 0 is a reflexive ideal. A ring R is said to be idempotent reflexive if and only if xRe =0 implies eRx = 0 for all x and $e^2= e \in R$ [12]. A ring R is reflexive if and only if $r_R(xR) = l_R(Rx)$ for all x∈ R[12]. Every abelian ring is an idempotent reflexive [12](i.e. $r_R(eR)= \ell_R(Re)$ for all $e^2= e \in R$) .

Proposition1.12. Every strongly Rickart ring is reflexive.

Proof : Suppose that R is right strongly Rickart ring then $r_R(aR)=r_R(a) = eR = Re = \ell_R(a)= \ell_R(aR)$. ◘

Recall that a left ideal I of a ring R is said to be GW-ideal (generalized weak ideal), if for all x∈I, $x^nR \subseteq I$ for some n >0 [18]

Proposition 1.13. Let R be a strongly Rickart ring .Then for every element a∈R, $r_R(a)$ and $\ell_R(a)$ are GW-ideals.

Proof Let a∈R, then $r_R(a)=eR$ for some $e^2=e\in S_\ell(R)$. If $x\in r_R(a)$, then ax= axR = 0.Thus $xR \subseteq r_R(a)$. So for n=1, $r_R(a)$ is GW- ideal. In the same way, one can show that $\ell_R(a)$ is a GW-ideal. ◘

Before we give another characterization for a strongly Rickart ring we needed to give the following lemma which appears in [18, proposition 2.1].

Lemma1.14. For a ring R the following conditions are equivalent

1. R is an abelian.

2. $r_R(e)$ is a GW-ideal for all idempotent e ∈R.

3. $\ell_R(e)$ is a GW-ideal for all idempotent e ∈R.

Proposition 1.15. The following statements are equivalent for a ring R.

1. R is a strongly Rickart.

2. R is a right Rickart with $r_R(a)$) is a GW-ideal for all idempotent e ∈R.

3. R is a left Rickart with $\ell_R(a)$ is a GW-ideal for all idempotent e ∈R.

Proposition 1.16. An isomorphic image of strongly Rickart ring is strongly Rickart.

Proof Let f: R →S be a ring isomorphism and R be a strongly Rickart ring. Let x ∈S. Since f is an epimorphism, then x = f (a) for some a∈ R. So $r_R(a)$ =cR for some c^2= c∈ S_ℓ (R). We claim that $r_S(x)$ =eS, where e=f(c). First we needed to show that $f(r_R(a)) \subseteq r_S(x)$. For that let y =f(n) ∈ $f(r_R(a))$, where an =0 and n=cn. Now, xy=f(a) f(n) =f(an)=f(acn) = f(can)= f(0)= 0 since R is an abelian. Hence y ∈ $r_S(x)$. But f ($r_R(a)$) = f (cR)= f (c) f (R) =f (c)S = eS . Hence eS ⊆ $r_S(x)$, one can show that e^2 =e ∈ S_ℓ(S). Then eS = f($r_R(a)$) ⊆r_S(f(a)) .Now, let n∈ $r_S(x)$ .So 0 =xn =f(a) f(r) =f(ar) for some r∈R such that n=f(r). But f is monomorphism and f (ar) =0, so ar =0. Hence r ∈ $r_R(a)$ =cR. Thus r=cr and this implies n=f (r) =f (cr) =f(c)f(r) ∈eS. Therefore $r_S(x)$ = eS for some e^2 =e ∈ S_ℓ(S). ◘

Remark 1.17. A homomorphic image of strongly Rickart needed not be strongly Rickart .For example: Let π: $Z \to \frac{Z}{4Z}$ be the nutral epimorphism . Z is strongly Rickart ring where Z is abelian Rickart ring. But $\frac{Z}{4Z} \approx Z_4$ is not strongly Rickart where $r_{Z_4}(\bar{2}) = \{\bar{0}, \bar{2}\} \nleq^{\oplus} Z_4$.

A module M has the summand sum property (SSP) if the sum of two summands is again summand of M[13]. A ring R is said to be has SSP if R_R has SSP property as a right R- module[14]. A module M has the summand intersection property (SIP) if the intersection of two summands is again summand of M. A ring R is said to be has SIP if R_R has SIP property as a right R- module. The following proposition assert that strongly Rickart rings have these two properties.

Proposition 1.18. Let R be a strongly Rickart ring .Then

1. R has the SSP.

2. R has the SIP.

Proof

1. Let A and B be direct summands right ideals in R. Let α and φ be endomorphisms of R with ker α=A and ker φ =B. Then A= ker α= eR and B=ker φ = fR for e and f ∈ S_ℓ(R). Hence A+B = eR + fR= (e+f-ef) R for some idempotent e and f of R. Now it's well known that $(e+f-ef)^2$= (e+f-ef) ∈ S_ℓ(R). Thus eR + fR is a direct summand of R.

2. In the same way of (1), let A=eR and B =fR be direct summands right ideals in R for some e and f ∈ S_ℓ(R). Then A∩B=eR ∩fR= efR. Since $(ef)^2$ =ef ∈ S_ℓ(R). Then A ∩B is direct summand in R. ◘

We don't know whether the subring of strongly Rickart ring is strongly Rickart ring. The following proposition gives a condition which make that is true.

Proposition 1.19. Let R be a strongly Rickart ring and B be subring of R containing all idempotent elements in R. Then B is a strongly Rickart ring.

Proof. Let B be a subring of a strongly Rickart ring R which containing all idempotent in R. Let x∈B. Then r_R (x)=eR for some e^2 =e∈ S_ℓ(R). But e∈B. Then $r_B(x)$= eR∩ B= eB. Hence B is strongly Rickart. ◘

In [11]Kaplansky shows that the center of Baer(resp.Rickart) ring is Baer(resp. Rickart). So, center of Baer ring is Rickart. It's well known that the center of Rickart ring is Rickart. For a Strongly Rickart ring we assert that this property is valid for Strongly Rickart ring.

Proposition 1.20. Center of strongly Rickart ring is strongly Rickart.

Corollary 1.21. Center of right Rickart (and hence Baer) ring is strongly Rickart.

G. Lee in[9, Remark2.3.11] give a result: If R is right Rickart ring then R is indecomposable if and only if R is domain. For strongly Rickart rings we have the following result .

Proposition 1.22. A strongly Rickart ring is indecomposable if and only if its center is a domain.

Proof ⇒) Suppose that a ring R is indecomposable. Since R is strongly Rickart ring so center of R is strongly Rickart (proposition 1.20). But R is abelian ring, and then the set of all idempotent in R and the set of all idempotent in center of R are coinciding. So center of R is indecomposable .Therefore center of R is a domain.

⇐) Conversely, since center R is domain, so center R is indecomposable Rickart ring[9, Remark2.3.11]. But R is Strongly Rickart ring and hence is abelian. Then R is an indecomposable. ◘

Corollary 1.23. A strongly Rickart is indecomposable if and only if its center is indecomposable .

The indecomposable condition in proposition(1.22) and corollary 1.23 cannot be dropped as the following example.

Example 1.24 Consider the ring R=Z_6. R is strongly Rickart but not indecomposable. Hence center of Z_6 is strongly Rickart which is not domain where $\overline{2}$ and $\overline{3}$ are nonzero divisor element in R.

From [11] if R is Baer ring (resp. right Rickart[15]) then so is eRe for $e^2=e \in$R.

Proposition 1.25. If R is a strongly Rickart ring then so is eRe for each idempotent e∈ R.

Proof. Since e is an idempotent in R, and then e is a central. But eRe is Rickart [15,proposition 2.3], hence eRe is strongly Rickart. ◘

Corollary1.26. If R is a strongly Rickart ring, then eR and (1-e)R are strongly Rickart module for each idempotent e ∈ R.

Corollary1.27. A direct summand of strongly Rickart ring is strongly Rickart.

Proof Let J be a direct summand of a strongly Rickart ring R. Hence J =eR for some $e^2 = e \in$ R. So by corollary (1.26) J is strongly Rickart. ◘

Baer rings in[4],quasi-Baer rings[6] and right Rickart rings[4] are closed under the direct product .By using the same tetchiness proof use, we show that this property is valid for strongly Rickart.

Proposition1.28. Direct product of Strongly Rickart rings is Strongly Rickart ring.

Proof Let a=(a_α)∈R= $\prod_{\alpha\in I} R_\alpha$. If each R_α is strongly Rickart ring, then $r_{R_\alpha}(a_\alpha) = e_\alpha R_\alpha$ for some e_α ∈R. Let b=(b_γ)∈ $r_{R_\alpha}(a_\alpha)$. Hence ab=0 if and only if $a_\alpha b_\gamma$ =0 for all α, γ∈ I. It's clear that $b_\gamma=e_\alpha b_\gamma$ and $a_\alpha e_\alpha$= 0. Now, if we Put e=(e_α) then eb =(e_α)(b_γ)= ($e_\alpha b_\gamma$) = (b_γ)= b. It is easy to see that e= (e_α) $\in S_\ell(R_\alpha)$. Also, ae= (a_α)(e_α)= ($a_\alpha e_\alpha$)=0. Thus give us r_R(a)= eR for some e∈S_ℓ(R) and so R is strongly Rickart ring.

Conversely, suppose that R= $\prod_{\alpha\in I} R_\alpha$ is strongly Rickart ring and $a_\alpha \in R_\alpha$ .Then each of a ∈ R is a_α in α^{th} component and 1 other wise. Hence r_R(a) =eR for some e ∈S_ℓ(R). Then e = (e_α) is also e_α in α^{th} component and 1 otherwise. So $r_{R_\alpha}(a_\alpha)= e_\alpha R_\alpha$, for some $e_\alpha \in S_\ell(R_\alpha)$. Therefore R_α is strongly Rickart ring. ◘

We end this section by summarize all the previous concepts in the following proposition.

Proposition 1.29. A ring R is strongly Rickart if and only if

1.R is a right Rickart and reduced

2. R is a right Rickart and has IFP

3. R is a right Rickart and abelian .

4. R is a right Rickart and reflexive.

5. R is a right Rickart and symmetric.

6. R is a right Rickart with r_R(a)) is a GW-ideal for all idempotent e ∈R.

7. R is a left Rickart with ℓ_R(a) is a GW-ideal for all idempotent e ∈R.

2. Strongly Rickart and p.q.-Baer rings

In this part we will discusses the relation between the two concepts strongly Rickart rings and p.q.-Baer rings. It's well known that the concept of Rickart and p.q.-Baer are different of each other's: Conceder the ring R= $\begin{pmatrix} Z & Z \\ 0 & Z \end{pmatrix}$ where Z is the ring of integers. Then R is right p.q.-Baer ring which is neither left nor right Rickart ring [6, example 1.3]. Berkenmaier in [6, example 1.6] shows that there is a Rickart ring which is not p.q.-Baer ring.

Firstly, we assert that the class of strongly Rickart rings is contained in the class of p.q.-Baer ring.

Proposition 2.1. Every strongly Rickart ring R is right (left) p.q –Baer ring.

Proof. Assume that R is strongly Rickart and aR is a principle right ideal in R for a∈R. So, we have $r_R(a)$= eR for some e^2 =e ∈ $S_\ell(R)$. Then from proposition 1.4 R has IFP, and hence $r_R(a)$ =$r_R(aR)$= eR for e^2 =e ∈R. Thus R is right p.q –Baer. ◘

The converse of the proposition (2.1) is not true in general see the previous example.

The following proposition gives a characterization of strongly Rickart rings using p.q.-Baer rings.

Proposition 2.2. A ring R is Strongly Rickart if and only if R is right p. q.–Baer with IFP.

Proof. ⟹) Clear from the proposition (2.1) and proposition (1.4).

⟸) If R is a p.q.-Baer ring and satisfy IFP, then R is Rickart and abelian. [6,corollary 1.15]. So from proposition (1.5) R is strongly Rickart ring. ◘

Following [6, exampale1.5] there is an example of strongly Rickart ring which is not quasi-Baer ring. Using the IFP concept we can give a relation between these two concepts.

Corollary 2.3. If R is a Quasi-Baer ring satisfies the IFP then R is strongly Rickart.

Proof. Since R has IFP then $r_R(a)$= $r_R(Ra)$ =$r_R(RaR)$ where R has IFP. Since R is quasi-Baer ring then $r_R(a)$ =$r_R(RaR)$ = eR for some e^2=e ∈ R. But R is abelian (where R has IFP), then e is central idempotent and hence is left semicentral idempotent in R.

Birkenmeier, Kim and Park proved in [6, proposition(1.9)] the following lemma.

Lemma 2.4. A ring R is right p.q.-Baer if and only if whenever I is a principal ideal of R there exists e^2=e ∈ $S_r(R)$ such that I ⊆ Re and $r_R(I)$∩ Re = (1-e)Re.

Here we give analogous result to lemma (2.4) for a strongly Rickart ring.

Proposition 2.5. A ring R is strongly Rickart if and only if $r_R(a)$∩Re =(1-e)Re for some e^2=e∈ $S_r(R)$ and Ra ≤Re.

Proof. ⇒) Let a∈ R. So $r_R(a)$ =fR for f^2= f ∈ $S_\ell(R)$. It follows a∈ $\ell_R(fR)$ = R(1-f). Put e=1-f ∈ $S_r(R)$. $r_R(a)$∩Re = (1-e)R ∩ Re =(1-e)Re. For the other condition, Ra ⊆ $\ell_R(r_R(Ra))$ = $\ell_R(r_R(a))$ = $\ell_R(fR)$=R(1-f)= Re. So, Ra ⊆ Re.

⇐) Let a∈ R. Then by hypothesis ∃ e ∈ $S_r(R)$ such that $r_R(a)$ ∩Re =(1-e) Re. Clear that (1-e) R⊆ $r_R(a)$,for that since Ra ⊆ Re, so $r_R(Ra)$= $r_R(a)$ ⊇ $r_R(Re)$ = (1-e)R. Hence (1-e)R ⊆ $r_R(a)$. Now, let x∈$r_R(a)$. Then xe=exe +(1-e)xe ∈ $r_R(a)$∩Re =(1-e)Re. Since exe = ex, where e ∈ $S_r(R)$ and xe ∈ (1-e)Re. Then exe ∈ e(1-e)Re =0. So, ex =0. Thus x ∈$l_R(e)$ = (1-e)R. Hence $r_R(a)$ =(1-e)R. Now, since e^2=e∈ $S_r(R)$ then $(1-e)^2$=(1-e)∈ $S_\ell(R)$ i.e. $r_R(a)$ is generated by left semicentral idempotent (1-e) ∈ R. Hence R is strongly Rickart ring. ◘

Corollary 2.6. The following statements are equivalent for a ring R.

1. For all principle ideal I of a ring R there exists e^2= e∈ $S_r(R)$ such that I ⊆ Re and $r_R(I)$∩Re =(1-e)Re with IFP condition.

2. For all a ∈R, Ra ⊆Re and $r_R(a)$∩Re =(1-e)Re for some e^2= e∈ $S_r(R)$.

Proof. 1⇒2) Let a∈R then $r_R(a)$= $r_R(Ra)$. But R has IFP, so $r_R(a)$=$r_R(RaR)$

and hence $r_R(a)$ ∩ Re = (1-e)Re for some e^2= e ∈ $S_r(R)$. Clear that Ra ⊆Re.

2⇒1) Let I be a principal ideal in a ring R. So there is an element a∈R such that I=RaR ⊆ ReR. But e^2= e∈ $S_r(R)$, so Re is an ideal in R[6]. Hence I ⊆ Re. Now, $r_R(I)$ ∩ Re ⊇ $r_R(Re)$ ∩ Re =(1-e)R ∩ Re = (1-e)Re. It's clear that $r_R(I)$ ∩ Re ⊆ (1-e)Re. That gives $r_R(I)$ ∩Re =(1-e)Re. ◘

3. Strongly regular ring and strongly Rickart ring.

We continue the study of strongly Rickart rings. In this section we investigate the relation between strongly Rickart rings and strongly regular rings. It's well known that every regular ring is Rickart ring[4]. Note that regular property is not sufficient to strongly Rickart property for a ring R. In fact regular and strongly Rickart concepts are different. The ring of integers Z is strongly Rickart ring which is not regular (where if one take x=2 ∈ Z. ∄ y∈Z such that 2= 2y2). Berkenmaier in [6, example 1.6] proved that if

$$R=\begin{pmatrix} \Pi_{n=1}^{\infty}\mathrm{Fn} & \oplus_{n=1}^{\infty}\mathrm{Fn} \\ \oplus_{n=1}^{\infty}\mathrm{Fn} & \langle \oplus_{n=1}^{\infty}\mathrm{Fn}, 1\rangle \end{pmatrix}$$

where F field $F=F_n$ and $\langle \oplus_{n=1}^{\infty}\mathrm{Fn}, 1\rangle$is the F-algebra generated by $\oplus_{n=1}^{\infty}\mathrm{Fn}$ and 1. Then R is a regular ring neither left nor right p.q.-Baer ring and hence not strongly Rickart.

The following proposition gives a characterization for strongly regular ring using strongly Rickart ring. Before that let us recall the following lemma which appears in [16].

Lemma 3.1. [1, 3.11, p.21]. For a ring R with unit, the following properties are equivalent:

a. R is strongly regular

b. R is regular and contains no nonzero nilpotent elements.

c. Every left (right) principle ideal is generated by central idempotent.

d. R is regular and every left (right) ideal is an ideal.

Firstly, we prove that the class of strongly regular rings is contained in the class of strongly Rickart rings.

Proposition 3.2. Every strongly regular ring is strongly Rickart rings.

Proof. Suppose that R is strongly regular ring and let a∈ R. Then R is a regular ring and so is Rickart ring [9,corollary (2.2.21)]. Hence, $r_R(a)$ =eR for some e^2 =e∈ R. From lemma (3.1) (c), $r_R(a)$ =eR for some e∈B(R). Hence R is strongly Rickart ring. ◘

Recall that a ring R is said to be right SS-ring if and only if every right ideal of R generated by idempotent element is stable and hence every right ideal of R generated by idempotent element is fully invariant [17]. A ring R is said to be a right C_2 if every right ideal of R isomorphic to a direct summand eR where e^2 = e ∈ R is direct summand right ideal in R[21]. Also in [9], G. Lee proved that a ring R is a (right) Rickart ring satisfying C_2 condition as a right R-module if and only if R is a Von Neumann regular ring.

Proposition 3.3. A ring R is strongly Rickart right C_2 ring if and only if R is Von Neumann regular right SS-ring.

Proof ⇐) From [9,corollary (2.2.21)] every Von Neumann regular ring is right Rickart right C_2- ring. But R is right SS-ring so is strongly Rickart ring.

⇒) From [9,corollary (2.2.21)] R is regular ring. Now, to show that R is right SS-ring let I = eR for e^2=e ∈ R be a direct summand right ideal in R. Then there is an endomorphism φ of R with ker φ=I. But R is strongly Rickart ring hence I is a two sided ideal in R. So R is a right SS-ring. ◘

Now we available to give the main two results in this section

Theorem 3.4. The following statements are equivalents for a ring R:

1. Strongly regular.

2. Strongly Rickart right C_2.

3. Von Neumann regular right SS-ring.

4. Right Rickart right C_2 right SS-ring.

Proof. 1⇒2) From proposition (3.2) and the fact that the condition C_2 is always contained in a regular ring [9,corollary (2.2.21)]

2⇒1) From corollary (1.9) R is a reduced Rickart ring. But R is right C_2 ring, then R is regular ring [9, corollary (2.2.21)]. Hence from lemma (3.1) R is a strongly Regular.

2⇔3) From proposition (3.3).

3⇔4) [9,corollary (2.2.21)] ◘

Before we give another relation between strongly Rickart and strongly regular, let us recall the following lemma which appear in [7].

Lemma 3.5. [7,corollary2.6]. Let R be a ring satisfying $S_\ell(R)= B(R)$. If there is an injective right ideal eR with $e^2 =e\in$ R and $(1-e)R_R$ is semisimple, then R is right self-injective.

Before the following proposition let us recall that a ring R is strongly regular if and only if R is reduced right p-injective[13,theorem 47, p.22].

Proposition 3.6. For a ring R the following statements are equivalent

1. R is right self–injective strongly regular ring.

2. R is strongly Rickart ring can be decomposed into an injective right ideal eR and a semisimple right ideal $(1-e)R_R$ for $e^2= e\in$ R.

Proof. (1⇒2) Follows from [7, corollary(2.7)] and proposition 3.2.

(2⇒1) Since R is strongly Rickart then R is semiprime and hence $S_\ell(R)= B(R)$. So by hypothesis and lemma (3.5), R is self injective and hence p-injective. But R is strongly Rickart, and then R is reduced. Therefore R is strongly regular. ◘

G.F .Birkenmeier, D.R Huynh, J.Y. Kim and J.K. parle give in [7] a ring R=$\begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ Such that F a field. If e=$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ ∈R. Then eR is an injective as a right R–module and $(1-e)R_R$ is semisimple . But R is not p-injective and hence not self-injective.

4. Some types of extensions of strongly Rickart ring.

The polynomial extension of a Rickart ring R have been studied by Armendariz in [3] and he proved that this property is not inherited to the polynomial rings R[X] unless R is reduced ring. In this section we study the relationship between strongly Rickart ring R and the polynomial extension of it. Also we study special types of extensions of strongly Rickart rings as Dorroh extension and the idealization of a module M on R.

Firstly, we prove the polynomial ring R[X] is strongly Rickart if and only if the ring R is strongly Rickart.

Proposition 4.1. A ring R is a strongly Rickart if and only if R[X] is a strongly Rickart ring.

Proof. Suppose that R is a strongly Rickart ring and $f(x) = a_0+ a_1x+\ldots+ a_nx^n \in R[X]$. Then for each i = 0,1,…,n, $a_i \in R$. Since R is strongly Rickart then $r_R(a_i) =e_iR$ for all i=0,1,…n and $e_i^2= e_i\in S_\ell(R)$. Hence there is a left semicentral idempotent $e = \sum_{i=0}^{n} e_i - e_0e_1\ldots e_n \in R$ such that $r_R(\{a_0, a_1,\ldots, a_n\})= eR$. Now by [3, corollary2], $r_{R[X]}= r_R(\{a_0,a_1,\ldots,a_n\})[X]= eR[X]$. Hence R[X] is a strongly Rickart ring.

Conversely, if R[X] is a strongly Rickart ring and a ∈R, then $r_{R[X]}(a) = e(x) R[X]$ for some $e(x)^2 = e(x) \in S_\ell(R[X])$. Hence it's well known that $e_0\in S_\ell(R)$ such that $e(x)R[X] = e_0 R[X]$ where e_0 is the constant term of e(x)[5]. Then $r_R(a) = r_{R[X]}(a) \cap R = e_0 R[X] \cap R = e_0 R$ and so R is a strongly Rickart ring. ◘

Let R be any ring and Z be the ring of integer's numbers. Then Dorroh extension of R by Z (simply D(R, Z)) can be defined by D(R, Z) = {(r, n)| r∈R, n∈Z}. D(R,Z) is a ring with the addition defined as $(r_1,n_1) +(r_2,n_2) = (r_1+r_2 , n_1+n_2)$ and multiplication $(r_1,n_1) .(r_2 ,n_2) =(r_1r_2+ r_1n_2+ r_2n_1, n_1n_2)$ where $r_i\in R$, $n_i \in Z$ for i= 1,2[8].

A number of researchers studied these types of extensions of some known rings see [8] and [18]. Here we studied these types on Rickart rings and then translate them to strongly Rickart rings.

Before we give the first essentially result of this section we need to give the following facts

Lemma 4.2. Let (e, n) be an idempotent of the Dorroh extension D(R, Z). Then there are two types of idempotent in D(R, Z):

1. (e,0) is an idempotent in D(R,Z) if and only if e is an idempotent in R.

2. (e,1) is an idempotent in D(R,Z) if and only if e +1is an idempotent in R where 1is used an integer 1and as an identity 1 of R if there is no confusion.

Proof. Let (e, n) be an idempotent in D(R,Z) .Then $(e, n)^2 = (e^2+2ne, n^2) = (e, n)$ .We get $n^2 = n$ and $e^2+ 2ne =e$.

Case I. n =0. Then $e^2 =e$. Hence (e, 0) is an idempotent in D(R,Z).

Case II n=1. Then $e^2+2e =e$. So $(e+1)^2= e^2+ 2e+1 = e + 1$ is an idempotent in R and $(e,1)^2= (e^2+ 2e,1) = (e,1)$is an idempotent in D (R,Z).The rest is clear and this completes the proof . ◘

Lemma 4.3. Let R be a ring and D=D(R,Z) be the Dorroh extension from lemma (4.2) there are two types of left(right) semicentral idempotent in D then the following holds.

1. (e,0) is a left(right) semicentral idempotent in D if and only if e is a left(right) semicentral idempotent in R.

2. (e,1) is a left(right) semicentral idempotent in D if and only if e+1is a left(right) semicentral idempotent in R where 1is used an integer 1and as an identity 1 of R if there is no confusion.

3. (e,0) is a central idempotent in D if and only if e is a central idempotent in R.

4. (e,1) is a central idempotent in D if and only if e+1is a central idempotent in R where 1is used an integer 1and as an identity 1 of R if there is no confusion.

Proof.

1. Let $(e,0)^2 = (e, 0) \in S_\ell(D)$. Then by lemma (4, 2), $e^2=e \in R$. Let $x \in R$. Then (xe,0) = (x,0) (e,0) =(e,0)(x,0)(e,0)=(exe,0). Hence xe = exe. So $e \in S_\ell(R)$.

Conversely, let $e^2=e \in S_\ell(R)$. Then by lemma (4.2), $(e,0)^2 = (e, 0) \in D$. Let $(x, n) \in D$. Then (e, 0) (x, n) (e, 0) = (e,0) (xe+ 2ne, 0) = $(exe+2ne^2,0)$ = (xe+2ne,0) =(x, n) (e, 0). Hence (x, n) (e, 0) = (e, 0) (x, n) (e, 0) and so $(e, 0) \in S_\ell(D)$.

2. Let (e, 1) is an idempotent element in $S_\ell(D)$. Then e+1 is an idempotent element in R (lemma (4.2)). Let $x \in R$, then (x(e+1),0) = (x,0) (e+1,0) = (x, 0) (e,1) = (e, 1) (x,0) (e, 1) = (e,1) (x(e+1),0) = (ex(e+1)+x(e+1),0) = ((e+1)x(e+1),0). Hence x(e+1) = (e+1)x (e+1). So $(e+1)\in S_\ell(R)$.

Conversely, let $(e+1)^2= e+1 \in S_\ell(R)$, then from lemma (4.2) (e, 1) is an idempotent element in D. Let $(x, n) \in D$, then (e, 1) (x, n) (e, 1) = (e, 1) (xe+ x+ en, n) =(e,1) (x(e+1) +en, n) = $(ex(e+1) +e^2n + en + en + x(e+1), n)$ = $(e+1)x(e+1)+(e^2+2e)n$, n) = ((e+1)x(e+1)+en, n) = (x(e+1)+en, n) = (xe+ x.1 + en, n) = =(x,n) (e,1) where $e^2 + 2e= e$. Hence $(e, 1) \in S_\ell(D)$.

3. Follows from(1).

4. Follows from (2)

I n the same way one can prove the lemma (4.3) for a right semicentral idempotent element. ◘

Lemma 4.4. Let $(a,0) \in D(R,Z)$ =D. If $r_R(a) =eR$ for some $e^2= e\in R$. Then $r_D(a,0)= (e -1,1)D$.

Proof : Let $(b, m)\in r_D(a,0)$.Then ab +am=0 or e(b+ m)= b+ m equivalently, (1-e)(b+m)=0 where m denotes m1 for short. Hence $r_D(a,0) =\{(b,m)|(1-e) (b+ m) =0\}$. Claim: $r_D (a,0)=r_D(1-e,0)$.

For that, let $(b, m) \in r_D(a,0)$. Then a(b+m)=0. Since $r_R(a)= eR$, so e(b+ m) =b+ m or (1-e)(b+ m) = 0. Hence $(b, m)\in r_D(1-e, 0)$, so $r_D(a,0) \subseteq r_D((1-e),0)$. For the converse, let $(b, m) \in r_D((1-e), 0)$. Then (1-e) (b+ m)= 0. Hence $(b,m) \in r_D(a, 0)$. So $r_D(a, 0) = r_D((1-e), 0)$. Since $((1-e),0)^2 = ((1-e), 0)$ and (0,1) is the identity of D and (0,1) - (1-e,0)= (e-1,1) is an idempotent, $r_D(1-e,0)=(e-1,1)D$. Since ((e-1),1) is an idempotent in D, this completes the proof . ◘

Theorem 4.5. A ring R is a right (left) Rickart ring if and only if the Dorroh extension D =D(R, Z) of R by the ring Z is a right (left) Rickart ring.

Proof Let $(0, 0) \neq (a, n) \in D$ for $a \in R$ and $n \in Z$. We divide the proof into two cases:

Case I. n =0. Then $r_D(a,0)$ is generated by an idempotent .This case is proved in Lemma 4. 4 .

Case II. $n\neq 0$. Let $(a, n) \in D$. Suppose $(h, m)\in r_D (a,n)$.Then (a,n) (h, m) = (ah+ nh+ am, nm)=(0,0) implies m =0 since Z is an integral domain and $n \neq 0$ and ah+ nh+am =ah+ nh =(a+ n)h=0. So $h \in r_R(a+n.1)$. Since R is Rickart, there exists an idempotent d in R such that $r_R(a+n.1_R) = dR$. That gives us $(h,m) \in (d,0)(R,0)$. So $r_D(a, n) \subseteq$

(d,0)(R,0). For the reverse inclusion let (d,0) (r,0) ∈ (d, 0)(R,0).Then(a, n)(d,0)(r,0) = (ad+ nd ,0)(r,0) = ((a+n)d,0) (r,0) =(0,0) (0,0) implies (d,0)(r,0) ∈r_D(a, n) or (d,0)(R, 0) ⊆ r_D(a, n). Thus r_D(a, n) = (d,0)(R,0).

Conversely, assume that D(R, Z) is Rickart ring and let a∈R. Then the annihilator of a in D is generated by an idempotent in D, say r_D(a,0) =(e,n)D .We discuss two cases as in Lemma 4.2.

Case I. n =0 .Then r_D(a,0)=(e,0)D. (e,0)is an idempotent in D .By Lemma 4.2, e is an idempotent in R, We want to prove that r_R(a)=eR in this case . Since (a,0)(e,0)=(0,0) we get ae =0. So eR⊆r_R (a). Let t∈r_R(a).Then at = 0 and so (a,0)(t,0)=(0,0). Hence (t,0)∈r_D(a,0)=(e,0)D. It implies that (e,0) (t, 0) = (t,0) or et=t∈eR. That is r_R(a)=eR and e is an idempotent in this case .

Case II n=1. Then r_D (a,0)=(e,1)D where (e,1) is an idempotent in D. By lemma 4.2, e+1 is an idempotent in R .We want to prove that r_R(a)=(e+1)R in this case. Since (a,0) (e,1) =(0,0), we get a(e+1)=0. So (e+1)R⊆r_R(a). Let t ∈ r_R(a), then at =0 and so (a,0)(t,0)=(0,0). Hence (t,0)∈r_D(a,0)=(e+1,0)D. It implies that (e+1,0) (t,0) =(t,0) or (e+1)t=t ∈(e+1)R. That is r_R (a)⊆(e+1)R. Thus r_R(a) =(e+1)R and e+1is an idempotent in this case also . This completes the proof. ◘

Proposition 4.6. A ring R is strongly Rickart if and only if the Dorroh extension D=D (R, Z) is strongly Rickart ring .

Proof . Follows theorem (4. 5) and lemma (4. 3). ◘

Let R be a Commutative ring and M be an R-module. Then the idealization of a module M (simply R(+)M) is a Commutative ring with multiplication (r_1,m_1) (r_2,m_2) = (r_1r_2, r_1m_2+r_2m_1) [2].

Before we give the second essentially result of this section, we needed to the following lemma which appears in [2].

Lemma 4.7.[2,Theorem 3.7] Let R be a commutative ring and M be an R-module then the idempotent of R(+)M are Id(R(+)M) = Id(R)(+)0.

Proposition 4.8. Let R be a commutative ring. If R(+)M is a Rickart then So is R .

Proof. Let a∈R. Since R can be embedded in R(+)M then (a, 0) ∈R(+)M. But R(+)M is Rickart . Hence $r_{R(+)M}$ (a,0) is generated by idempotent element in R(+)M . So from lemma (4.7), $r_{R(+)M}$ (a,0) = (e,0)(R,0). We claim that r_R(a) =eR. Suppose that b ∈r_R(a). Hence ab =0 and so (b,0) ∈ $r_{R(+)M}$(a,0)= (e,0) (R,0) where (a,0) (b,0) =(ab,0)=0. So (b,0)=(e,0) (b,0) =(eb,0) . Thus b=eb. Hence b∈ eR. Then r_R(a) ⊆eR. Now, let x= er ∈eR. Then (x,0) ∈ (e,0) (R,0) =$r_{R(+)M}$ (a,0). Thus (a,0) (x,0)= (ax,0) =0 . So ax=0. Hence x ∈r_R(a). That give us eR ⊆r_R(a). Therefore r_R(a) = eR. Hence R is Rickart. ◘

Since a commutative ring R is strongly Rickart ring if and only if R is Rickart ring. So we have the following corollary.

Corollary 4.9. Let R be a commutative ring. Then if R(+)M is strongly Rickart then So is R .

Remark 4.10.Since the module M is isomorphic to the nilpotent ideal 0(+)M in R(+)M of index 2 [2], then if R is strongly Rickart that does not mean R(+)M is strongly Rickart unless that M=0.

The idealization of strongly Rickart ring needed not strongly Rickart as the following examples

Examples 4.11.

1. The idealization ring Z(+)Z of Strongly Rickart ring R= Z has a nonzero nilpotent element (0, n) with index 2 for all n ∈ Z. So Z(+)Z is not reduced and hence is not Strongly Rickart.

2. If R = Z(+)Q/Z, then R is a commutative P-injective ring and so is C_2-ring. But R is not strongly Rickart ring. For that, suppose R is strongly Rickart and then by theorem (3.5) R is Von Neumann regular ring and hence R semiregular ring this a contradiction where R is not semiregular [21].

Recall that from [2] there is an isomorphism between R(+)M and R[x](+)M[x] for a ring R and R-module M . Using this fact, we end this paper by give the following two results.

Corollary 4.10. If R[x](+)M[x] is Rickart then so is R.

Proof. From proposition (4.8) R[x] is Rickart and hence R is Rickart [3, Theorem A] . ◘

Corollary 4.11. If R[x] (+) M[x] is strongly Rickart then so is R.

Proof. From corollary (4.9) R[x] is strongly Rickart and hence by proposition (4.1) R is strongly Rickart.◘

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