A note on Bayesian One-Way Repeated Measurements Model

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Abstract

In this paper, we consider the linear one- way repeated measurements model which has only one within units factor and one between units factor incorporating univariate random effects as well as the experimental error term. Bayesian approach based on Markov Chain Monte Carlo is employed to making inferences on the one-way repeated measurements model.

Keywords: Repeated Measurements, ANOVA, Bayesian inference, Prior density, posterior density, Bayes factor.

1. Introduction

Repeated measurements analysis is widely used in many fields, for example , in the health and life science, epidemiology, biomedical, agricultural, industrial, psychological, educational researches and so on.

Repeated measurements is a term used to describe data in which the response variable for each experimental units is observed on multiple occasions and possible under different experimental conditions . Repeated measures data is a common form of multivariate data, and linear models with correlated error which are widely used in modeling repeated measures data. Repeated measures is a common data structure with multiple measurements on a single unit repeated over time. Multivariate linear models with correlated errors have been accepted as one of the primary modeling methods for repeated measures data [1],[2] ,[7],[9] .

Repeated measures designs involving two or more independent groups are among the most common experimental designs in a variety of research settings. Various statistical procedures have been suggested for analyzing data from split- plot designs when parametric model assumptions are violated [1],[2].

In the Bayesian approach to inference, all unknown quantities contained in a probability model for the observed data are treated as random variables. Specifically, the fixed but unknown parameters are viewed as random variables under the Bayesian approach. Bayesian techniques based on Markov chain Monte Carlo provide what we believe to be the most satisfactory approach to fitting complex models as well as the direction that model is most likely to take in the future [3],[4],[5],[6],[8],[10],[11] .

In this paper, we consider the linear one- way repeated measurements model which has only one within units factor and one between units factor incorporating univariate random effects as well as the experimental error term. Bayesian approach based on Markov Chain Monte Carlo is employed to making inferences on the one-way repeated measurements model. We investigate the posterior density and identify the analytic form of the Bayes factor .

2.Repeated Measurements Model and Prior Distribution

Consider the model

$$y_{ijk} = \mu + \tau_j + \delta_{i(j)} + \gamma_k + (\tau\gamma)_{jk} + e_{ijk} \tag{1}$$

Where

i=1,…., n is an index for experimental unit within group j ,

j=1,…,q is an index for levels of the between-units factor (Group) ,

k=1,…,p is an index for levels of the within-units factor (Time) ,

y_{ijk} is the response measurement at time k for unit i within group j,

μ is the overall mean ,

τ_j is the added effect for treatment group j ,

$\delta_{i(j)}$ is the random effect for due to experimental unit i within treatment group j ,

γ_k is the added effect for time k ,

$(\tau\gamma)_{jk}$ is the added effect for the group j × time k interaction ,

e_{ijk} is the random error on time k for unit i within group j ,

For the parameterization to be of full rank, we imposed the following set of conditions

$\sum_{j=1}^{q}\tau_{j=0}$, $\sum_{k=1}^{p}\gamma_{k=0}$, $\sum_{j=1}^{q}(\tau\gamma)_{jk=0}$ for each k=1,…,p

$\sum_{k=1}^{p}(\tau\gamma)_{jk=0}$ for each j=1,…,q

And we assumed that the e_{ijk} and $\delta_{i(j)}$ are indepndent with

$e_{ijk} \sim$ i.i.d $N(0,\sigma_e^2)$, $\delta_{i(j)} \sim$ i.i.d $N(0,\sigma_\delta^2)$.

Sum of squares due to groups, subjects(group), time, group*time and residuals are then defined respectively as follows:

$$SS_G = np\sum_{j=1}^{q}\left(\bar{y}_{.j.} - \bar{y}_{...}\right)^2, \qquad SS_{U(G)} = p\sum_{i=1}^{n}\sum_{j=1}^{q}(\bar{y}_{ij.} - \bar{y}_{.j.})^2$$

$$SS_{time} = nq\sum_{k=1}^{p}(\bar{y}_{..k} - \bar{y}_{...})^2, \quad SS_{G\times time} = n\sum_{j=1}^{q}\sum_{k=1}^{p}(\bar{y}_{.jk} - \bar{y}_{.j.} - \bar{y}_{..k} + \bar{y}_{...})^2$$

$$SS_E = \sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk} - \bar{y}_{.jk} - \bar{y}_{ij.} + \bar{y}_{.j.})^2$$

Where

$$\bar{y}_{...} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}y_{ijk}}{nqp}$$ is the overall mean.

$$\bar{y}_{.j.} = \frac{\sum_{i=1}^{n}\sum_{k=1}^{p}y_{ijk}}{np}$$ is the mean for group j.

$$\bar{y}_{ij.} = \frac{\sum_{k=1}^{p}y_{ijk}}{p}$$ is the mean for the i^{th} subject in group j.

$$\bar{y}_{..k} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}y_{ijk}}{nq}$$ is the mean for time k.

$\bar{y}_{.jk} = \frac{\sum_{i=1}^{n} y_{ijk}}{n}$ is the mean for group j at time k.

ANOVA table for one-way Repeated measures model

Source of variation	d.f	S.S	M.S	E(M.S)
Group	$q-1$	$S.S_G$	$\frac{S.S_G}{q-1}$	$\frac{np}{(q-1)}\sum_{j=1}^{q}\tau_j^2 + p\sigma_\delta^2 + \sigma_e^2$
Unit (Group)	$q(n-1)$	$S.S_{U(G)}$	$\frac{S.S_{U(G)}}{q(n-1)}$	$p\sigma_\delta^2 + \sigma_e^2$
Time	$p-1$	$S.S_{time}$	$\frac{S.S_{time}}{p-1}$	$\frac{nq}{(p-1)}\sum_{k=1}^{p}\gamma_k^2 + \sigma_e^2$
Group*Time	$(q-1)(p-1)$	$S.S_{G\times time}$	$\frac{S.S_{G\times time}}{(q-1)(p-1)}$	$\frac{n}{(p-1)(q-1)}\sum_{j=1}^{q}\sum_{k=1}^{p}(\tau\gamma)_{jk}^2 + \sigma_e^2$
Residual	$q(p-1)(n-1)$	$S.S_E$	$\frac{S.S_E}{q(p-1)(n-1)}$	σ_e^2

We assume that the prior distribution on one-way repeated measurements model coefficients as following

$$\mu \sim N(0,\sigma_\mu^2) \quad , \quad \tau_j \sim N(0,\sigma_\tau^2) \quad , \quad \gamma_k \sim N(0,\sigma_\gamma^2)$$

$$(\tau\gamma)_{jk} \sim N(0,\sigma_{(\tau\gamma)}^2)\ , \quad \sigma_\delta^2 \sim IG(\alpha_\delta,\beta_\delta) \quad , \quad \sigma_e^2 \sim IG(\alpha_e,\beta_e) \qquad (2)$$

3.Posterior Calculation

The likelihood function for the model (1) can derive as follows

$$L(y|\mu,\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2) \propto \prod_{i=1}^{n}\prod_{j=1}^{q}\prod_{k=1}^{p}\frac{1}{\sqrt{2\pi\sigma_e^2}} \exp\left[\frac{-(y_{ijk}-\mu-\tau_j-\delta_{i(j)}-\gamma_k-(\tau\gamma)_{jk})^2}{2\sigma_e^2}\right]$$

$$\to L(y|\mu,\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2) \propto (2\pi(\sigma_e^2))^{-\frac{nqp}{2}} \times \exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\mu-\tau_j-\delta_{i(j)}-\gamma_k-(\tau\gamma)_{jk})^2}{2\sigma_e^2}\right]$$

since

$$\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\mu-\tau_j-\delta_{i(j)}-\gamma_k-(\tau\gamma)_{jk})^2 = \sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}[y_{ijk}+y_{i.k}-y_{i.k}+y_{..k}-y_{..k}+y_{ij.}-y_{ij.}+y_{i..}-y_{i..}-\mu-\tau_j-\delta_{i(j)}-\gamma_k-(\tau\gamma)_{jk}]^2$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}\left[(y_{ijk}-\mu)^2+(y_{i.k}-\tau_j)^2+(y_{..k}-\delta_{i(j)})^2+(y_{ij.}-\gamma_k)^2+(y_{i..}-(\tau\gamma)_{jk})^2-(y_{i.k}+y_{..k}+y_{ij.}+y_{i..})^2\right].$$

Then

$$L\left(y|\mu,\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2\right) \propto \left(2\pi(\sigma_e^2)\right)^{-\frac{nqp}{2}}\exp[-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\mu)^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{i.k}-\tau_j)^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{..k}-\delta_{i(j)})^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ij.}-\gamma_k)^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{i..}-(\tau\gamma)_{jk})^2}{2(\sigma_e^2)}+\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{i.k}+y_{..k}+y_{ij.}+y_{i..})^2}{2(\sigma_e^2)}]. \quad (3)$$

Then we have the posterior density of one-way repeated measurements model coefficients and the variances (σ_e^2) and (σ_δ^2) as follows

$$\pi_1\left(\mu|\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2\right) \propto L\left(y|\mu,\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2\right)\pi_0(\mu),$$

where π_0 and π_1 represents prior and posterior density respectively, then

The posterior of μ ($\mu|\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2$) is

$$\pi_1\left(\mu|\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2\right) \propto \left(2\pi(\sigma_e^2)\right)^{-\frac{nqp}{2}}\exp[-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\mu)^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{i.k}-\tau_j)^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{..k}-\delta_{i(j)})^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ij.}-\gamma_k)^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{i..}-(\tau\gamma)_{jk})^2}{2(\sigma_e^2)}+\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{i.k}+y_{..k}+y_{ij.}+y_{i..})^2}{2(\sigma_e^2)}]$$

$$\times (2\pi\sigma_\mu^2)^{-\frac{1}{2}}\exp\left[\frac{-\mu^2}{2\sigma_\mu^2}\right]$$

$$\propto \left(2\pi(\sigma_e^2)\right)^{-\frac{nqp}{2}}(2\pi\sigma_\mu^2)^{-\frac{1}{2}}\exp\left[-\frac{1}{2}\mu^2\left(\frac{nqp}{\sigma_e^2}+\frac{1}{\sigma_\mu^2}\right)+\mu\left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}y_{ijk}}{\sigma_e^2}\right)\right]$$

$$\propto \exp\left[-\frac{1}{2}\left(\mu^2-2\mu\frac{\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}y_{ijk}}{\sigma_e^2}}{\frac{nqp}{\sigma_e^2}+\frac{1}{\sigma_\mu^2}}\right)\times\left(\frac{nqp}{\sigma_e^2}+\frac{1}{\sigma_\mu^2}\right)\right]$$

$$= \exp\left[\frac{-\frac{1}{2}\left(\mu-\frac{\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}y_{ijk}}{\sigma_e^2}}{\frac{nqp}{\sigma_e^2}+\frac{1}{\sigma_\mu^2}}\right)^2}{\frac{1}{\frac{nqp}{\sigma_e^2}+\frac{1}{\sigma_\mu^2}}}\right]$$

$$\therefore \mu|\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2 \sim N\left[\frac{\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}y_{ijk}}{\sigma_e^2}}{\frac{nqp}{\sigma_e^2}+\frac{1}{\sigma_\mu^2}},\frac{1}{\frac{nqp}{\sigma_e^2}+\frac{1}{\sigma_\mu^2}}\right]. \quad (4)$$

The posterior of τ_j $(\tau_j|\mu,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2)$ is

$$\pi_1\left(\tau_j|\mu,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2\right) \propto L\left(y|\mu,\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2\right)\pi_0(\tau_j)$$

$$\pi_1\left(\tau_j|\mu,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2\right) \propto \left(2\pi(\sigma_e^2)\right)^{-\frac{nqp}{2}}\exp[-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\mu)^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{i.k}-\tau_j)^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{..k}-\delta_{i(j)})^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ij.}-\gamma_k)^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{i..}-(\tau\gamma)_{jk})^2}{2(\sigma_e^2)}+$$

$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{i.k}+y_{..k}+y_{ij.}+y_{i..})^2}{2(\sigma_e^2)}]\times(2\pi\sigma_\tau^2)^{-\frac{q}{2}}\exp\left[\frac{-\sum_{j=1}^{q}\tau_j{}^2}{2\sigma_\tau^2}\right]$$

$$\rightarrow \pi_1\left(\tau_j \middle| \mu, \delta_{i(j)}, \gamma_k, (\tau\gamma)_{jk}, \sigma_\delta^2, \sigma_e^2\right) \propto \left(2\pi(\sigma_e^2)\right)^{-\frac{nqp}{2}} (2\pi\sigma_\tau^2)^{-\frac{q}{2}} \times$$
$$\exp\left[-\frac{1}{2}\sum_{j=1}^{q} \tau_j^2 \left(\frac{np}{\sigma_e^2} + \frac{1}{\sigma_\tau^2}\right) + \sum_{j=1}^{q} \tau_j \left(\frac{\sum_{i=1}^{n}\sum_{k=1}^{p} y_{i.k}}{\sigma_e^2}\right)\right]$$

$$\propto \exp\left[-\frac{1}{2}\left(\sum_{j=1}^{q} \tau_j^2 - 2\sum_{j=1}^{q} \tau_j \frac{\frac{\sum_{i=1}^{n}\sum_{k=1}^{p} y_{i.k}}{\sigma_e^2}}{\frac{np}{\sigma_e^2}+\frac{1}{\sigma_\tau^2}}\right) \times \left(\frac{np}{\sigma_e^2)} + \frac{1}{\sigma_\tau^2}\right)\right]$$

$$= \exp\left[\frac{-\frac{1}{2}\left(\sum_{j=1}^{q} \tau_j - \frac{\sum_{i=1}^{n}\sum_{k=1}^{p} y_{i.k} + \sigma_e^2}{\frac{np}{\sigma_e^2}+\frac{1}{\sigma_\tau^2}}\right)^2}{\frac{1}{\frac{np}{\sigma_e^2}+\frac{1}{\sigma_\tau^2}}}\right]$$

$$\therefore\ \tau_j | \mu, \delta_{i(j)}, \gamma_k, (\tau\gamma)_{jk}, \sigma_\delta^2, \sigma_e^2 \sim N\left[\frac{\frac{\sum_{i=1}^{n}\sum_{k=1}^{p} y_{i.k}}{\sigma_e^2}}{\frac{np}{\sigma_e^2}+\frac{1}{\sigma_\tau^2}}, \frac{1}{\frac{np}{\sigma_e^2}+\frac{1}{\sigma_\tau^2}}\right]. \quad (5)$$

The posterior of γ_k $\left(\gamma_k \middle| \mu, \tau_j, \delta_{i(j)}, (\tau\gamma)_{jk}, \sigma_\delta^2, \sigma_e^2\right)$ is

$$\pi_1\left(\gamma_k \middle| \mu, \tau_j, \delta_{i(j)}, (\tau\gamma)_{jk}, \sigma_\delta^2, \sigma_e^2\right) \propto L\left(y \middle| \mu, \tau_j, \delta_{i(j)}, \gamma_k, (\tau\gamma)_{jk}, \sigma_\delta^2, \sigma_e^2\right) \pi_0(\gamma_k)$$

$$\pi_1\left(\gamma_k \middle| \mu, \tau_j, \delta_{i(j)}, (\tau\gamma)_{jk}, \sigma_\delta^2, \sigma_e^2\right) \propto \left(2\pi(\sigma_e^2)\right)^{-\frac{nqp}{2}} \exp[-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\mu)^2}{2(\sigma_e^2)} - \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{i.k}-\tau_j)^2}{2(\sigma_e^2)}$$
$$- \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{..k}-\delta_{i(j)})^2}{2(\sigma_e^2)} - \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ij.}-\gamma_k)^2}{2(\sigma_e^2)} - \frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{i..}-(\tau\gamma)_{jk})^2}{2(\sigma_e^2)} +$$
$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{i.k}+y_{..k}+y_{ij.}+y_{i..})^2}{2(\sigma_e^2)}] \times (2\pi\sigma_\gamma^2)^{-\frac{p}{2}} \exp\left[\frac{-\sum_{k=1}^{p}{\gamma_k}^2}{2\sigma_\gamma^2}\right]$$

$$\rightarrow \pi_1\left(\gamma_k \middle| \tau_j, \delta_{i(j)}, \mu, (\tau\gamma)_{jk}, \sigma_\delta^2, \sigma_e^2\right) \propto$$
$$\left(2\pi(\sigma_e^2)\right)^{-\frac{nqp}{2}} (2\pi\sigma_\gamma^2)^{-\frac{p}{2}} \times \exp\left[-\frac{1}{2}\sum_{k=1}^{p} \gamma_k^2 \left(\frac{nq}{\sigma_e^2} + \frac{1}{\sigma_\gamma^2}\right) + \sum_{k=1}^{p} \gamma_k \left(\frac{\sum_{i=1}^{n}\sum_{j=1}^{q} y_{ij.}}{\sigma_e^2}\right)\right]$$

$$\propto \exp\left[-\frac{1}{2}\left(\sum_{k=1}^{p} \gamma_k^2 - 2\sum_{k=1}^{p} \gamma_k \frac{\frac{\sum_{i=1}^{n}\sum_{j=1}^{q} y_{ij.}}{\sigma_e^2}}{\frac{nq}{\sigma_e^2}+\frac{1}{\sigma_\gamma^2}}\right) \times \left(\frac{nq}{\sigma_e^2} + \frac{1}{\sigma_\gamma^2}\right)\right]$$

$$= \exp\left[\frac{-\frac{1}{2}\left(\sum_{k=1}^{p} \gamma_k - \frac{\frac{\sum_{i=1}^{n}\sum_{j=1}^{q} y_{ij.}}{\sigma_e^2}}{\frac{nq}{\sigma_e^2}+\frac{1}{\sigma_\gamma^2}}\right)^2}{\frac{1}{\frac{nq}{\sigma_e^2}+\frac{1}{\sigma_\gamma^2}}}\right]$$

$$\therefore\ \gamma_k\ | \mu, \tau_j\ , \delta_{i(j)}, (\tau\gamma)_{jk}, \sigma_\delta^2, \sigma_e^2 \sim N\left[\frac{\frac{\sum_{i=1}^{n}\sum_{j=1}^{q} y_{ij.}}{\sigma_e^2}}{\frac{nq}{\sigma_e^2}+\frac{1}{\sigma_\gamma^2}}, \frac{1}{\frac{nq}{\sigma_e^2}+\frac{1}{\sigma_\gamma^2}}\right]. \quad (6)$$

The posterior of $(\tau\gamma)_{jk}$, $\left((\tau\gamma)_{jk}\middle|\mu,\tau_j,\delta_{i(j)},\gamma_k,\sigma_\delta^2,\sigma_e^2\right)$ is

$$\pi_1\left((\tau\gamma)_{jk}\middle|\mu,\tau_j,\delta_{i(j)},\gamma_k,\sigma_\delta^2,\sigma_e^2\right) \propto L\left(y\middle|\mu,\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2\right)\pi_0\left((\tau\gamma)_{jk}\right)$$

$$\pi_1\left((\tau\gamma)_{jk}\middle|\mu,\tau_j,\delta_{i(j)},\gamma_k,\sigma_\delta^2,\sigma_e^2\right) \propto \left(2\pi(\sigma_e^2)\right)^{-\frac{nqp}{2}}\exp[-\frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p(y_{ijk}-\mu)^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p(y_{i.k}-\tau_j)^2}{2(\sigma_e^2)}$$
$$-\frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p(y_{..k}-\delta_{i(j)})^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p(y_{ij.}-\gamma_k)^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p(y_{i..}-(\tau\gamma)_{jk})^2}{2(\sigma_e^2)}+$$
$$\frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p(y_{i.k}+y_{..k}+y_{ij.}+y_{i..})^2}{2(\sigma_e^2)}]\times\left(2\pi\sigma_{(\tau\gamma)}^2\right)^{-\frac{qp}{2}}\exp\left[\frac{-\sum_{j=1}^q\sum_{k=1}^p(\tau\gamma)_{jk}^2}{2\sigma_{(\tau\gamma)}^2}\right]$$

$$\propto\left(2\pi(\sigma_e^2)\right)^{-\frac{nqp}{2}}\left(2\pi\sigma_{(\tau\gamma)}^2\right)^{-\frac{qp}{2}}\times\exp\left[-\frac{1}{2}\sum_{j=1}^q\sum_{k=1}^p\left((\tau\gamma)_{jk}\right)^2\left(\frac{n}{\sigma_e^2}+\frac{1}{\sigma_{\tau\gamma}^2}\right)+\sum_{j=1}^q\sum_{k=1}^p(\tau\gamma)_{jk}\left(\frac{\sum_{i=1}^n y_{i..}}{\sigma_e^2}\right)\right]$$

$$\propto\exp\left[-\frac{1}{2}\left(\sum_{j=1}^q\sum_{k=1}^p\left((\tau\gamma)_{jk}\right)^2-2\sum_{j=1}^q\sum_{k=1}^p(\tau\gamma)_{jk}\frac{\frac{\sum_{i=1}^n y_{i..}}{\sigma_e^2}}{\frac{n}{\sigma_e^2}+\frac{1}{\sigma_{\tau\gamma}^2}}\right)\times\left(\frac{n}{(\sigma_e^2)}+\frac{1}{\sigma_{\tau\gamma}^2}\right)\right]$$

$$=\exp\left[\frac{-\frac{1}{2}\left(\sum_{j=1}^q\sum_{k=1}^p(\tau\gamma)_{jk}-\frac{\frac{\sum_{i=1}^n y_{i..}}{\sigma_e^2}}{\frac{n}{\sigma_e^2}+\frac{1}{\sigma_{\tau\gamma}^2}}\right)^2}{\frac{1}{\frac{n}{\sigma_e^2}+\frac{1}{\sigma_{\tau\gamma}^2}}}\right]$$

$$\therefore(\tau\gamma)_{jk}\,|\mu,\tau_j,\delta_{i(j)},\gamma_k,\sigma_\delta^2,\sigma_e^2\sim N\left[\frac{\frac{\sum_{i=1}^n y_{i..}}{\sigma_e^2}}{\frac{n}{\sigma_e^2}+\frac{1}{\sigma_{\tau\gamma}^2}},\frac{1}{\frac{n}{\sigma_e^2}+\frac{1}{\sigma_{\tau\gamma}^2}}\right]. \quad (7)$$

The posterior of $\delta_{i(j)}$, $\left(\delta_{i(j)}\middle|\mu,\tau_j,\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2\right)$ is

$$\pi_1\left(\delta_{i(j)}\middle|\mu,\tau_j,\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2\right)\propto L\left(y\middle|\mu,\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2\right)\pi_0\left(\delta_{i(j)}\right)$$

$$\pi_1\left(\delta_{i(j)}\middle|\mu,\tau_j,\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2\right)\propto\left(2\pi(\sigma_e^2)\right)^{-\frac{nqp}{2}}\exp[-\frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p(y_{ijk}-\mu)^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p(y_{i.k}-\tau_j)^2}{2(\sigma_e^2)}$$
$$-\frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p(y_{..k}-\delta_{i(j)})^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p(y_{ij.}-\gamma_k)^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p(y_{i..}-(\tau\gamma)_{jk})^2}{2(\sigma_e^2)}+$$
$$\frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p(y_{i.k}+y_{..k}+y_{ij.}+y_{i..})^2}{2(\sigma_e^2)}]\times\left(2\pi\sigma_\delta^2\right)^{-\frac{nq}{2}}\exp\left[\frac{-\sum_{i=1}^n\sum_{j=1}^q\delta_{i(j)}^2}{2\sigma_\delta^2}\right]$$

$$\rightarrow\pi_1\left(\delta_{i(j)}\middle|\mu,\tau_j,\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2\right)\propto$$
$$\left(2\pi(\sigma_e^2)\right)^{-\frac{nqp}{2}}\left(2\pi\sigma_\delta^2\right)^{-\frac{nq}{2}}\exp\left[-\frac{1}{2}\sum_{i=1}^n\sum_{j=1}^q\delta_{i(j)}^2\left(\frac{p}{\sigma_e^2}+\frac{1}{\sigma_\delta^2}\right)+\sum_{i=1}^n\sum_{j=1}^q\delta_{i(j)}\left(\frac{\sum_{k=1}^p y_{..k}}{\sigma_e^2}\right)\right]$$

$$\propto\exp\left[-\frac{1}{2}\left(\sum_{i=1}^n\sum_{j=1}^q\delta_{i(j)}^2-2\sum_{i=1}^n\sum_{j=1}^q\delta_{i(j)}\frac{\frac{\sum_{k=1}^p y_{..k}}{\sigma_e^2}}{\frac{p}{\sigma_e^2}+\frac{1}{\sigma_\delta^2}}\right)\times\left(\frac{p}{\sigma_e^2}+\frac{1}{\sigma_\delta^2}\right)\right]$$

$$= \exp\left[\frac{-\frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{q}\delta_{i(j)}-\frac{\frac{\sum_{k=1}^{p}y_{..k}}{\sigma_e^2}}{\frac{p}{\sigma_e^2}+\frac{1}{\sigma_\delta^2}}\right)^2}{\frac{1}{\frac{p}{\sigma_e^2}+\frac{1}{\sigma_\delta^2}}}\right]$$

$$\therefore \delta_{i(j)}\ |\mu,\tau_j\ ,\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2 \sim N\left[\frac{\frac{\sum_{k=1}^{p}y_{..k}}{\sigma_e^2}}{\frac{p}{\sigma_e^2}+\frac{1}{\sigma_\delta^2}},\frac{1}{\frac{p}{\sigma_e^2}+\frac{1}{\sigma_\delta^2}}\right]. \tag{8}$$

The posterior of σ_δ^2, $\left(\sigma_\delta^2\middle|\mu,\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_e^2\right)$ is

$$\pi_1\left(\sigma_\delta^2\middle|\mu,\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_e^2\right) \propto L\left(y\middle|\mu,\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2\right)\pi_0(\sigma_\delta^2)$$

$$\pi_1\left(\sigma_\delta^2\middle|\mu,\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_e^2\right) \propto \left(2\pi(\sigma_e^2)\right)^{-\frac{nqp}{2}}\exp[-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\mu)^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{i.k}-\tau_j)^2}{2(\sigma_e^2)}$$
$$-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{..k}-\delta_{i(j)})^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ij.}-\gamma_k)^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{i..}-(\tau\gamma)_{jk})^2}{2(\sigma_e^2)}+\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{i.k}+y_{..k}+y_{ij.}+y_{i..})^2}{2(\sigma_e^2)}]$$
$$(\sigma_\delta^2)^{-(\alpha_\delta+1)}\ \frac{\beta_\delta^{\alpha_\delta}}{\Gamma(\alpha_\delta)}\exp\left[\frac{-\beta_\delta}{\sigma_\delta^2}\right]$$

$$\propto (\sigma_\delta^2)^{-(\alpha_\delta+1)}\ \frac{\beta_\delta^{\alpha_\delta}}{\Gamma(\alpha_\delta)}\exp\left[\frac{-\beta_\delta}{\sigma_\delta^2}\right]$$

$$\therefore \sigma_\delta^2|\mu,\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_e^2 \sim IG[\alpha_\delta,\beta_\delta]. \tag{9}$$

The posterior of σ_e^2, $\left(\sigma_e^2\middle|\mu,\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2\right)$ is

$$\pi_1\left(\sigma_e^2\middle|\mu,\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2\right) \propto L\left(y\middle|\mu,\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2\right)\pi_0(\sigma_e^2)$$

$$\pi_1\left(\sigma_e^2\middle|\mu,\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2\right) \propto \left(2\pi(\sigma_e^2)\right)^{-\frac{nqp}{2}}\exp[-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-\mu)^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{i.k}-\tau_j)^2}{2(\sigma_e^2)}$$
$$-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{..k}-\delta_{i(j)})^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ij.}-\gamma_k)^2}{2(\sigma_e^2)}-\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{i..}-(\tau\gamma)_{jk})^2}{2(\sigma_e^2)}+$$

$$\frac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{i.k}+y_{..k}+y_{ij.}+y_{i..})^2}{2(\sigma_e^2)}]\times(\sigma_e^2)^{-(\alpha_e+1)}\ \frac{\beta_e^{\alpha_e}}{\Gamma(\alpha_e)}\exp\left[\frac{-\beta_e}{\sigma_e^2}\right]$$

$$\propto (\sigma_e^2)^{-(\alpha_e+\frac{nqp}{2}+1)}\ \frac{\beta_e^{\alpha_e}}{\Gamma(\alpha_e)}\exp\left[\frac{-\left(\frac{RSS}{2}\right)-\beta_e}{\sigma_e^2}\right]$$

Where $RSS=\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}\left[(y_{ijk}-\mu)^2+(y_{i.k}-\tau_j)^2+(y_{..k}-\delta_{i(j)})^2+(y_{ij.}-\gamma_k)^2+(y_{i..}-(\tau\gamma)_{jk})^2-(y_{i.k}+y_{..k}+y_{ij.}+y_{i..})^2\right]$

$$\therefore \sigma_e^2|\mu,\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2 \sim IG\left(\alpha_e+\frac{nqp}{2},\beta_e+\frac{RSS}{2}\right). \tag{10}$$

4. Model checking and Bayes factors

We would like to choose between a Bayesian mixed repeated measurements model and its fixed counterpart by the criterion of the Bayes factor for tow hypotheses :

H_0: $y_{ijk} = \mu + \tau_j + \gamma_k + (\tau\gamma)_{jk} + e_{ijk}$ versus

$$H_1:\ y_{ijk} = \mu + \tau_j + \delta_{i(j)} + \gamma_k + (\tau\gamma)_{jk} + e_{ijk} \tag{11}$$

We compute the Bayes factor, B_{01}, of H_o relative to H_1 for testing problem (11) as following

$$B_{01}(y_{ijk}) = \frac{m(y_{ijk}|H_o)}{m(y_{ijk}|H_1)}\ , \tag{12}$$

where $m(y_{ijk}|H_i)$ is the predictive (marginal) density of y_{ijk} under model $H_i, i = 0, 1$.

We have

$$m(y_{ijk}|H_o) = \frac{1}{(2\pi(\sigma_\mu^2+\sigma_\tau^2+\sigma_\gamma^2+\sigma_{(\tau\gamma)}^2+\sigma_e^2))^{\frac{1}{2}}} exp\left[\frac{-\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p y_{ijk}^2}{2(\sigma_\mu^2+\sigma_\tau^2+\sigma_\gamma^2+\sigma_{(\tau\gamma)}^2+\sigma_e^2)}\right],$$

and

$$m(y_{ijk}|H_1) = \frac{1}{(2\pi(\sigma_\mu^2+\sigma_\tau^2+\sigma_\delta^2+\sigma_\gamma^2+\sigma_{(\tau\gamma)}^2+\sigma_e^2))^{\frac{1}{2}}} exp\left[\frac{-\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p y_{ijk}^2}{2(\sigma_\mu^2+\sigma_\tau^2+\sigma_\delta^2+\sigma_\gamma^2+\sigma_{(\tau\gamma)}^2+\sigma_e^2)}\right],$$

$$\therefore B_{01}(y_{ijk}) = \frac{m(y_{ijk}|H_o)}{m(y_{ijk}|H_1)} = \frac{\sqrt{(\sigma_\mu^2+\sigma_\tau^2+\sigma_\delta^2+\sigma_\gamma^2+\sigma_{(\tau\gamma)}^2+\sigma_e^2)}}{\sqrt{(\sigma_\mu^2+\sigma_\tau^2+\sigma_\gamma^2+\sigma_{(\tau\gamma)}^2+\sigma_e^2)}} \frac{\exp\left[\frac{-\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p y_{ijk}^2}{2(\sigma_\mu^2+\sigma_\tau^2+\sigma_\gamma^2+\sigma_{(\tau\gamma)}^2+\sigma_e^2)}\right]}{\exp\left[\frac{-\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p y_{ijk}^2}{2(\sigma_\mu^2+\sigma_\tau^2+\sigma_\delta^2+\sigma_\gamma^2+\sigma_{(\tau\gamma)}^2+\sigma_e^2)}\right]}. \tag{13}$$

4.Conclusions

1-The likelihood function of one-way repeated measurement model is

$$L(y|\mu,\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2) \propto (2\pi(\sigma_e^2))^{-\frac{nqp}{2}} \exp[-\frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p(y_{ijk}-\mu)^2}{2(\sigma_e^2)} - \frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p(y_{i.k}-\tau_j)^2}{2(\sigma_e^2)} - \frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p(y_{..k}-\delta_{i(j)})^2}{2(\sigma_e^2)} - \frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p(y_{ij.}-\gamma_k)^2}{2(\sigma_e^2)} - \frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p(y_{i..}-(\tau\gamma)_{jk})^2}{2(\sigma_e^2)} + \frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p(y_{i.k}+y_{..k}+y_{ij.}+y_{i..})^2}{2(\sigma_e^2)}].$$

2-The posterior density of μ is $\mu|\tau_j,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2 \sim N\left[\frac{\frac{\sum_{i=1}^n\sum_{j=1}^q\sum_{k=1}^p y_{ijk}}{\sigma_e^2}}{\frac{nqp}{\sigma_e^2}+\frac{1}{\sigma_\mu^2}}, \frac{1}{\frac{nqp}{\sigma_e^2}+\frac{1}{\sigma_\mu^2}}\right]$.

3-The posterior density of τ_j is $\tau_j|\mu,\delta_{i(j)},\gamma_k,(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2 \sim N\left[\frac{\frac{\sum_{i=1}^n\sum_{k=1}^p y_{i.k}}{\sigma_e^2}}{\frac{np}{\sigma_e^2}+\frac{1}{\sigma_\tau^2}}, \frac{1}{\frac{np}{\sigma_e^2}+\frac{1}{\sigma_\tau^2}}\right]$.

4- The posterior density of $\gamma_k$is $\gamma_k\ |\mu,\tau_j\ ,\delta_{i(j)},(\tau\gamma)_{jk},\sigma_\delta^2,\sigma_e^2 \sim N\left[\frac{\frac{\sum_{i=1}^n\sum_{j=1}^q y_{ij.}}{\sigma_e^2}}{\frac{nq}{\sigma_e^2}+\frac{1}{\sigma_\gamma^2}}, \frac{1}{\frac{nq}{\sigma_e^2}+\frac{1}{\sigma_\gamma^2}}\right]$.

5- The posterior density of $(\tau\gamma)_{jk}$ is $(\tau\gamma_{jk)}\ |\mu, \tau_j\ , \delta_{i(j)}, \gamma_k, \sigma_\delta^2, \sigma_e^2 \sim \mathrm{N}\left[\frac{\frac{\sum_{i=1}^{n} y_{i..}}{\sigma_e^2}}{\frac{n}{\sigma_e^2}+\frac{1}{\sigma_{\tau\gamma}^2}}, \frac{1}{\frac{n}{\sigma_e^2}+\frac{1}{\sigma_{\tau\gamma}^2}}\right]$.

6- The posterior density of δ_{ij} is $\delta_{i(j)}\ |\mu, \tau_j\ , \gamma_k, (\tau\gamma)_{jk}, \sigma_\delta^2, \sigma_e^2 \sim \mathrm{N}\left[\frac{\frac{\sum_{k=1}^{p} y_{..k}}{\sigma_e^2}}{\frac{p}{\sigma_e^2}+\frac{1}{\sigma_\delta^2}}, \frac{1}{\frac{p}{\sigma_e^2}+\frac{1}{\sigma_\delta^2}}\right]$.

7- The posterior density of σ_δ^2 is $\sigma_\delta^2|\mu, \tau_j, \delta_{i(j)}, \gamma_k, (\tau\gamma)_{jk}, \sigma_e^2 \sim \mathrm{IG}[\alpha_\delta, \beta_\delta]$.

8- The posterior density of σ_e^2 is $\sigma_e^2|\mu, \tau_j, \delta_{i(j)}, \gamma_k, (\tau\gamma)_{jk}, \sigma_\delta^2 \sim \mathrm{IG}\left(\alpha_e + \frac{nqp}{2}, \beta_e + \frac{RSS}{2}\right)$.

9-The Bayes factor for checking the Bayesian repeated measurements model is

$$B_{01}(\mathrm{y_{ijk}}) = \frac{\sqrt{(\sigma_\mu^2+\sigma_\tau^2+\sigma_\delta^2+\sigma_\gamma^2+\sigma_{(\tau\gamma)}^2+\sigma_e^2)}}{\sqrt{(\sigma_\mu^2+\sigma_\tau^2+\sigma_\gamma^2+\sigma_{(\tau\gamma)}^2+\sigma_e^2)}} \frac{\exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p} y_{ijk}^2}{2(\sigma_\mu^2+\sigma_\tau^2+\sigma_\gamma^2+\sigma_{(\tau\gamma)}^2+\sigma_e^2)}\right]}{\exp\left[\frac{-\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p} y_{ijk}^2}{2(\sigma_\mu^2+\sigma_\tau^2+\sigma_\delta^2+\sigma_\gamma^2+\sigma_{(\tau\gamma)}^2+\sigma_e^2)}\right]}.$$

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