Analytical approximate solutions for linear and nonlinear Volterra integral and integro-differential equations and some applications for the Lane-Emden equations using a power series method

M. A. AL-Jawary , H. R. AL-Qaissy
Department of Mathematics, College of Education Ibn-AL-Haithem, Baghdad University, Baghdad, Iraq
E-mail: Majeed.a.w@ihcoedu.uobaghdad.edu.iq;

Abstract.
In the present paper, we have presented a recursive method namely the Power Series Method (PSM) to solve first the linear and nonlinear Volterra integral and integro-differential equations. The PSM is employed then to solve resulting equations of the nonlinear Volterra integral and integro-differential forms of the Lane-Emden equations. The Volterra integral and integro-differential equations forms of the Lane-Emden equation overcome the singular behavior at the origin $x = 0$. Some examples are solved and different cases of the Lane-Emden equations of first kind are presented. The results demonstrate that the method has many merits such as being derivative-free, and overcoming the difficulty arising in calculating Adomian polynomials to handle the nonlinear terms in Adomian Decomposition Method (ADM). It does not require to calculate Lagrange multiplier as in Variational Iteration Method (VIM) and no need to construct a homotopy as in Homotopy Perturbation Method (HPM). The results prove that the present method is very effective and simple and does not require any restrictive assumptions for nonlinear terms. The software used for the calculations in this study was MATHEMATICA ® 8.0.

Keywords: Power series method, Volterra integral equation, Lane-Emden equations, Singular boundary value problem

1 Introduction

A variety of problems in physics, chemistry and biology have their mathematical setting as integral equations. Many methods have been developed to solve integral equations, especially nonlinear, which are receiving increasing attention.

Many attempts have been made to develop analytic and approximate methods to solve the linear and nonlinear Volterra integral and integro-differential equations, see [1–10]. Moreover, Chebyshev polynomials are applied for solving of nonlinear Volterra integral [11]. Although such methods have been successfully applied but some difficulties have appeared, for examples, construct a homotopy in HPM and solve the corresponding algebraic equations, in calculating Adomian polynomials to handle the nonlinear terms in ADM and calculate Lagrange multiplier in VIM, respectively.

Recently Tahmasbi and Fard [11, 12] have proposed a new technique namely the Power Series Method (PSM) for solving nonlinear and system of the second kind of Volterra integral equations, respectively. The PSM converges to the exact solution, if it exists, through simple calculations . However for concrete problems, a few approximations can be used for numerical purposes with high degree of accuracy. The PSM is simple to understand and easy to implement using computer packages and does not require any restrictive assumptions for nonlinear terms. In this paper, the applications of the PSM for both the linear and nonlinear Volterra integral and integro-differential equations and the resulting equations of the nonlinear Volterra integral and integro-differential equations forms of the Lane-Emden equations will be presented.

The results obtained in this paper are compared with those obtained by other iterative methods such as ADM [13], and the VIM [14].

The present paper has been organized as follows. In section 2 the power series method (PSM) is explained. In section 3 the analytical approximate solutions for linear and nonlinear Volterra integral equations by PSM are presented. In section 4 solving linear and nonlinear Volterra integro-differential equations by using PSM are given. In section 5 the applications of PSM for the Lane-Emden equation of first kind and how to convert it to Volterra integral and integro-differential equations (of first and second orders) and some illustrative cases are solved and finally in section 6 the conclusion is presented.

2 The power series method (PSM)

Consider the nonlinear Volterra integral equation of second kind

$$u(x) = f(x) + \int_0^x K(x,t)F(u(t))dt. \quad x \in [0,1] \tag{1}$$

the kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions, and $F(u(x))$ is a nonlinear function of $u(x)$ such as $u^2(x)$, $\sin(u(x))$ and $e^{u(x)}$.
Suppose the solution of Eq.(1) be as

$$u(x) = c_0 + c_1 x, \tag{2}$$

where $c_0 = f(0) = u(0)$ and c_1 is a unknown parameter. By Substituting Eq.(2) into Eq.(1) with simple calculations, we get

$$(ac_1 - b)x + Q(x^2) = 0. \tag{3}$$

where $Q(x^2)$ is a polynomial of order greater than one. By neglecting $Q(x^2)$, we have linear equation of c_1 in the form,

$$ac_1 = b. \tag{4}$$

the parameter c_1 of x in Eq.(2) is then obtained. In the next step, we assume that the solution of Eq.(1) is

$$u(x) = c_0 + c_1 x + c_2 x^2 \tag{5}$$

where c_0 and c_1 both are known and c_2 is unknown parameter. By Substituting Eq.(5) into Eq.(1), we get

$$(ac_2 - b)x^2 + Q(x^3) = 0, \tag{6}$$

where $Q(x^3)$ is a polynomial of order greater than two. By neglecting $Q(x^3)$, we have linear equation of c_2 in the form,

$$ac_2 = b \tag{7}$$

the unknown parameter c_2 of x^2 in Eq.(5) is then obtained. Having repeated the above procedure for m iterations, a power series of the following form is derived:

$$u(x) = c_0 + c_1 x + c_2 x^2 + ... + c_m x^m \tag{8}$$

Eq.(8) is an approximation for the exact solution $u(x)$ of Eq.(1) in the interval [0, 1].

Theorem 2.1 [11]
Let $u = u(x)$ be the exact solution of the following volterra integral eqution

$$u(x) = f(x) + \int_0^x K(x,t)[u(t)]^p dt \tag{9}$$

Then, the proposed method obtains the Taylor expansion of $u(x)$.

Corollary 2.2 [11]

If the exact solution to Eq.(9) be a polynomial, then the proposed method will obtain the real solution.

3 Analytical approximate solutions for linear and nonlinear Volterra integral equations by using PSM

In this section, the PSM is applied to solve the linear and nonlinear Volterra integral equations.

3.1 Linear Volterra integral equations

The standard form of the linear Volterra integral equation of second kind is given by [7, 15]

$$u(x) = f(x) + \int_0^x K(x,t)u(t)dt \tag{10}$$

Where, $K(x,t)$ and $f(x)$ are given functions, and $u(x)$ is unknown function occurs to the first power under the integral sign.
The PSM can be applied by following the same procedure as in section 2 to the following example:

Example 1: Consider the following linear Volterra integral equations [7]:

$$u(x) = 1 + x + \frac{x^3}{6} - \int_0^x (x-t)u(t)dt \tag{11}$$

$c_0 = u(0) = f(0) = 1$ as the initial condition, suppose the solution of Eq.(11) be

$$u(x) = c_0 + c_1 x = 1 + c_1 x \tag{12}$$

Substitute Eq.(12) in Eq.(11) we get

$$1 + c_1 x = 1 + x + \frac{x^3}{6} - \int_0^x (x-t)(1+c_1 t)dt$$

By integrating and solving we get (13)

$(c_1 - 1)x - (-\frac{x^2}{2} + (\frac{1}{6} - \frac{1}{6}c_1)x^3) = 0,$ by neglecting $(-\frac{x^2}{2} + (\frac{1}{6} - \frac{1}{6}c_1)x^3),$ therefore $c_1 = 1$

Substitute $c_1 = 1$ in Eq.(12) we get

$$u(x) = 1 + x. \tag{14}$$

Suppose the solution of Eq.(11) be as

$$u(x) = c_0 + c_1 x + c_2 x^2 = 1 + x + c_2 x^2 \tag{15}$$

Substitute Eq.(15) in Eq.(11) we get

$1 + x + c_2x^2 = 1 + x + \frac{x^3}{6} - \int\limits_0^x (x - t)(1 + t + c_2t^2)dt$

By integrating and solving we get (16)

$(c_2 + \frac{1}{2})x^2 + (c_2\frac{x^4}{12}) = 0,$ by neglecting $c_2\frac{x^4}{12},$ therefore $c_2 = -\frac{1}{2}$

Substitute $c_2 = -\frac{1}{2}$ in Eq.(15) we get

$$u(x) = 1 + x - \frac{x^2}{2}. \tag{17}$$

Suppose the solution of Eq.(11) be as

$u(x) = c_0 + c_1x + c_2x^2 + c_3x^3 = 1 + x - \frac{x^2}{2} + c_3x^3$ (18)

Substitute Eq.(18) in Eq.(11) we get

$1 + x - \frac{x^2}{2} + c_3x^3 = 1 + x + \frac{x^3}{6} - \int\limits_0^x (x - t)(1 + t - \frac{t^2}{2} + c_3t^3)dt$

By integrating and solving we get (19)

$c_3 - (\frac{1}{24}x^4 - c_3\frac{x^5}{20}) = 0,$ by neglecting $(\frac{1}{24}x^4 - c_3\frac{x^5}{20}),$ therefore $c_3 = 0$

Substitute $c_3 = 0$ in Eq.(18) we get

$$u(x) = 1 + x - \frac{x^2}{2}. \tag{20}$$

Continue by this way we will get series of the form $u(x) = x + (1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + ...,)$ so that gives the exact solution $u(x) = x + \cos x$ [7].

3.2 Nonlinear Volterra integral equation

The standard form of the nonlinear Volterra integral equation (VIE) of the second kind is given by [7, 15]

$$u(x) = f(x) + \int\limits_0^x K(x,t)F(u(t))dt \tag{21}$$

where, $F(u(t))$ is a nonlinear function of $u(t)$.
The algorithm of power series method is given by the same procedure as in section 2. The PSM will be applied to the following example:

Example 2: Consider the following nonlinear Volterra integral equation [7]:

$$u(x) = \frac{1}{4} + \frac{x}{2} + e^x - \frac{e^{2x}}{4} + \int\limits_0^x (x - t)u^2(t)dt. \tag{22}$$

$c_0 = u(0) = f(0) = 1$ as the initial condition, Suppose the solution of Eq.(22) be as

$$u(x) = c_0 + c_1 x = 1 + c_1 x \tag{23}$$

substitute Eq.(23) in Eq.(22) we get

$$1 + c_1 x = \tfrac{1}{4} + \tfrac{x}{2} + e^x - \tfrac{e^{2x}}{4} + \int_0^x (x-t)(1+c_1 t)^2 dt$$

By integrating and solving we get

$$(c_1 - 1)x - (\tfrac{1}{4} + e^x - \tfrac{e^{2x}}{4} + \tfrac{x^2}{2} + \tfrac{c_1 x^3}{3}) = 0, \text{ by neglecting } \quad (\tfrac{1}{4} + e^x - \tfrac{e^{2x}}{4} + \tfrac{x^2}{2} + \tfrac{1}{3}), \text{ therefore } c_1 = 1 \tag{24}$$

substitute $c_1 = 1$ in Eq.(23) we get

$$u(x) = 1 + x. \tag{25}$$

Suppose the solution of Eq.(22) be as

$$u(x) = c_0 + c_1 x + c_2 x^2 = 1 + x + c_2 x^2 \tag{26}$$

substitute Eq.(26) in Eq.(22) we get

$$1 + x + c_2 x^2 = 1 + c_1 x = \tfrac{1}{4} + \tfrac{x}{2} + e^x - \tfrac{e^{2x}}{4} + \int_0^x (x-t)(1+t+c3t^3)^2 dt$$

By integrating and solving we get

$$(c_2 - \tfrac{1}{2})x^2 - (\tfrac{1}{4} + e^x - \tfrac{e^{2x}}{4} + x + \tfrac{x^2}{2} + \tfrac{c_1 x^3}{6} + (\tfrac{1}{48} + \tfrac{c_2}{6})x^4) + \tfrac{c_2 x^5}{20} + \tfrac{c_2^2 x^6}{30}) = 0. \tag{27}$$

By neglecting $-(\frac{1}{4} + e^x - \frac{e^{2x}}{4} + x + \frac{x^2}{2} + \frac{c_1 x^3}{6} + (\frac{1}{48} + \frac{c_2}{6})x^4 + \frac{c_2 x^5}{20} + \frac{c_2^2 x^6}{30})$, therefore $c_2 = \frac{1}{2}$

Substitute $c_2 = \frac{1}{2}$ in Eq.(26) we get

$$u(x) = 1 + x + \frac{x^2}{2}. \tag{28}$$

Continue by this way we will get series of the form $u(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + ...$, so that gives the exact solution $u(x) = e^x$ [7].

4 Solving linear and nonlinear Volterra integro-differential equations by PSM

In this section some linear and nonlinear Volterra integro-differential equations will be solved by PSM to show effectiveness of PSM .

4.1 Solving linear Volterra integro-differential equations by PSM

Let us consider the following linear Volterra integro-differential equation of kth order.

$$u^k(x) = f(x) + \int_0^x K(x,t)u(t)dt. \tag{29}$$

where $u^{(k)}(x) = \frac{d^k}{dx^k}$. Because the resulted equation in (29) combines the differential operator and the integral operator, then it is necessary to define initial conditions $u(0), u'(0), ..., u^{(k-1)}(0)$ for the determination of the particular solution $u(x)$ of the Volterra integro-differential equation (29). Without loss of generality, we may assume a Volterra integro-differential equation of the second kind given by

$$u''(x) = f(x) + \int_0^x K(x,t)u(t)dt, \quad u(0) = \alpha_0, u'(0) = \alpha_1 \tag{30}$$

By integrating both sides of Eq.(30) twice from 0 to x and use initial the conditions we get,

$$u(x) = \alpha_0 + \alpha_1 x + \int_0^x \int_0^x f(x)dxdx + \int_0^x \int_0^x \int_0^x K(x,t)u(t)dtdxdx \tag{31}$$

The algorithm of power series method is given by the same procedure as in section 2. The PSM will be applied to the following example:

Example 3: Consider the following linear Volterra integro-differential equation of the second order:

$$u''(x) = 1 + x + \int_0^x (x-t)u(t)dt, \quad u(0) = 1, u'(0) = 1. \tag{32}$$

By integrating both sides of Eq.(32) twice, we get

$$u(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \int_0^x \int_0^x \int_0^x (x-t)u(t)dtdxdx. \tag{33}$$

$c_0 = u(0) = f(0) = 1$ as the initial condition,

Suppose the solution of Eq.(33) be

$$u(x) = c_0 + c_1 x = 1 + c_1 x \tag{34}$$

Substitute Eq.(34) in Eq.(33) we get

$$1 + c_1 x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \int_0^x \int_0^x \int_0^x (x-t)(1 + c_1 t)dtdxdx$$

By integrating and solving we get

$$(c_1 - 1)x - \left(\frac{x^2}{2} + \left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{24} + c_1\frac{x^5}{120}\right)\right) = 0. \tag{35}$$

By neglecting $\frac{x^2}{2} + (\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{24} + c_1\frac{x^5}{120})$, therefore $c_1 = 1$

Substitute $c_1 = 1$ in Eq.(34) we get

$$u(x) = 1 + x. \tag{36}$$

Suppose the solution of Eq.(33) be

$$u(x) = c_0 + c_1 x + c_2 x^2 = 1 + x + c_2 x^2 \tag{37}$$

Substitute Eq.(37) in Eq.(33) we get

$$1+x+c_2x^2 = 1+x+\frac{x^2}{2}+\frac{x^3}{6}+\int_0^x\int_0^x\int_0^x (x-t)(1+t+c_2t^2)dtdxdx$$

By integrating and solving we get (38)

$$(c_2-\frac{1}{2})x^2+(\frac{x^3}{3}+\frac{x^4}{24}+(\frac{x^5}{120}+(\frac{x^6}{360}c_2))=0.$$

By neglecting $(\frac{x^3}{3}+\frac{x^4}{24}+(\frac{x^5}{120}+(\frac{x^6}{360}c_2)$, therefore $c_2=\frac{1}{2}$.

Substitute $c_2=\frac{1}{2}$ in Eq.(37) we get

$$u(x) = 1+x+\frac{1}{2}x^2. \tag{39}$$

Continue by this way we will get series of the form $u(x)=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+...$, so that gives the exact solution $u(x)=e^x$.

4.2 Solving nonlinear Volterra integro-differential equations by PSM

Let us consider the following integro-differential equation of kth order.

$$u^k(x) = f(x)+\int_0^x K(x,t)F(u(t))dt, \tag{40}$$

where $u^{(k)}(x)=\frac{d^k}{dx^k}$. Because the resulted equation in (40) combines the differential operator and the integral operator, then it is necessary to define initial conditions $u(0), u'(0), ..., u^{(k-1)}(0)$ for the determination of the particular solution $u(x)$ of the Volterra integro-differential equation (40). Without loss of generality, we may assume a Volterra integro-differential equation of the second kind given by

$$u''(x) = f(x)+\int_0^x K(x,t)F(u(t))dt, \quad u(0)=\alpha_0, u'(0)=\alpha_1 \tag{41}$$

By integrating both sides of Eq.(41) twice from 0 to x and use the initial conditions, we get

$$u(x) = \alpha_0+\alpha_1 x+\int_0^x\int_0^x f(x)dxdx+\int_0^x\int_0^x\int_0^x K(x,t)F(u(t))dtdxdx \tag{42}$$

The algorithm of power series method is given by the same procedure as in section 2. The PSM will be applied to the following example:

Example 4: Consider the following nonlinear Volterra integro-differential equation of the first order:

$$u''(x) = -x^2-\frac{1}{4}x^5+xu(x)+\int_0^x xu(t)^3dt, \quad u(0)=0, u'(0)=1. \tag{43}$$

By integrating both sides of Eq.(43) twice, we get

$$u(x) = x-\frac{1}{8}x^4-\frac{1}{168}x^7+\int_0^x\int_0^x xu(x)dxdx+\int_0^x\int_0^x\int_0^x xu(t)^3dtdxdx. \tag{44}$$

$c_0=u(0)=f(0)=0$ as the initial condition, Suppose the solution of Eq.(44) be

$$u(x) = c_0 + c_1 x = c_1 x \tag{45}$$

Substitute Eq.(45) in Eq.(44) we get

$$c_1 x = x - \tfrac{1}{8}x^4 - \tfrac{1}{168}x^7 + \int_0^x\int_0^x x(c_1 x)dxdx + \int_0^x\int_0^x\int_0^x x(c_1 t)^3 dtdxdx.$$

By integrating and solving we get

$$(c_1 - 1)x - (\tfrac{1}{6}x^3 + (-\tfrac{1}{24} + c_1\tfrac{1}{12})x^4 + 3c_1x^5\tfrac{1}{40} + c_1^2x^6\tfrac{1}{30} + (-\tfrac{1}{168} + c_1^3\tfrac{1}{168})x^7) = 0$$

By neglecting $-((\tfrac{1}{6}x^3 + (-\tfrac{1}{24} + c_1\tfrac{1}{12})x^4 + 3c_1x^5\tfrac{1}{40} + c_1^2x^6\tfrac{1}{30} + (-\tfrac{1}{168} + c_1^3\tfrac{1}{168})x^7)$, therefore $c_1 = 1$. (46)

Substitute $c_1 = 1$ in Eq.(45) we get

$$u(x) = x. \tag{47}$$

Suppose the solution of Eq.(44) be

$$u(x) = c_0 + c_1 x + c_2 x^2 = x + c_2 x^2 \tag{48}$$

Substitute Eq.(48) in Eq.(44) we get

$$x + c_2x^2 = x - \tfrac{1}{8}x^4 - \tfrac{1}{168}x^7 + \int_0^x\int_0^x x(x + c_2x^2)dxdx + \int_0^x\int_0^x\int_0^x x(t + c_2t^2)^3 dtdxdx.$$

By integrating and solving we get (49)

$$c_2x^2 + (-\tfrac{1}{24}x^4 + \tfrac{c_2x^5}{20} + \tfrac{3c_2x^8}{280} + \tfrac{c_2^2x^9}{144} + \tfrac{c_2^3x^{10}}{630}) = 0.$$

By neglecting $-\tfrac{1}{24}x^4 + \tfrac{c_2x^5}{20} + \tfrac{3c_2x^8}{280} + \tfrac{c_2^2x^9}{144} + \tfrac{c_2^3x^{10}}{630}$, therefore $c_2 = 0$

Substitute $c_2 = 0$ in Eq.(48) we get

$$u(x) = x. \tag{50}$$

Note that $c_j = 0$ for $j \geq 2$, so that gives the exact solution $u(x) = x$ [16].

5 Applications

In this section the applications of the PSM for resulting equations of the nonlinear volterra integral and integro-differential equations forms of the Lane-Emden equation of first kind are presented.

5.1 The Lane-Emden equation of first kind

The Lane-Emden equation of the first kind in standard form [13]:

$$u'' + \frac{k}{x}u' + f[u(t)] = 0, u(0) = \alpha, u'(0) = 0, k \geq 0. \tag{51}$$

where $f[u(t)] = u^m$, Eq.(51) is a basic equation in the theory of stellar structure for $k = 2$. It is a useful equation in astrophysics for computing the structure of interiors of polytropic stars. This equation describes the temperature variation of a spherical gas could under the mutual attraction of its molecules and subject to the of laws of thermodynamics [13]. This equation is linear for $m = 0, 1$ and nonlinear otherwise. We convert the differential equation to integral equation for several reasons, through the integration we can rid of the singularity formula which is a problem in differential equations, and also in the integral equation we get more stable results [17]. Moreover, as indicated before, the power series method worked perfectly for the resulting integral equation rather than the original ODE, since it is required to convert the ODE to integral equation.

5.2 Convert the Lane-Emden equation of shape factor of 2 to volterra integral equation

The Lane-Emden equation of shape factor of 2 reads

$$u'' + \frac{2}{x}u' + f[u(t)] = 0, u(0) = \alpha, u'(0) = 0, k \geq 0. \tag{52}$$

To convert Eq. (52) to Volterra integral form, we follow the same procedure as in [13], however the prove is given here:

$$(x^2u')' = -x^2 f(u(t)) \tag{53}$$

$$u' = -\frac{1}{x^2}\int_0^x t^2 f[u(t)]dt$$

$$\int_0^x u'dx = -\int_0^x \frac{1}{x^2}\int_0^x t^2 f[u(t)]dtdx,$$

By integrating both sides of Eq.(53) twice

$$u(x) - u(0) = \int_0^x\int_0^x t^2 f[u(t)]dtd(\frac{1}{x})$$

then by integrating by part, we get,

$$= \left[\int_0^x t^2 f[u(t)]dt\frac{1}{x}\right]_0^x - \int_0^x \frac{1}{x}d\left(\int_0^x t^2 f[u(t)]dt\right)$$

$$= \int_0^x t^2 f[u(t)]dt\frac{1}{x} - \int_0^x \frac{1}{x}x^2 f[u(x)]dx$$

by using Leibnitz rule, we obtain

$$= \int_0^x \frac{t^2}{x} f[u(t)]dt - \int_0^x tf[u(t)]dt$$

$$u(x) - \alpha = \int_0^x t(\frac{t}{x} - 1)f[u(t)]dt$$

then the Lane-Emden equation of shape factor of 2 is

$$u(x) = \alpha + \int_0^x t(\frac{t}{x} - 1)f[u(t)]dt \tag{54}$$

By differentiating Eq.(54) twice, using Leibnitz rule, gives

$$u'(x) = -\int_0^x (\frac{t^2}{x^2}) f[u(t)]dt \tag{55}$$

$$u''(x) = -f[u(x)] + \int_0^x 2(\frac{t^2}{x^3}) f[u(t)]dt \tag{56}$$

Multiplying $u'(x)$ by $\frac{2}{x}$ adding the result to $u''(x)$ gives the Lane-Emden equation of shape factor of 2 in Eq.(52) .
This shows that the Volterra integral form Eq.(54) is the equivalent integral form for the Lane-Emden equation of shape factor of 2 [13].

5.3 The Lane-Emden equation of shape factor of k

The Lane-Emden equation of shape factor of k reads

$$u'' + \frac{k}{x}u' + f[u(t)] = 0, u(0) = \alpha, u'(0) = 0, k \geq 0, \tag{57}$$

If $k \neq 1$, by the same way that we got through it on the Eq.(54) we will get Eq. (59):

$$u(x) = \alpha - \frac{1}{k-1}\int_0^x t(1 - \frac{t^{k-1}}{x^{k-1}}) f[u(t)]dt \tag{58}$$

By differentiating Eq.(58) twice, using Leibnitz rule, gives

$$u'(x) = -\int_0^x (\frac{t^k}{x^k}) f[u(t)]dt \tag{59}$$

$$u''(x) = -f[u(x)] + \int_0^x (\frac{t^k}{x^{k+1}}) f[u(t)]dt \tag{60}$$

Multiplying $u'(x)$ by $\frac{k}{x}$ adding the result to $u''(x)$ gives the generlaized Lane-Emden equation Eq.(58). This shows that the Volterra integral form Eq.(58) is the equivalent integral form for the generlaized Lane-Emden equation [13].

If $k = 1$ the integral form is :

$$u(x) = \alpha + \int_0^x t(\ln \frac{t}{x}) f[u(t)]dt, \tag{61}$$

which can be obtained in limit as $k \to 1$ in Eq. (58) in fact, we have by L'Hospital's rule.

$$\lim_{k\to 1}[\frac{1}{k-1} - \{\frac{1}{(k-1)}(\frac{t}{x})^{k-1}\}]$$

$$= \lim_{k\to 1}\frac{1}{k-1} - \lim_{k\to 1}\frac{1}{k-1}(\frac{t}{x})^{k-1}$$

Applying L'Hospital's rule on each part, the first part

$$\lim_{k\to 1}\frac{1}{k-1} = 0$$

The second part is also applied by the same rule we obtain $f(k) = (\frac{t}{x})^{k-1}$ and $g(k) = \frac{1}{k-1}$

$f(k) = (\frac{t}{x})^{k-1}$ then $f'(k) = (\frac{t}{x})^{k-1} \ln \frac{t}{x}$, bcause $\left[\frac{d}{dx}(a^x) = a^x \ln a\right]$ and $g'(k) = 1$

Then:

$\lim_{k\to 1} \frac{1}{k-1} - \lim_{k\to 1} \frac{(\frac{t}{x})^{k-1}}{k-1}$

$= 0 - \lim_{k\to 1} \frac{f'(k)}{g'(k)}$

$= 0 - \frac{\lim_{k\to 1} (\frac{t}{x})^{k-1} \ln \frac{t}{x}}{1}$

$= 0 - \lim_{k\to 1} (\frac{t}{x})^{k-1} \ln (\frac{t}{x})$

$= - \ln \frac{t}{x}$

Therefore, the Volterra integral forms for the Lane-Emden equation is give by [13].

$$u(x) = \begin{cases} \alpha + \int_0^x t(\ln \frac{t}{x}) f[u(t)]dt, & \text{for} \quad k = 1, \quad f[u(t)] = u^m(t) \\ \alpha - \frac{1}{k-1} \int_0^x t(1 - \frac{t^{k-1}}{x^{k-1}}) f[u(t)]dt, & \text{for} \quad k > 0, k \neq 1 \end{cases} \tag{62}$$

The Volterra integral equation Eq.(62) and integro-differential equations Eqs.(59)-(60) forms of the Lane-Emden equation overcomes the singular behavior at the origin $x = 0$. The PSM will be applied to the following examples:

Example 5: Consider the following Volterra integral forms for the Lane-Emden equation when $k = 2, m = 0$:

$$u(x) = 1 - \int_0^x t(1 - \frac{t}{x})dt \tag{63}$$

$c_0 = u(0) = f(0) = 1$ as the initial condition,

Suppose the solution of Eq.(63) be as

$$u(x) = c_0 + c_1 x = 1 + c_1 x. \tag{64}$$

Substitute Eq.(64) in Eq.(63) we get

$1 + c_1 x = 1 - \int_0^x t(1 - \frac{t}{x})dt$

By integrating and solving we get (65)

$c_1 x + \frac{x^2}{6} = 0$, by neglecting $\frac{x^2}{6}$, therefore $c_2 = 0$.

Substitute $c_1 = 0$ in Eq.(64) we get

$$u(x) = 1. \tag{66}$$

Suppose the solution of Eq.(63) be as

$$u(x) = c_0 + c_1x + c_2x^2 = 1 + c_2x^2. \tag{67}$$

Substitute Eq.(67) in Eq.(63) we get

$$1 + c_2x^2 = 1 - \int_0^x t(1 - \tfrac{t}{x})dt$$

By integrating and solving we get (68)

$(c_2 + \frac{1}{6})x^2 = 0,$ by neglecting $\frac{x^2}{6}$, therefore $c_2 = -\frac{1}{6}$.

Substitute $c_2 = -\frac{1}{6}$ in Eq.(67) we get

$$u(x) = 1 - \frac{x^2}{6}. \tag{69}$$

Note that $c_j = 0$ for $j \geq 3$, so that the exact solution $u(x) = 1 - \frac{x^2}{6}$.

Example 6: Consider the following Volterra integral forms for the Lane-Emden equation when $k = 2, m = 1$:

$$u(x) = 1 - \int_0^x t(1 - \frac{t}{x})u(t)dt \tag{70}$$

$c_0 = u(0) = f(0) = 1$ as the initial condition,

Suppose the solution of Eq.(70) be as

$$u(x) = c_0 + c_1x = 1 + c_1x. \tag{71}$$

Substitute Eq.(71) in Eq.(70) we get

$$1 + c_1x = 1 - \int_0^x t(1 - \tfrac{t}{x})(1 + c_1t)dt$$

By integrating and solving we get (72)

$c_1x - \frac{x^2}{6} - c_1\frac{x^3}{12} = 0,$ by neglecting $\frac{x^2}{6} - c_1\frac{x^3}{12}$, therefore $c_1 = 0$.

Substitute $c_1 = 0$ in Eq.(71) we get

$$u(x) = 1. \tag{73}$$

Suppose the solution of Eq.(70) be as

$$u(x) = c_0 + c_1x + c_2x^2 = 1 + c_2x^2. \tag{74}$$

Substitute Eq.(74) in Eq.(70) we get

$$1 + c_2x^2 = 1 - \int_0^x t(1 - \tfrac{t}{x})(1 + c_2t^2)dt.$$

By integrating and solving we get (75)

$$(c_2 + \tfrac{x^2}{6})x^2 - c_2\tfrac{x^4}{20} = 0, \text{ by neglecting } \tfrac{x^2}{6}, \text{therefore } c_2 = -\tfrac{1}{6}.$$

Substitute $c_2 = -\frac{1}{6}$ in Eq.(74) we get

$$u(x) = 1 - \frac{x^2}{6}. \quad (76)$$

Suppose the solution of Eq.(70) be as

$$u(x) = c_0 + c_1x + c_2x^2 + c_3x^3 = 1 - \tfrac{x^2}{6} + c_3x^3. \quad (77)$$

Substitute Eq.(77) in Eq.(70) we get

$$1 - \tfrac{x^2}{6} + c_3x^3 = 1 - \int_0^x t(1 - \tfrac{t}{x})(1 - \tfrac{t^2}{6} + c_3t^3)dt.$$

By integrating and solving we get

$$c_3x^3 + (-\tfrac{x^2}{6} + \tfrac{x^2}{6} + \tfrac{x^4}{24} - c_3\tfrac{x^5}{6} + ...) = 0. \text{ By neglecting}(-\tfrac{x^2}{6} + \tfrac{x^2}{6} + \tfrac{x^4}{24} - c_3\tfrac{x^5}{6} + ...), \text{therefore } c_3 = 0. \quad (78)$$

Substitute $c_3 = 0$ in Eq.(77) we get

$$u(x) = 1 - \frac{x^2}{6}. \quad (79)$$

Continue by this way we will get series of the form $u(x) = 1 - \frac{x^2}{6} + \frac{1}{120}x^4 + ...,$, so that the exact solution $u(x) = \frac{\sin x}{x}$.

Example 7: Consider the following Volterra integral forms for the Lane-Emden equation when $k = 2, m = 5$:

$$u(x) = 1 - \int_0^x t(1 - \frac{t}{x})u(t)^5dt \quad (80)$$

$c_0 = u(0) = f(0) = 1$ as the initial condition,

Suppose the solution of Eq.(80) be as

$$u(x) = c_0 + c_1x = 1 + c_1x. \quad (81)$$

Substitute Eq.(81) in Eq.(80) we get

$$1 + c_1 x = 1 - \int_0^x t(1 - \tfrac{t}{x})(1 + c_1 t)^5 dt.$$

By integrating and solving we get

$$c_1 x - (-\tfrac{x^2}{6} - c_1 \tfrac{5x^3}{12} - c_1^2 \tfrac{x^4}{2} - c_1^3 \tfrac{x^5}{3} - c_1^4 \tfrac{5x^6}{42} - c_1^5 \tfrac{x^7}{56}) = 0. \quad (82)$$

By neglecting $-(-\tfrac{x^2}{6} - c_1 \tfrac{5x^3}{12} - c_1^2 \tfrac{x^4}{2} - c_1^3 \tfrac{x^5}{3} - c_1^4 \tfrac{5x^6}{42} - c_1^5 \tfrac{x^7}{56})$, therefore $c_1 = 0$.
Substitute $c_1 = 0$ in Eq.(81) we get

$$u(x) = 1. \quad (83)$$

Suppose the solution of Eq.(80) be

$$u(x) = c_0 + c_1 x + c_2 x^2 = 1 + c_2 x^2. \quad (84)$$

Substitute Eq.(84) in Eq.(80) we get

$$1 + c_2 x^2 = 1 - \int_0^x t(1 - \tfrac{t}{x})(1 + c_2 t^2)^5 dt.$$

By integrating and solving we get

$$(c_2 + \tfrac{1}{6})x^2 - (-c_2 \tfrac{42x^4}{84} + ...) = 0, \text{ by neglecting } \quad c_2 \tfrac{42x^4}{84} + ..., \text{therefore } c_2 = -\tfrac{1}{6}. \quad (85)$$

Substitute $c_2 = -\frac{1}{6}$ in Eq.(84) we get

$$u(x) = 1 - \frac{x^2}{6}. \quad (86)$$

Suppose the solution of Eq.(80) be

$$u(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 = 1 - \tfrac{x^2}{6} + c_3 x^3. \quad (87)$$

Substitute Eq.(87) in Eq.(80) we get

$$1 - \tfrac{x^2}{6} + c_3 x^3 = 1 - \int_0^x t(1 - \tfrac{t}{x})(1 - \tfrac{t^2}{6} + c_3 t^3)^5 dt.$$

By integrating and solving we get

$$c_3 x^3 - (-\tfrac{x^2}{6} + \tfrac{x^4}{24} - c_3 \tfrac{x^5}{6} + ...) = 0, \text{ by neglecting } \quad (-\tfrac{x^2}{6} + \tfrac{x^4}{24} - c_3 \tfrac{x^5}{6} + ...), \text{therefore } c_3 = 0. \quad (88)$$

Substitute $c_3 = 0$ in Eq.(87) we get

$$u(x) = 1 - \frac{x^2}{6}. \quad (89)$$

Continue by this way we will get series of the form $u(x) = 1 - \frac{x^2}{6} + \frac{x^4}{24} - \frac{5x^6}{432} + ...$, the approximate solution is a series form, which is the same as the result obtained by ADM in [13].

Example 8: Consider the following Volterra integral forms for the Lane-Emden equation when $k = 2, m = 2$:

$$u(x) = 1 - \int_0^x t(1 - \frac{t}{x})u(t)^2 dt \tag{90}$$

$c_0 = u(0) = f(0) = 1$ as the initial condition,

Suppose the solution of Eq.(90) be as

$$u(x) = c_0 + c_1 x = 1 + c_1 x. \tag{91}$$

Substitute Eq.(91) in Eq.(90) we get

$$1 + c_1 x = 1 - \int_0^x t(1 - \tfrac{t}{x})(1 + c_1 t)^2 dt.$$

By integrating and solving we get (92)

$$c_1 x - (-\tfrac{x^2}{6} - c_1 \tfrac{x^3}{6} - c_1^2 \tfrac{x^4}{20}) = 0, \text{ by neglecting } (-\tfrac{x^2}{6} - c_1 \tfrac{x^3}{6} - c_1^2 \tfrac{x^4}{20}), \text{therefore } c_1 = 0.$$

Substitute $c_1 = 0$ in Eq.(91) we get

$$u(x) = 1. \tag{93}$$

Suppose the solution of Eq.(90) be

$$u(x) = c_0 + c_1 x + c_2 x^2 = 1 + c_2 x^2. \tag{94}$$

Substitute Eq.(94) in Eq.(90) we get

$$1 + c_2 x^2 = 1 - \int_0^x t(1 - \tfrac{t}{x})(1 + c_2 t^2)^2 dt.$$

By integrating and solving we get

$$(c_2 + \tfrac{1}{6})x^2 - (-\tfrac{x^2}{6} - c_2 \tfrac{x^4}{10} - c_2^2 \tfrac{x^6}{42}) = 0, \text{ by neglecting } -\tfrac{x^2}{6} - c_2 \tfrac{x^4}{10} - c_2^2 \tfrac{x^6}{42}, \text{therefore } c_2 = -\tfrac{1}{6}. \tag{95}$$

Substitute $c_2 = -\frac{1}{6}$ in Eq.(94) we get

$$u(x) = 1 - \frac{x^2}{6}. \tag{96}$$

Suppose the solution of Eq.(90) be

$$u(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 = 1 - \tfrac{x^2}{6} + c_3 x^3. \tag{97}$$

Substitute Eq.(97) in Eq.(90) we get

$1-\frac{x^2}{6}+c_3x^3=1-\int\limits_0^x t(1-\frac{t}{x})(1-\frac{t^2}{6}+c_3t^3)^2dt.$

By integrating and solving we get (98)

$c_3x^3-(-\frac{x^2}{6}+\frac{x^4}{60}-c_3\frac{x^5}{15}+...)=0.$ By neglecting $(-\frac{x^2}{6}+\frac{x^4}{60}-c_3\frac{x^5}{15}+...)$, therefore $c_3=0$.

Substitute $c_3=0$ in Eq.(97) we get

$$u(x)=1-\frac{x^2}{6}. \tag{99}$$

Continue by this way we will get series of the form $u(x)=1-\frac{x^2}{6}+\frac{x^4}{60}-\frac{11x^6}{7560}+...$, the approximate solution is a series form which is the same as the result obtained by ADM in [13].

5.4 Solving Volterra integro-differential forms of the Lane-Emden equations

In this section we will discuss Volterra integro-differential forms of the Lane-Emden equations of first and second order given in Eq.(55) and Eq. (56), respectively and will be solved by PSM.

5.4.1 Solving Volterra integro-differential forms of the Lane-Emden equations of first order

We will solve the Volterra integro-differential forms of the Lane-Emden equations given in Eq.(55) by PSM, let us consider the following integro-differential equation of first order.

$$u'(x)=-\int\limits_0^x(\frac{t^k}{x^k})u^m(t)dt, \quad u(0)=\alpha \tag{100}$$

By integrating both sides of Eq.(100) and using the initial conditions, we get

$$\int\limits_0^x u'(x)dx=-\int\limits_0^x\int\limits_0^x(\frac{t^k}{x^k})u^m(t))dtdx \tag{101}$$

$$u(x)=\alpha-\int\limits_0^x\int\limits_0^x(\frac{t^k}{x^k})u^m(t)dtdx \tag{102}$$

The algorithm of power series method is given by the same procedure as in section 2. The PSM will be applied to the following example:

Example 9: Consider the following nonlinear Volterra integro-differential forms for the Lane-Emden equation of the first order when $m=1,k=2$:

$$u'(x)=-\int\limits_0^x\frac{t^2}{x^2}u(t)dt, \quad u(0)=1. \tag{103}$$

By integrating both sides of Eq.(103), we get

$\int\limits_0^x u'(x)dx=\int\limits_0^x\int\limits_0^x\frac{t^2}{x^2}u(t)dtdx,$

$$u(x) = 1 - \int_0^x \int_0^x \frac{t^2}{x^2} u(t) dt dx. \quad (104)$$

$c_0 = u(0) = f(0) = 1$ as the initial condition,

Suppose the solution of Eq.(104) be as

$$u(x) = c_0 + c_1 x = 1 + c_1 x. \quad (105)$$

Substitute Eq.(105) in Eq.(104) we get

$$1 + c_1 x = 1 - \int_0^x \int_0^x \frac{t^2}{x^2}(1 + c_1 t) dt dx.$$

By integrating and solving we get (106)

$$c_1 x + (\tfrac{x^2}{6} + c_1 \tfrac{x^3}{12}) = 0, \text{ by neglecting } \tfrac{x^2}{6} + c_1 \tfrac{x^3}{12}, \text{therefore } c_1 = 0.$$

Substitute $c_1 = 0$ in Eq.(105) we get

$$u(x) = 1. \quad (107)$$

Suppose the solution of Eq.(104) be as

$$u(x) = c_0 + c_1 x + c_2 x^2 = 1 + c_2 x^2. \quad (108)$$

Substitute Eq.(108) in Eq.(104) we get

$$1 + c_2 x^2 = 1 - \int_0^x \int_0^x \frac{t^2}{x^2}(1 + c_2 t^2) dt dx.$$

By integrating and solving we get (109)

$$(c_2 - \tfrac{1}{6})x^2 + \tfrac{c_2 x^4}{20} = 0, \text{ by neglecting } \tfrac{c_2 x^4}{20}, \text{therefore } c_2 = -\tfrac{1}{6}.$$

Substitute $c_2 = -\frac{1}{6}$ in Eq.(108) we get

$$u(x) = 1 - \frac{1}{6}x^2. \quad (110)$$

Suppose the solution of Eq.(104) be as

$$u(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 = 1 - \tfrac{1}{6}x^2 + c_3 x^3. \quad (111)$$

Substitute Eq.(111) in Eq.(104) we get

$$1 - \tfrac{1}{4}x^2 + c_3 x^3 = 1 - \int_0^x \int_0^x \frac{t^2}{x^2}(1 - \tfrac{1}{4}t^2 + c_3 t^3) dt dx.$$

By integrating and solving we get (112)

$$c_3 x^3 + (\tfrac{x^2}{6} - \tfrac{x^4}{120} + c_3 \tfrac{x^5}{30}) = 0, \text{ by neglecting } \tfrac{x^2}{6} - \tfrac{x^4}{120} + c_3 \tfrac{x^5}{30}, \text{therefore } c_3 = 0.$$

Substitute $c_3 = 0$ in Eq.(111) we get

$$u(x) = 1 - \frac{1}{6}x^2. \tag{113}$$

Continue by this way we will get series of the form $u(x) = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - ...$, so the exact solution is $\frac{\sin x}{x}$, which is the same result when we solved the Volterra integral equation in example 6.

5.4.2 Solving Volterra integro-differential forms of the Lane-Emden equations of second order

We will solve Volterra integro-differential forms of the Lane-Emden equations of second order given in (56) by PSM.

$$u''(x) = -f(u(x)) + \int_0^x k(\frac{t^k}{x^{k+1}})f(u(t))dt, u(0) = \alpha, u'(0) = 0 \tag{114}$$

By integrating both sides of Eq.(114) twice and using the initial conditions, we get

$$\int_0^x u'(x)dx = -\int_0^x f(u(x))dx + \int_0^x\int_0^x k(\frac{t^k}{x^{(k+1)}})u(t)dtdx, \tag{115}$$

$$u(x) = \alpha - \int_0^x\int_0^x f(u(x))dxdx + \int_0^x\int_0^x\int_0^x k(\frac{t^k}{x^{k+1}})f(u(t))dt \tag{116}$$

The algorithm of power series method is given by the same procedure as in section 2. The PSM will be applied to the following example:

Example 10: Consider the following nonlinear Volterra integro-differential forms for the Lane-Emden equation of the second order where $m = 1, k = 2$:

$$u''(x) = -f(u(x) + \int_0^x 2\frac{t^2}{x^3}u(t)dt, \quad u(0) = 1, u'(0) = 0. \tag{117}$$

By integrating both sides of Eq.(117) twice, we get

$$u(x) = 1 - \int_0^x\int_0^x f(u(x))dxdx + \int_0^x\int_0^x\int_0^x 2(\tfrac{t^2}{x^3})u(t)dtdxdx. \tag{118}$$

$c_0 = u(0) = f(0) = 1$ as the initial condition,

$$u(x) = c_0 + c_1x = 1 + c_1x. \tag{119}$$

Substitute Eq.(119) in Eq.(118) we get

$$1 + c_1x = 1 - \int_0^x\int_0^x f(u(x))dxdx + \int_0^x\int_0^x\int_0^x 2(\tfrac{t^2}{x^3})1 + c_1tdtdxdx.$$

By integrating and solving we get (120)

$$c_1x + (\tfrac{x^2}{6} + c_1\tfrac{x^3}{12}) = 0, \text{ by neglecting } \tfrac{x^2}{6} - c_1\tfrac{x^3}{12}, \text{therefore } c_1 = 0.$$

Substitute $c_1 = 0$ in Eq.(119) we get

$$u(x) = 1. \tag{121}$$

$$u(x) = c_0 + c_1x + c_2x^2 = 1 + c_2x^2. \tag{122}$$

Substitute Eq.(122) in Eq.(118) we get

$$1 + c_2x^2 = 1 - \int_0^x\int_0^x f(u(x))dxdx + \int_0^x\int_0^x\int_0^x 2(\tfrac{t^2}{x^3})(1 + c_2t^2)dtdxdx.$$

By integrating and solving we get (123)

$(c_2 - \frac{1}{6})x^2 + \frac{c_2x^4}{20} = 0,$ by neglecting $\frac{c_2x^4}{20}$, therefore $c_2 = -\frac{1}{6}$.

Substitute $c_2 = -\frac{1}{6}$ in Eq.(122) we get

$$u(x) = 1 - \frac{1}{6}x^2. \tag{124}$$

$$u(x) = c_0 + c_1x + c_2x^2 + c_3x^3 = 1 - \tfrac{1}{6}x^2 + c_3x^3. \tag{125}$$

Substitute Eq.(125) in Eq.(118) we get

$$1 + c_3x^3 = 1 - \int_0^x\int_0^x f(u(x))dxdx + \int_0^x\int_0^x\int_0^x 2(\tfrac{t^2}{x^3})(1 + c_3t^3)dtdxdx.$$

By integrating and solving we get (126)

$c_3x^3 + (\frac{x^2}{6} - \frac{x^4}{120} + c_3\frac{x^5}{30}) = 0,$ by neglecting $\frac{x^2}{6} - \frac{x^4}{120} + c_3\frac{x^5}{30}$, therefore $c_3 = 0$.

Substitute $c_3 = 0$ in Eq.(125) we get

$$u(x) = 1 - \frac{1}{6}x^2. \tag{127}$$

Continue by this way we will get series of the form $u(x) = 1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - ...$, so the exact solution is $\frac{\sin x}{x}$, which is the same result when we solved the Volterra integral equation in example 6.

It can also be clearly seen that from examples 6, 9 and 10 the solutions of the Volterra integral equation Eq.(62) and integro-differential equations Eqs.(59)-(60) forms of the Lane-Emden equations are equivalent.

6 Conclusion

In this paper, we implement the power series method (PSM) to obtain an analytical approximate solutions for solving linear and nonlinear Volterra integral or integro-differential equations. Then the applications of PSM for solving Volterra integral or integro-differential forms of Lane-Emden equations also given. The proposed Volterra integral or integro-differential forms facilitates the computational work and overcomes the difficulty of the singular behavior at $x = 0$ of the original initial value problem of the Lane-Emden equations. Moreover, the results obtained in current paper are in a complete agreement with the results by ADM [13] and VIM [14]. Furthermore, the PSM is simple to understand and easy to implement and does not require any restrictive assumptions for nonlinear terms as required by some existing techniques. Also, this method reproduces the analytical solution when the exact solutions are polynomial. It is economical in terms of computer power/memory and does not involve tedious calculations. It is worth to mention here, by solving some examples, it is seems that the PSM appears to be very accurate to employ with reliable results.

References

[1] Krystyna S., Tadeusz S. (2000). A higher-order finite difference method for solving a system of integro-differential equations. Journal of Computational and Applied Mathematics 126, 33-46.

[2] K. Maleknejad, F. Mirzaee and S. Abbasbandy (2004). Solving linear integro-differential equations system by using rationalized Haar functions method. Applied Mathematics and Computation 155, 317-328.

[3] K. Maleknejad, F. Mirzaee and S. Abbasbandy (2004). Solving linear integro-differential equation system by Galerkin methods with hybrid functions. Applied Mathematics and Computation 159, 603-612.

[4] Elcin Yusufoǧlu (Agadjanov) (2007). An efficient algorithm for solving integro-differential equations system. Applied Mathematics and Computation 192, 51-55.

[5] J.S.Nadjafi, M.Tamamgar. (2008). The variational iteration method: A highly promising method for solving the system of integro-differential equations. Computers and Mathematics with Applications 56, 346-351.

[6] S. Abbasbandy, A. Taati. (2009). Numerical solution of the system of nonlinear Volterra integro-differential equations with nonlinear differential part by the operational Tau method and error estimation. Journal of Computational and Applied Mathematics 231,106-113.

[7] Wazwaz, A.M. (2010).The variational iteration method for solving linear and nonlinear Volterra integral and integro-differential equations. International Journal of Computer Mathematics 87, 1131-1141.

[8] J.H. He.(2000).Variational iteration method for autonomous ordinary differential systems. Applied Mathematics and Computation 114, 115-123.

[9] J.H. He.(2007).Variational iteration method – Some recent results and new interpretations. Journal of Computational and Applied Mathematics 207, 3-17.

[10] J.H. He, X.H. Wu. (2007). Variational iteration method: New development and applications. Computers & Mathematics with Applications 54, 881-894.

[11] A.Tahmasbi, O.S.Fard.(2008). Numerical Solution of Nonlinear Volterra Integral Equations of the Second Kind by Power Series. Journal of Information and Computing Science 3, 57-61.

[12] A.Tahmasbi, O.S.Fard. (2008). Numerical solution of linear Volterra integral equations system of the second kind. Applied Mathematics and Computation 201, 547-552.

[13] Wazwaz, A.M., Randolph R., Jun-Sheng D.(2013). Adomian decomposition method for solving the Volterra integral form of the LaneEmden equations with initial values and boundary conditions. Applied Mathematics and Computation 219, 5004-5019.

[14] Wazwaz, A.M. (2014). The variational iteration method for solving the Volterra integro-differential forms of the Lane-Emden equations of the first and the second kind. Journal of Mathematical Chemistry, 52, 613-626.

[15] Wazwaz, A.M. (2011). Linear and Nonlinear Integral Equations Methods and Applications. Beijing and Springer-Verlag Berlin Heidelberg.

[16] B. Basirat, M.A. Shahdadi.(2013). Numerical Solution of Nonlinear Integro-Differential Equations with Initial Conditions by Bernstein Operational Matrix of Derivative. International Journal of Modern Nonlinear Theory and Application, 2,141-149.

[17] Chambers, L.I.G.(1976). Integral Equation: A short Course. International textbook company limited.