Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.4, No.9, 2014

On best one-sided approximation by multivariate Lagrange Interpolation in weighted spaces

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Abstract:

Degree of best one-sided approximation of multivariate function f that lies in weighted space ($L_{p,\alpha}$-space) by construct an operator $\boldsymbol{L}_{\boldsymbol{n}}^{\mp}$ which is dependent on multivariate Lagrange Interpolation L_n that satisfies:

$$\|f - L_n\|_{\infty,\alpha(\mathrm{X})} \leq \sum_{\mu\in\Lambda_\mu} P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu) \int_{R^d} D_{x-x_{x^\mu}^{(n)}} D_{x^\mu}^n f(y) M(y|x^\mu, x)dy$$

Where the domain $\mathrm{X} = [-1,1]^d$ has been studied in this work. The result which we end in it for the function $f \in W_{p,\alpha}^{\Lambda}(\Omega), \Omega \subseteq \mathrm{X}$(i.e. f lies in Sobolov Space) that the degree of best one-sided approximation of derivative of a function f which lies in the Sobolev space $W_p^{\Lambda}(\Omega)$ is minimized when the domain of f is restricted into rectangle domain Q in X^d where:

$Q = \prod_{i=1}^{d}[a_i - b_i]$ and $\delta_i = b_i - a_i > 0$,direct theorem is proved in this work by construct a new form of modulus of continuity and with benefit from properties of simplex spline $M(y|v^0, \dots \dots . v^n)$. Finally we shall prove invers theorem of best one sided approximation of the function f by using an operator $\boldsymbol{L}_n^{\mp}(x)$, the same result of best approximation of the function f and the relation between best and best one sided approximation.

1. Introduction:-

Throughout this paper, we use the weight function $\omega_\alpha(x) = e^{-\alpha \prod_{i=1}^{d} x_i}$ which is non-negative measurable function on R^d.For $1 \leq p \leq \infty$ the weighted space is define by:

$L_{p,\alpha=} \{f | f: X \longrightarrow R , |f(x)\omega_\alpha(x)| \leq M, \alpha \geq 1\}$ Such that for the function$f$,

$$\|f\|_{p,\alpha} = \left[\int_a^b |f(x)|^p e^{-\alpha p x} \, dx\right]^{\frac{1}{p}} < \infty, \text{ also}$$

$\|f\|_{\infty,\alpha} = \{sup\{|f(x)e^{-\alpha x}|, x \in X\}\} < \infty,\ X = [-1,1]^d, f: R^d \longrightarrow R$ and degree of best one sided approximation of the function $f \in L_{p,\alpha(X)}$ is defined as:

$\tilde{E}_n(f)_{p,\alpha[-1,1]^d} = \underbrace{inf}_{P_n^{\mp} \in \boldsymbol{P}_n^d} \|P_n^+ + P_n^-\|_{p,\alpha(X)}$ (Where $\boldsymbol{P}_n^d$ is the space of all polynomials of degree n and d variables and $P_n^-(x) \le f(x) \le P_n^+(x)$) and degree of best approximation of is defined as:

$E_n(f)_{p,\alpha[-1,1]^d} = \underbrace{inf}_{P_n \in \boldsymbol{P}_n^d} \|f - P_n\|_{p,\alpha(X)}$. For $\alpha \in R^d$ we have $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots \dots x_d^{\alpha_d}$ .Now for $n \in N_0$ there are $r_n^d = \binom{n+d-1}{n}$ monomials x^α which have total degree n ,a natural basis for $\boldsymbol{P}_n^d$ is formed by $\{x^\alpha, 0 \le |\alpha| \le n\}$.Let $Y = \{x_0, x_1, \dots\}$ be a sequence of pairwise distinct points in R^d also that $Y_N = \{x_0, x_1, \dots x_N\}$ if $N = \dim \boldsymbol{P}_n^d$ and if there is a unique polynomial $P \in \boldsymbol{P}_n^d$ such that:

$P(x_k) = f(x_k), 1 \le k \le N$ for each $f: R^d \longrightarrow R$ then we say that the Lagrange interpolation problem is poised with respect to Y_N in $\boldsymbol{P}_n^d$ and we denoted for the polynomial P by$L_n(f, x)$,we call the (possibly infinite) sequence Y poised in block if for any $n \in N_0$ the Lagrange interpolation problem is poised with respect to Y_N in $\boldsymbol{P}_n^d$ where $N = \dim \boldsymbol{P}_n^d$ .For $\boldsymbol{P}_n^d$ we have a whole block of monomials of degree n in several variables called the monomials $x^\alpha, |\alpha| = n$, if we arrange the multi-indices $|\alpha| = n$ in lexicographical order, we can number the monomials of total degree n as $q_1^{[n]}, \dots q_{r_n^d}^{[n]}$ and $\boldsymbol{P}^d$ is spanned by :

$$\{q_1^{[0]} | q_1^{[1]}, \dots, q_d^{[1]} \left| q_1^{[2]}, \dots, q_{r_2^d}^{[2]} \right| \dots \left| q_1^{[n]}, \dots, q_{r_n^d}^{[n]} \right| \dots\}$$

From this block wise viewpoint, it is only natural to group the interpolation points Y according to the structure and we rewrite them as:

$$Y = \{x_1^{[0]} | x_1^{[1]}, \dots, x_d^{[1]} \left| x_1^{[2]}, \dots, x_{r_2^d}^{[2]} \right| \dots \left| x_1^{[n]}, \dots, x_{r_n^d}^{[n]} \right| \dots\}$$

We refer that whenever Y is poised in block, the pointe $x_1, x_2, \dots$ can be arranged in such a way. If Y is poised in block, the nth Newton fundamental polynomial, denoted by $P_j^{[n]} \in \boldsymbol{P}_n^d$, $1 \le j \le r_n^d$ and is uniquely defined by the conditions:

$P_j^{[n]}\left(x_i^{(k)}\right) = 0, k < n \; and \; P_j^{[n]}\left(x_i^{[n]}\right) = \delta_{ij}, i = 1, \ldots, r_n^d$. Also we introduce the vectors:

$$P^n(x) = [x^\alpha]_{|\alpha| \le n}, P^n(x) \in R^N, N = \dim P_n^d$$

This means that for each level there are r_n^d Newton fundamental polynomials of degree exactly n which vanish on all points of lower level and all points of the n th level except the one which has the same index Next, we recall the definition and some properties of simplex spline as in **[1],** given $n + 1 \ge d + 1$ knots $v^0, \ldots v^n \in R^d$ the simplex spline $M(x|v^0, \ldots v^n)$ is defined by the condition:

$$\int_{R^d} f(x) M(x|v^0, \ldots v^n) dx = (n-d)! \int_{S_n} f(\propto_0 v^0 + \cdots \propto_n v^n) d \propto \ldots\ldots\ldots\ldots\ldots \text{(1-1)}$$

$f \in C(R^d)$ Where $S_n = \{\propto = (\propto_0, \ldots, \propto_d) : \propto_i \ge 0, \propto_0 + \cdots + \propto_d = 1\}$ .To exclude cases of degeneration which can be handled similarly, let us assume here that the convex hull of the knots, $[v^0, \ldots v^n]$ has dimension d. Then the simplex spline $M(x|v^0, \ldots v^n)$ is a nonnegative piecewise polynomial of degree $n - d$, supported on $[v^0, \ldots v^n]$. The order of differentiability depends on the position of the knots; if, e.g., the knots are in general position (i.e. any subset of $d + 1$ knots spans a proper simplex, then the simplex spline has maximal order of differentiability, namely $n - d + 1$. The most important property for our present purposes is the formula for directional derivatives namely, $D_y M(x|v^0, \ldots v^n) = \sum_{j=0}^n u_j M\left(x \middle| v^0, \ldots, v^{j-1}, v^{j+1}, \ldots, v^n\right)$ where $y = \sum_{j=0}^n u_j v^j, \sum_{j=0}^n u_j$ in particular for $0 \le i, j \le n$.

$D_{v^i - v^j} = M\left(x \middle| v^0, \ldots, v^{i-1}, v^{i+1}, \ldots, v^n\right) - M\left(x \middle| v^0, \ldots, v^{j-1}, v^{j+1}, \ldots, v^n\right)$.Now

let $x^0, x^1, \ldots$ be poised in blocked, we introduce the vectors

$X^n = [x_1^{(n)}, \ldots, x_{r_n^d}^{(n)}]$ And $x = [x^0, x^1, \ldots]$ so we define the finite difference in R^d which denoted by $\lambda_n [x^0, \ldots, x^{n-1}, x] f, x \in R^d$ as:

$$\lambda_0[x] f = f(x)$$

$$\lambda_{n+1} [x^0, \ldots, x^n, x] = \lambda_n [x^0, \ldots, x^{n-1}, x] f - \sum_{i=1}^{r_n^d} \lambda_n [x^0, \ldots, x^{n-1}, x_i^n] f . P_i^{[n]}(x).$$

If $d = 1$ and if we have ordered points $x_0 < x_1 < \cdots < x_n \in R$ then there exists one fundamental polynomial, given by:

$P_1^{[n]}(x) = \frac{(x-x_0)\ldots(x-x_{n-1})}{(x_n-x_0)\ldots(x_n-x_{n-1})}$, so the classical divided difference of a function of one variable is defined:

$\lambda_{n+1}\,[x^0, \ldots, x^n] = f[x^0, \ldots, x^n](x_n - x_0) \ldots (x_n - x_{n-1})$.Also in **[1]** we establish a representation of our finite difference in terms of simplex splines let:

$\Lambda_n = \{\mu = (\mu_0, \mu_{1,\ldots,}\mu_n) \in N^{n+1} : 1 \leq \mu_i \leq r_i^d, i = 0, \ldots, n\}$ be an index set each $\mu \in \Lambda_n$ define a path among the components of $x^0, \ldots, x^n$ which we denoted by x^μ, $x^\mu = \{x_{\mu_0}^{(0)}, \ldots, x_{\mu_n}^{(n)}\}$ we note that $\mu_0 = 1$ by definition thus the path described by x^μ starts from $x_1^{(0)}$,passes through $x_{\mu_1}^{(1)}, \ldots, x_{\mu_{n-1}}^{(n-1)}$ and ends at $x_{\mu_n}^{(n)}$,the collection $\{x^\mu : \mu \in \Lambda_\mu\}$ contains all paths from the sole point $x_1^{(0)}$of level 0 to all the pointe of level n .For any path $x^\mu, \mu \in \Lambda_\mu$ we define the nth directional derivative along that path as:

$\mathrm{D}_{x^\mu}^n = D_{x_{\mu_n}^{(n)} - x_{\mu_{n-1}}^{(n-1)}} D_{x_{\mu_{n-1}}^{(n-1)} - x_{\mu_{n-2}}^{(n-2)}} \ldots D_{x_{\mu_1}^{(1)} - x_{\mu_0}^{(0)}}, \mu \in \Lambda_\mu$ And we will need the values $\pi_\mu(x^\mu) = \prod_{i=0}^{n-1} P_{\mu_i}^{[i]}\left(x_{\mu_{i+1}}^{(i+1)}\right), \mu \in \Lambda_\mu$.

In this search we introduce a new form of modulus of smoothness in multivariate case as the following:

$\Delta_t^{\propto}(f,.) = \Delta_{t_1.1}^{\propto_1}.\Delta_{t_2.2}^{\propto_2} \ldots \Delta_{t_n.n}^{\propto_n}(f,.)$ Where $\Delta_{t_i}^{\propto_i} i$ is the usual $\propto_i$th forward difference of step length t_i with respect to x_i , $\propto \in Z_+^d, t \geq 0$.

Also, modulus of smoothness is defined as:

$$\omega_\propto(f,h,X) = \underbrace{sup}_{0 \leq t < h} \|\Delta_t^\propto(f,x)\|_{p,\alpha}.$$

In this search we define a new form of difference as:

$\Delta_{t_{S_n}}(f,x) = n!\,f(\propto_0 v^0 + \cdots \propto_n v^n) - d!\,f(\propto_0 v^0 + \cdots \propto_n v^n)$Where $n - d \leq t$

$$\begin{aligned}\omega_{\propto_{S_n}}(f,h,X) &= \sup\left\|\Delta_{t_{S_{n,\alpha}}}(f,x)\right\| \\ &= \sup[\int_{S_{n,\alpha}}(n! - d!)f(\propto_0 v^0 + \cdots \propto_n v^n)\omega_\alpha(\,x)dx]\end{aligned}$$

It is clear that $\int_{S_n} f(x)\omega_\alpha(x)dx = \|(f,x)\|_{S_n,\alpha} \le \|(f,x)\|_{1,\alpha} \le \|(f,x)\|_{p,\alpha}$.
In **[3]** suppose that $\Lambda \subseteq \boldsymbol{Z}_+^{\boldsymbol{d}}$(where $\boldsymbol{Z}_+$is positive integer numbers) be abounded and satisfy if $\sigma \in \Lambda$ and $\beta \le \sigma$ Then $\beta \in \Lambda$, we define $\boldsymbol{P}_\Lambda = span\{x^\sigma, \sigma \in \Lambda\}$ and we assume that Q is rectangle in R^d with side length δ_i in the i-th direction , $f \in L_{p,\alpha(Q)}$, $\Omega \subseteq Q$.
Also In **[3]** we define:
$\omega_\Lambda(f,h,X)_p = \sum_{\propto\in\partial\Lambda} \omega_\propto(f,h,X)_p$, $E_\Lambda(f,\Omega)_{p,\alpha} = \underbrace{inf}_{P\in P_\Lambda} \|f-P\|_{p,\alpha(\Omega)}$.
And If Ω Satisfies:
(if $\mathrm{x}, \mathrm{x}+\mathrm{h}e_i \in \Omega$ for some $\mathrm{h} > 0$ and some i, then $\mathrm{x}+\mathrm{t}e_i \in \Omega$ for all $0 \le \mathrm{t} \le \mathrm{h}$) and $\Omega \subseteq \bigcup_1^m Q_i$ where each Q_i is rectangle and for each i, there is a rectangle R_i with one of its vertices o such that:
$x \in Q_i \cap \Omega$ Implies $x + R_i \subset \Omega$ …………………………………………… (1-2).

2. <u>Auxiliary Results</u>:

<u>Theorem 2.1:</u>[1]

Let the interpolation problem is based on the points $x^0, \dots, x^n$ be poised. Then the Lagrange interpolation polynomial $L_n(f,x) \in \boldsymbol{P}_n^d$ is given by:

$$L_n(f,x) = \sum_{j=0}^{n} \sum_{i=1}^{r_j^d} \lambda_n[x^0, \dots, x^{n-1}, x_i^n]f . P_i^{[j]}(x) .$$

<u>Theorem 2.2:</u>[1]

For $n \in N$ and $f \in C^{n+1}(R^d)$ we have:

$$L_n(f,x) - f(x) = \sum_{\mu\in\Lambda_\mu} P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu) \int_{R^d} D_{x-x_{\mu_n}^{(n)}} D_{x^\mu}^n f(y) M(y|x^\mu . x)dy$$

<u>Theorem 2.3:</u>

For $f \in L_{p,\alpha(\mathrm{X})}$, $\mathrm{X} = [-1,1]^d$ we get:

$$\omega_\propto(D^\sigma f, \delta, \mathrm{X})_{p,\alpha} \le c_p \omega_\propto(f, \delta, \mathrm{X})_{\infty,\alpha}.$$

<u>Proof:</u>

By using properties of norm, properties of partial derivative and by

Assume $F(x) = \sum_{i=0}^{\propto}(-1)^{i+\propto}\binom{\propto}{i}D^{\sigma}f(x+it)$ we get:

$$\omega_{\propto}\left(D^{\sigma}f,\delta,\mathrm{X}\right)_{p,\alpha} = \underbrace{sup}_{0\le t<\delta}\{\left\|\Delta_t^{\propto}\left(D^{\sigma}f,x\right)\right\|_{p,\alpha}\}$$

$$\le c_p \underbrace{sup}_{0\le t<\delta}\{\|\Delta_t^{\propto}(D^{\sigma}f,x)\|_{1,\alpha}\}$$

$$= c_p \underbrace{sup}_{0\le t<\delta}\{\int_X |\sum_{i=0}^{\propto}(-1)^{i+\propto}\binom{\propto}{i}D^{\sigma}f(x+it)\,|e^{-\alpha x}dx\}$$

$$= c_p \underbrace{sup}_{0\le t<\delta}\{\int_X \sum_{i=0}^{\propto}(-1)^{i+\propto}\binom{\propto}{i}D^{\sigma}f(x+it)sing(F(x))\,e^{-\alpha x}dx\}$$

$$= c_p \underbrace{sup}_{0\le t<\delta}\{\int_X D^{\sigma}\sum_{i=0}^{\propto}(-1)^{i+\propto}\binom{\propto}{i}f(x+it)sing(F(x))\,e^{-\alpha x}dx$$

$$= c_p \underbrace{sup}_{0\le t<\delta}\{\sum_{i=0}^{\propto}(-1)^{i+\propto}\binom{\propto}{i}f(x+it)sing\big(F(x)\big)\,e^{-\alpha x}dx\}$$

$$= c_p \underbrace{sup}_{0\le t<\delta}\{|(\sum_{i=0}^{\propto}|(-1)^{i+\propto}\binom{\propto}{i}f(x+it)|\,e^{-\alpha x}\}$$

$$= c_p \underbrace{sup}_{0\le t<\delta}\{|\Delta_t^{\propto}(f,x)e^{-\alpha x}|\}$$

$$\le c_p \underbrace{sup}_{0\le t<\delta}\{\underbrace{sup}_{x\in X}|\Delta_t^{\propto}(f,x)e^{-\alpha x}|\}$$

$$= c_p \underbrace{sup}_{0\le t<\delta}\{\|\Delta_t^{\propto}(f,x)\|_{\infty,\alpha}\}$$

$$= c_p\omega_{\propto}(f,\delta,\mathrm{X})_{\infty,\alpha}.$$

Theorem 2.4:

For $n \in N$ and $f \in L_{p,\alpha(X)}$ we have:

$\sum_{\mu\in\Lambda_\mu}\int_{R^d} D_{x-x_{\mu_n}^{(n)}} D_{x^\mu}^{n} f(y) M(y|x^\mu.x)\omega_\alpha(y)dy \le c_p\,\omega\big(f(y|x_\mu),\delta,X\big)_{p,\alpha}.$

Proof:

Suppose that $D_{x-x_{\mu_n}^{(n)}} D_{x^\mu}^{n} f(y) = F(y)$ and by using (1-1) and theorem 2.3 we have:

$\sum_{\mu\in\Lambda_\mu}\int_{R^d} D_{x-x_{\mu_n}^{(n)}} D_{x^\mu}^{n} f(y) M(y|x^\mu.x)\omega_\alpha(y)dy =$
$\sum_{\mu\in\Lambda_\mu} F(y) M(y|x^\mu.x)\omega_\alpha(y)dy$

$= \sum_{\mu\in\Lambda_\mu}(n-d)!\int_{S_n} F(\propto_0 v^0 + \cdots \propto_n v^n)d\propto$

$= \sum_{\mu\in\Lambda_\mu}(n!\int_{S_n} F(\propto_0 v^0 + \cdots \propto_n v^n)d\propto - d!\int_{S_n} F(\propto_0 v^0 + \cdots \propto_n v^n)d\propto)$

$= \sum_{\mu\in\Lambda_\mu}\omega_{\propto_{S_n}}(F,\delta) = \sum_{\mu\in\Lambda_\mu}\omega_{\propto_{S_n}}\left(D_{x-x_{\mu_n}^{(n)}} D_{x^\mu}^{n} f(y),\delta\right)$

$\le c_p\sum_{\mu\in\Lambda_\mu}\omega_{\propto_{S_n}}(f(y|x_\mu),\delta,X).$

$\le cc_p\sum_{\mu\in\Lambda_\mu}\omega\big(f(y|x_\mu),\delta,X\big)_{p,\alpha}$

$= c_p\sum_{\mu\in\Lambda_\mu}\omega\big(f(y|x_\mu),\delta,X\big)_{p,\alpha}.$

Theorem 2.5:

For $f\in L_{p,\alpha}[-1,1]$ and $x_i\in X=[-1,1], i=1,2,\dots,n$ and $|x_i-x_{i+1}|=\frac{1}{n}$ we have:

$$P_1^{[n]}(x) = \frac{(x-x_0)\dots(x-x_{n-1})}{(x_n-x_0)\dots(x_n-x_{n-1})} \le (2n)^n.$$

Proof:

$$P_1^{[n]}(x) = \prod_{i=0}^{n-1}\frac{x-x_i}{x_j-x_i} \le \prod_{i=0}^{n-1}\frac{2}{x_j-x_i} \le \prod_{i=0}^{n-1}\frac{2}{\frac{1}{n}}$$

$$= \prod_{i=0}^{n-1} 2n = (2n)^n.$$

Now, we construct an operator $\boldsymbol{L}_n^{\mp}(f,x)$ as we mention before as:

For the polynomial $P_1^{[n]}(x) = \frac{(\boldsymbol{x}-\boldsymbol{x_0})\ldots(\boldsymbol{x}-\boldsymbol{x_{n-1}})}{(\boldsymbol{x_n}-\boldsymbol{x_0})\ldots(\boldsymbol{x_n}-\boldsymbol{x_{n-1}})}$ it is clear that $P_1^{[n]}(x)$ is algebraic polynomial of degree less than n .And for $P_j^{[n]}$

$P_j^{[n]}\left(x_i^{(k)}\right) = 0, k < n$ And $P_j^{[n]}\left(x_i^{(n)}\right) = \delta_{ij}, i = 1, \ldots \ldots, r_n^d, 1 \le j \le r_n^d$

Is the n Th Newton fundamental polynomial $P_j^{[n]} \in \boldsymbol{P}_n^d$ and:

$P_j^{[n]}\left(x_i^{(n)}\right) = \prod_{s=1}^d P_1^{[n]}(x_{i_s}^{(n)})$

Assume that $\Psi(x) = \prod_{s=1}^d \frac{c_{p,d}(x_{n_s}-x_{i_s})}{\left\|\sum_{\mu\in\Lambda_\mu} P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu)\right\|_{\infty,\alpha}(P_1^{[n]}\left(x_{i_s}^{(k)}\right))}, i = 1,2, \ldots$

Now, we define $\boldsymbol{L}_n^{\mp}(f,x)$-opearator as:

$\boldsymbol{L}_n^{\mp}(f,x) = L_n(f,x) \mp \sum_{m\in\boldsymbol{N}^d} \Psi_m(x)\|f(x) - L_n(f,x)\|_{\infty,\alpha}(X_m)$.

Where:

$\boldsymbol{N^d} = \{1,2, \ldots, n\}^d, m = (m_1, \ldots, m_d), X_m = \left[\boldsymbol{N}_{m_1}, \boldsymbol{N}_{m_1+1}\right] \times \ldots \times \left[\boldsymbol{N}_{m_{1-1}}, \boldsymbol{N}_{m_{1-1}+1}\right]$

<u>Theorem 2.6:</u>

$\Psi(x) \ge 1, x \in [-1,1]$.

<u>Proof:</u>

As in theorem (2.5) we have:

$$\Psi(x) = \prod_{s=1}^d \frac{c_{p,d}\prod_{i=1}^n(x_s - x_{i_s})}{\left\|\sum_{\mu\in\Lambda_\mu} P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu)\right\|_{\infty,\alpha}(P_1^{[n]}\left(x_{i_s}^{(k)}\right))}$$

$$= \prod_{s=1}^d \frac{c_{p,d}\prod_{i=0}^{n-1}(x_s - x_{i_s})}{\underbrace{\sup}_{x,x^\mu\in X} \left|\sum_{\mu\in\Lambda_\mu} P_{\mu_n}^{[n]}(x_\circ)\pi_\mu(X_\circ^\mu)\right|(P_1^{[n]}\left(x_{i_s}^{(k)}\right))}$$

$$= \prod_{s=1}^{d} \frac{c_{p,d} \prod_{i=0}^{n-1}(x_s - x_{i_s})}{c_{p,d} \frac{(x_s - x_{0_s})\dots(x_s - x_{n_s-1})}{(x_{n_s} - x_{0_s})\dots(x_{n_s} - x_{n_s-1})}}$$

$$\geq \prod_{s=1}^{d} \frac{\prod_{i=0}^{n-1}(x_s - x_{i_s})}{\frac{(x_s - x_{0_s})\dots(x_s - x_{n_s-1})}{2^n}})$$

$$= \prod_{s=1}^{d} \frac{1}{\frac{1}{2^n}} = \prod_{s=1}^{d} 2^n = 2^{nd} \geq 1.$$

Theorem 2.6:

For $f \in L_{p,\alpha[-1,1]^d}$ then $\mathbf{L}_{\boldsymbol{n}}^{-}(f,x) \leq f(x) \leq \mathbf{L}_{\boldsymbol{n}}^{+}(f,x)$, $x \in X = [-1,1]^d$.

Proof:

$$\mathbf{L}_{\boldsymbol{n}}^{+}(f,x) = L_{\boldsymbol{n}}(f,x) + \sum_{m \in N^d} \Psi_m(x) \|f - L_{\boldsymbol{n}}\|_{\infty,\alpha(X_m)}$$

$$\geq L_{\boldsymbol{n}} + \|f - L_{\boldsymbol{n}}\|_{\infty,\alpha(X_m)}$$

$$\geq L_{\boldsymbol{n}} + |f - L_{\boldsymbol{n}}|$$

$$= f$$

With the same way we can prove: $\mathbf{L}_{\boldsymbol{n}}^{-}(f,x) \leq f(x)$.

Then $\mathbf{L}_{\boldsymbol{n}}^{-}(f,x) \leq f(x) \leq \mathbf{L}_{\boldsymbol{n}}^{+}(f,x)$, $x \in X = [-1,1]^d$.

Theorem 2.7:

For the polynomial $P_j^{[n]}(x)$ we have:

$$\lim_{n\to\infty} \left\| \frac{c_{p,d} \prod_{i=0}^{n-1}(x_s - x_{i_s})}{(P_1^{[n]}(x_{i_s}^{(k)}))} \right\|_{p,\alpha} = 0.$$

Proof:

By using properties of limit we get:

$$\lim_{n\to\infty} \left\| \frac{c_{p,d} \prod_{i=0}^{n-1}(x_s - x_{i_s})}{(P_1^{[n]}\left(x_{i_s}^{(k)}\right))} \right\|_{p,\alpha} = \left\| \lim_{n\to\infty} \frac{c_{p,d} \prod_{i=0}^{n-1}(x_s - x_{i_s})}{(P_1^{[n]}\left(x_{i_s}^{(k)}\right))} \right\|_{p,\alpha}$$

$$\left\|\lim_{n\to\infty}\frac{c_{p,d}\prod_{i=0}^{n-1}(x_s-x_{i_s})}{\frac{(x-x_0)\dots(x-x_{n-1})}{(x_n-x_0)\dots(x_n-x_{n-1})}}\right\|_{p,\alpha}=\left\|\frac{\lim\limits_{n\to\infty}c_{p,d}\prod_{i=0}^{n-1}(x_s-x_{i_s})}{\lim\limits_{n\to\infty}\frac{(x-x_0)\dots(x-x_{n-1})}{(x_n-x_0)\dots(x_n-x_{n-1})}}\right\|_{p,\alpha}$$

$$\left\|\frac{c_{p,d}\lim_{n\to\infty}\prod_{i=0}^{n-1}(x_s-x_{i_s})}{\lim_{n\to\infty}\frac{(x-x_0)\dots(x-x_{n-1})}{(x_n-x_0)\dots(x_n-x_{n-1})}}\right\|_{p,\alpha}=\left\|\frac{c_{p,d}.\lim_{n\to\infty}(x_n-x_0)\dots(x_n-x_{n-1})\lim_{n\to\infty}\prod_{i=0}^{n-1}(x_s-x_{i_s})}{\lim_{n\to\infty}(x-x_0)\dots(x-x_{n-1})}\right\|_{p,\alpha}$$

$$=\left\|\frac{c_{p,d}\lim_{n\to\infty}\frac{c_1}{n^2}}{\lim_{n\to\infty}(x-x_0)\dots(x-x_{n-1})}\right\|_{p,\alpha}=0.$$

3. Main Results:

Theorem 3.1:

For the function $f\in L_{p,\alpha[-1,1]^d}$ we have:

$$\tilde{E}_n(f)_{p,\alpha[-1,1]^d}\le c\sum_{\mu\in\Lambda_\mu}\tau(f(x_\mu),\delta)_{p,\alpha}.$$

Proof:

By using theorems 2.6, 2.2, properties of norm and theorem 2.7 we get:

$$\tilde{E}_n(f)_{p,\alpha[-1,1]^d}\le\|\mathbf{L}_{\boldsymbol{n}}^{+}(f,x)-\mathbf{L}_{\boldsymbol{n}}^{-}(f,x)\|_{p,\alpha}$$

$$=2\left\|\sum_{m\in\boldsymbol{N}^d}\Psi_m(x)\|f-L_n(f,x)\|_{\infty,\alpha(X_m)}\right\|_{p,\alpha}$$

$$=2\left\|\sum_{m\in\boldsymbol{N}^d}\prod_{s=1}^{d}\frac{c_{p,d}\prod_{i=0}^{n-1}(x_s-x_{i_s})}{\left\|\sum_{\mu\in\Lambda_\mu}P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu)\right\|_{\infty,\alpha}(P_1^{[n]}\left(x_{i_s}^{(k)}\right))}\|f-L_n(f,x)\|_{\infty,\alpha(X_m)}\right\|_{p,\alpha}$$

$$=2\left\|\prod_{s=1}^{d}\sum_{m\in\boldsymbol{N}^d}\frac{c_{p,d}\prod_{i=0}^{n-1}(x_s-x_{i_s})}{\left\|\sum_{\mu\in\Lambda_\mu}P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu)\right\|_{\infty,\alpha}(P_1^{[n]}\left(x_{i_s}^{(k)}\right))}\left\|\sum_{\mu\in\Lambda_\mu}P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu)\int_{R^d}D_{x-x_{\mu_n}^{(n)}}D_{x^\mu}^{n}f(y)M(y|x^\mu,x)dy\right\|_{\infty,\alpha}\right.$$

$$= 2\left\|\prod_{s=1}^{d}\sum_{m\in \boldsymbol{N}^d}\frac{c_{p,d}\prod_{i=0}^{n-1}(x_s-x_{i_s})}{\left\|\sum_{\mu\in\Lambda_\mu}P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu)\right\|_{\infty,\alpha}\left(P_1^{[n]}\left(x_{i_s}^{(k)}\right)\right)}\underbrace{\boldsymbol{sup}}_{\boldsymbol{x\in X}}\left|\sum_{\mu\in\Lambda_\mu}P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu)\int_{R^d}D_{x-x_{\mu_n}^{(n)}}D_{x^\mu}^{n}f(y)M(y|x^\mu,x)dy\right|\right\|_{p}$$

$$\leq \boldsymbol{2c_1}\left\|\prod_{s=1}^{d}\sum_{m\in \boldsymbol{N}^d}\frac{c_{p,d}\prod_{i=0}^{n-1}(x_s-x_{i_s})}{\left\|\sum_{\mu\in\Lambda_\mu}P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu)\right\|_{\infty,\alpha}\left(P_1^{[n]}\left(x_{i_s}^{(k)}\right)\right)}\underbrace{\boldsymbol{sup}}_{\boldsymbol{x\in X}}\sum_{\mu\in\Lambda_\mu}\left|P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu)\int_{R^d}D_{x-x_{\mu_n}^{(n)}}D_{x^\mu}^{n}f(y)M(y|x^\mu,x)dy\right|$$

$$\leq \boldsymbol{2c_2}\left\|\prod_{s=1}^{d}\sum_{m\in \boldsymbol{N}^d}\frac{c_{p,d}\prod_{i=0}^{n-1}(x_s-x_{i_s})}{\left\|\sum_{\mu\in\Lambda_\mu}P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu)\right\|_{\infty,\alpha}\left(P_1^{[n]}\left(x_{i_s}^{(k)}\right)\right)}\sum_{\mu\in\Lambda_\mu}\underbrace{\boldsymbol{sup}}_{\boldsymbol{x\in X}}\left|P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu)\int_{R^d}D_{x-x_{\mu_n}^{(n)}}D_{x^\mu}^{n}f(y)M(y|x^\mu,x)dy\right|\right\|_{\boldsymbol{p,\alpha}}$$

$$\leq \boldsymbol{2c_3}\left\|\prod_{s=1}^{d}\sum_{m\in \boldsymbol{N}^d}\frac{c_{p,d}\prod_{i=0}^{n-1}(x_s-x_{i_s})}{\left\|\sum_{\mu\in\Lambda_\mu}P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu)\right\|_{\infty,\alpha}\left(P_1^{[n]}\left(x_{i_s}^{(k)}\right)\right)}\sum_{\mu\in\Lambda_\mu}\underbrace{\boldsymbol{sup}}_{\boldsymbol{x\in X}}\left|P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu)\right|.\underbrace{\boldsymbol{sup}}_{\boldsymbol{x\in X}}\left|\int_{R^d}D_{x-x_{\mu_n}^{(n)}}D_{x^\mu}^{n}f(y)M(y\right.$$

$$\leq \boldsymbol{2c_3}\left\|\prod_{s=1}^{d}\sum_{m\in \boldsymbol{N}^d}\frac{c_{p,d}\prod_{i=0}^{n-1}(x_s-x_{i_s})}{\left\|\sum_{\mu\in\Lambda_\mu}P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu)\right\|_{\infty,\alpha}\left(P_1^{[n]}\left(x_{i_s}^{(k)}\right)\right)}\sum_{\mu\in\Lambda_\mu}\left\|\boldsymbol{P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu)}\right\|_{\infty,\alpha}\left\|\int_{\boldsymbol{R^d}}\boldsymbol{D_{x-x_{x^\mu}^{(n)}}D_{x^\mu}^{n}f(y)M(y|x^\mu,x)d}\right.$$

$$= \boldsymbol{2c_3}\left\|\prod_{s=1}^{d}\sum_{m\in \boldsymbol{N}^d}\frac{c_{p,d}\prod_{i=0}^{n-1}(x_s-x_{i_s})}{\left\|\sum_{\mu\in\Lambda_\mu}P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu)\right\|_{\infty,\alpha}\left(P_1^{[n]}\left(x_{i_s}^{(k)}\right)\right)}\sum_{\mu\in\Lambda_\mu}\left\|\boldsymbol{P_{\mu_n}^{[n]}(x)\pi_\mu(x^\mu)}\right\|_{\infty,\alpha}.\sum_{\mu\in\Lambda_\mu}\left\|\int_{\boldsymbol{R^d}}\boldsymbol{D_{x-x_{x^\mu}^{(n)}}}\right.$$

$$\leq \boldsymbol{2c_2}\left\|\prod_{s=1}^{d}\sum_{m\in \boldsymbol{N}^d}\frac{c_{p,d}\prod_{i=0}^{n-1}(x_s-x_{i_s})}{\left(P_1^{[n]}\left(x_{i_s}^{(k)}\right)\right)}\sum_{\mu\in\Lambda_\mu}\left\|\int_{\boldsymbol{R^d}}\boldsymbol{D_{x-x_{x^\mu}^{(n)}}D_{x^\mu}^{n}f(y)M(y|x^\mu,x)dy}\right\|_{\infty,\alpha}\right\|_{\boldsymbol{p,\alpha}}$$

$$\leq 2c_2\left\|\prod_{s=1}^{d}\sum_{m\in \boldsymbol{N}^d}\frac{c_{p,d}\prod_{i=0}^{n-1}(x_s-x_{i_s})}{\left(P_1^{[n]}\left(x_{i_s}^{(k)}\right)\right)}\right\|_{p,\alpha}\sum_{\mu\in\Lambda_\mu}\left\|\int_{\boldsymbol{R^d}}\boldsymbol{D_{x-x_{x^\mu}^{(n)}}D_{x^\mu}^{n}f(y)M(y|x^\mu,x)dy}\right\|_{\infty,\boldsymbol{\alpha}}$$

$$\leq c_{p,d}\sum_{\mu\in\Lambda_\mu}\|\omega(f,\delta)\|_{\infty,\alpha}$$

$\leq c_{p,d} \sum_{\mu \in \Lambda_\mu} \tau(f,\delta)_{\infty,\alpha}$

$= c_{p,d} \sum_{\mu \in \Lambda_\mu} \tau(f,\delta)_{p,\alpha}$.

Now, we discuss the best on sided approximate of derivative of the function $f \in W_{p,\alpha}^{\Lambda}[-1,1]^d$ and minimize degree of this approximation when domain of the function f is restrict to rectangle domain Q in X^d where $Q = \prod_{i=1}^{d}[a_i, b_i]$ and $\delta_{i=}b_{i-}a_i > 0$.

Theorem 3.2:[3]

If Ω satisfies (1-2) and $\Omega \subset Q$ is rectangle with side length vector δ then for $f \in L_{p,\alpha}(\Omega)$:

$\|f - g\|_{p,\alpha(\Omega)} + \sum_{\propto \in \partial\Lambda} \delta^{\propto} \|D^{\propto} g\|_{p,\alpha}(\Omega) \leqq c\omega_{\Lambda}(f,\delta,\Omega)_{p,\alpha}$.

Theorem 3.3:

For $f \in W_{p,\alpha}^{\Lambda}[-1,1]^d$ we have:

$\tilde{E}_n(D^\alpha f)_{p,\alpha[-1,1]^d} \leq n^\alpha \tau(f,\delta)_{p,\alpha}$ *where* $\delta \leq \frac{1}{n^{\alpha+1}}$.

Proof:

Since $\mathbf{L}_{\boldsymbol{n}}^{-}(f,x) \leq f(x) \leq \mathbf{L}_{\boldsymbol{n}}^{+}(f,x)$, $x \in X = [-1,1]^d$

Then either $D^\alpha \mathbf{L}_{\boldsymbol{n}}^{-}(f,x) \leq D^\alpha f(x) \leq D^\alpha \mathbf{L}_{\boldsymbol{n}}^{+}(f,x)$

Or, $D^\alpha \mathbf{L}_{\boldsymbol{n}}^{-}(f,x) \leq D^\alpha f(x) \leq D^\alpha \mathbf{L}_{\boldsymbol{n}}^{+}(f,x)$ where $\alpha \in R^d$

So, by using Bernstein equality in multivariate case and theorem 3.1 we get:

$$\tilde{E}_n(D^\alpha f)_{p,\alpha[-1,1]^d} \leq \|D^\alpha \mathbf{L}_{\boldsymbol{n}}^{+}(f,x) - D^\alpha \mathbf{L}_{\boldsymbol{n}}^{-}(f,x)\|_{p,\alpha}$$

$$= \|D^\alpha (\mathbf{L}_{\boldsymbol{n}}^{+}(f,x) - \mathbf{L}_{\boldsymbol{n}}^{-}(f,x))\|_{p,\alpha}$$

$$\leq cn^\alpha \|\mathbf{L}_{\boldsymbol{n}}^{+}(f,x) - \mathbf{L}_{\boldsymbol{n}}^{-}(f,x)\|_{p,\alpha}$$

$$\leq cn^\alpha \sum_{\mu \in \Lambda_\mu} \tau(f,\delta)_{p,\alpha}.$$

The other cases:

And since f lies in weighted space then $D^{\alpha}(f)$ lies in weighted space.

So, by using properties of norm and theorems 3.2.

There exist $g \in W_{p,\alpha}^{\Lambda}[-1,1]^d$ such that:

$$\begin{aligned}\tilde{E}_n(D^{\alpha}f)_{p,\alpha(Q)} = E_n(D^{\alpha}f)_{p,\alpha(Q)} &\le \|D^{\alpha}f - D^{\alpha}\mathbf{L}_{\boldsymbol{n}}^{+}(f,x)\|_{p,\alpha} \\ &= \|D^{\alpha}f - D^{\alpha}\mathbf{L}_{\boldsymbol{n}}^{+}(f,x) - g(x) + g(x)\|_{p,\alpha} \\ &\le \|D^{\alpha}f - g(x)\|_{p,\alpha} + \|g(x) - D^{\alpha}\mathbf{L}_{\boldsymbol{n}}^{+}(f,x)\|_{p,\alpha} \\ &\le c(\|D^{\alpha}f - g(x)\|_{p,\alpha} + E_{\Lambda}(g,\Omega)_{p,\alpha}) \\ &\le c(\|D^{\alpha}f - g(x)\|_{p,\alpha} + \textstyle\sum_{\alpha\in\partial\Lambda} t^{\alpha}\,\|D^{\alpha}(g(x))\|_{p,\alpha}) \\ &\le c\omega_{\Lambda}(D^{\alpha}f,t,Q)_{p,\alpha}\,.\end{aligned}$$

In this section we try to estimate invers theorem of one sided in multivariate case by benefit from invers theorem in multivariate case and relation between best approximation and best one approximation.

Theorem 3.4:[4]

For $f \in L_{p,\alpha[-1,1]}$, $1 \le p \le \infty$ we have:

$$E_n(f)_{\infty,\alpha} \le \tilde{E}_n(f)_{\infty,\alpha} \le 2E_n(f)_{\infty,\alpha}.$$

It's easy to prove above theorem in multivariate case

Theorem 3.5:[2](Invers Theorem)

Let ω_{q_k} be Freud weights for $k = 1,\dots,d$, $d \ge 1$ be an integer then for $0 < \delta \le 1$ then:

$$K_r(q,p,f,\delta) \le c\delta^r\{\left\|\omega_{q_k}\right\|_p + \textstyle\sum_{0\le m\le\delta^{-1}}(m+1)^{r-1}\,E_m(f,q)_p\}.$$

Theorem 3.6:

For $f \in L_{p,\alpha[-1,1]^d}$ we have:

$$K_r(q,p,f,\delta) \le c\delta^r\{\left\|\omega_{q_k}\right\|_p + \textstyle\sum_{0\le m\le\delta^{-1}}(m+1)^{r-1}\,\tilde{E}_m(f,q)_p\}.$$

Proof:

By using theorems 3.4 and 3.5 we get:

$E_n(f)_{\infty,\alpha} \leq \tilde{E}_n(f)_{\infty,\alpha}$ Then:

$$c\delta^r \left\{ \left\|\omega_{q_k}\right\|_p + \sum_{0\leq m\leq \delta^{-1}} (m+1)^{r-1} E_m(f,q)_p \right\}$$
$$\leq c\delta^r \{ \left\|\omega_{q_k}\right\|_p + \sum_{0\leq m\leq \delta^{-1}} (m+1)^{r-1} \tilde{E}_m(f,q)_p \}$$

So, we have:

$K_r(q,p,f,\delta) \leq c\delta^r \{ \left\|\omega_{q_k}\right\|_p + \sum_{0\leq m\leq \delta^{-1}} (m+1)^{r-1} \tilde{E}_m(f,q)_p \}.$

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