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Central Moments of Traffic Delay at a Signalized Intersection

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Abstract

Traffic delay model was developed under a basis of deterministic and stochastic delay components. The latter component was put under D/D/1 framework and therein mean and its variance derived. While the stochastic component was put under the M/G/1 framework, mean and variance derived. Extension on stochastic component and M/G/1 framework was discussed with the usage of compressed queueing processes. Harmonization of the moments of deterministic and stochastic components to obtain the overall central moments of traffic delay has been discussed. Simulation was performed using Matlab for traffic intensities ranging from 0.1 to 1.9. The simulated results indicate that both deterministic and stochastic components are incompatible as the traffic intensity approaches capacity. Furthermore the results indicated that oversaturated conditions and random delay renders the stochastic component in traffic delay models unrealistic. Also, with the ability to estimate the variance of overall traffic delay, it is feasible to integrate the concept of reliability into design and analysis of a signalized intersection.

Keywords: D/D/1 model, M/G/1 model, compressed queueing processes, simulation.

1.0 Introduction

Traffic delays and queues are principal measures of performance that determine the level of service (LoS) at signalized intersections. They also evaluate the adequacy of the lane lengths and the estimation of fuel consumption and emissions. Quantifying these delays accurately at an intersection is critical for planning, design and analysis of traffic lights. Signalized intersection referred herein, is a road junction controlled by a traffic light. Traffic lights were implemented for the purpose of reducing or eliminating congestions at intersections. These congestions exist because an intersection is an area shared among multiple traffic streams, and the role of the traffic light is to manage the shared usage of the area. Traffic models in an intersection are always subjected to both uniform and random properties of traffic flows. As a result of these properties, vehicle travel times in an urban traffic environment are highly time dependant.

Models that incorporate both deterministic and stochastic components of traffic performance are very appealing in the signalized intersection since they are applied in a wide range of traffic intensities as well as to various types of traffic lights. They simplify theoretical models with delay terms that are numerically inconsequential. Of the various queueing models, D/D/1 and $M_{+\Delta}/G_{+\Delta}$ /1 were used in this study. The D/D/1 model assumed that the arrivals and departures were uniform and one service channel (traffic light) existed. This model is quite intuitive and easily solvable. Using this form of queueing with an arrival rate, denoted by λ and a service rate, denoted by μ, certain useful values regarding the consequences of queues were computed. The $M_{+\Delta}/G_{+\Delta}$ /1 model used implied that the vehicles arrived at an intersection in a Poisson process with rate λ and were treated in the order of arrival with inter arrival times following exponential distribution with parameter μ. The service times were treated as independent identically distributed with an arbitrary distribution. Similarly, one service channel (traffic light) was considered in this model.

2.0 Traffic Problem

The main traffic problem here is to develop overall traffic delay model using D/D/1 and compressed $M_{+\Delta}/G_{+\Delta}$ /1 queueing systems. With D/D/1 system, the inter-arrivals and service times are deterministic while in $M_{+\Delta}/G_{+\Delta}$ /1 system, the Markovian arrivals and iid service times follow a general distribution. A single service channel (traffic light) was used. The model will be used for estimating the mean of the time delay and its variance at a signalized intersection.

3.0 Compressed $M_{+\Delta}/G_{+\Delta}/1$ Queueing System

The service times in the stochastic delay component can be analyzed effectively using the compressed $M_{+\Delta}/G_{+\Delta}/1$ queueing system because of its distribution. The system is drawn from compressed queueing processes theory so as to estimate statistical measures of traffic delay in case of large variations of service times. In this model, $M_{+\Delta}$ represents the exponential shifted distribution for the inter arrival times, $G_{+\Delta}$ represents the general shifted distribution of service times and 1 implies a single service channel (traffic light). The level of service in this model is basically described by the mean and variance of the service time spent by a vehicle in the queue. The compressed queueing processes used in this study are based on two assumptions:

i) The service time rate for a compressed model, denoted by μ' and given by

$$\mu' = \frac{\mu}{1-\mu\Delta}. \tag{1}$$

ii) The arrival rate for the compressed model, denoted by λ' and given by

$$\lambda' = \frac{\lambda}{1-\lambda\Delta}. \tag{2}$$

4.0 Problem Formulation

Consider a cumulative arrival and departure of vehicles in a signalized intersection for the time interval $\left[0,T\right]$. The time taken by a vehicle in the queue herein referred to as overall traffic delay is denoted by D_t. Here, D_t comprises of deterministic and stochastic delay components and can be broken as follows:

$$D_t = D_{t_1} + D_{t_2}, \tag{3}$$

where D_{t_1} is the deterministic delay component representing a delay that is incurred by a vehicle with uniform arrival times and departures within the time interval $\left[t, t+c_y\right]$ while D_{t_2} is the stochastic delay component representing the delay that is caused by random queues resulting from the random nature of arrivals.
To solve the stochastic Equation (3), we make the following assumptions:

a) The intersection consists of only a single lane controlled by a fixed-time signal and unlimited space for queueing;
b) The vehicles' arrival at the intersection is either uniform or random variable following a Poisson process and no initial queue is present at the time when a prediction is performed;
c) The vehicle time prediction horizon is assumed to be equal to the signal cycle time.

4.1 Deterministic Delay Component

Deterministic delay component as described in (3) is denoted by D_{t_1}. In this section, we shall be interested in the computation of the mean and variance of D_{t_1}. The mean and variance of the deterministic delay component is estimated by deterministic queueing model D/D/1, In Figure (1) below, we present a diagrammatic description of deterministic delay process.

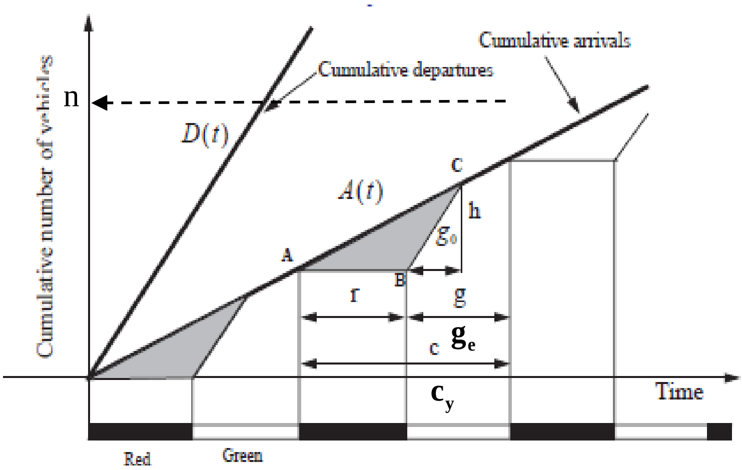


Figure 1: Deterministic component of overall traffic delay.

From the Figure, *D(t)* and *A(t)* represents the cumulative departures and arrivals, respectively. The area under cross-sectional area covered by triangle *ABC* represents the total deterministic delay at the intersection. We can determine the statistical measures: mean and variance.

4.1.1 Mean

To compute the mean, we assume that vehicle arrivals and departures are uniformly distributed with rates λ and μ, respectively. The figure shows a typical cumulative arrival/departure graph against time for uniform arrival rate approach to an intersection. The slope of the cumulative arrival line is the uniform arrival rate in vehicles per unit time, denoted by λ. The slope of the cumulative departure line is sometimes zero (when the light is red) and sometimes ρ (when the light is green); where ρ is the traffic intensity obtained as $\rho=\lambda/\mu$. Upon utilizing D/D/1 queueing system and the theory behind it, we compute the mean. Notice that the duration of c_y at the signalized intersection is given by

$$c_y = r + g_e, \tag{4}$$

From Figure (1), we note that g_o denotes the time necessary for the queue to dissipate. Here, the queue must dissipate before the end of g_e. But if the queue doesn't dissipate before the end of g_e, the queue would escalate indefinitely. From this statement, we deduce that

$$g_o \le g_e. \tag{5}$$

Condition (5) is satisfied if the total number of vehicle arrivals during c_y is less than or equal to the total number of vehicle departures during g_e. That is,

$$\frac{\lambda}{\mu} \le \frac{g_e}{c_y}. \tag{6}$$

Also from Figure (1), we can deduce that vehicles arrive during time period $\left(r+g_0\right)$ and depart during the time period $\frac{c_y}{g_e}.g_o$. Since the total number of vehicle arrivals equals the total number of vehicle departures, we have that

$$\left(\mu\frac{c_y}{g_e}-\lambda\right)g_o = \lambda r. \tag{7}$$

The time period g_o required for queue to dissipate is

$$g_o = \frac{\rho . r}{\left(\frac{c_y}{g_e} - \rho\right)}. \tag{8}$$

From the figure, it can be seen that D_{t_1} is given by

$$D_{t_1} = \sum_{i=1}^{n} d(i),$$

where *d(i)* is the shaded cross-sectional area in Figure (1). Assuming that *n* is large enough so that the discrete sum of *d(i)* is equal to the area of the cross-sectional area covered by triangle *ABC* in the figure the following can be written:

$$D_{t_1} = \frac{1}{2} h\left(c_y - g_e\right).$$

And here, h can be easily determined by noting that

$$h = \lambda\left(r + g_o\right).$$

Hence,

$$D_{t_1} = \frac{\lambda r^2}{2\left(1 - \frac{g_e}{c_y}\rho\right)}. \tag{9}$$

To obtain the expected deterministic delay, we divide D_{t_1} by the total number of vehicles in a cycle, that is, λc_y to give

$$E\left[D_{t_1}\right] = \frac{c_y\left(1 - \frac{g_e}{c_y}\right)^2}{2\left(1 - \frac{g_e}{c_y}\rho\right)} \tag{10}$$

as the mean of the deterministic component, D_{t_1}.

4.1.2 Variance

The conventional way of computing the $Var\left[D_{t_1}\right]$ is

$$Var\left[D_{t_1}\right] = E\left[D_{t_1}{}^2\right] + \left(E\left[D_{t_1}\right]\right)^2. \tag{11}$$

Since (10) provides us with $E\left[D_{t_1}\right]$, we compute for $E\left[D_{t_1}{}^2\right]$. To begin with, we compute $D_{t_1}{}^2$. Again, we assume *n* large enough so that the discrete sum of $d(i)^2$ is equal to the volume of the cross-sectional area covered by triangle *ABC* in the figure, that is

$$D_{t_1}{}^2 = \sum_{i=1}^{n} d(i)^2$$

$$D_{t_1}{}^2 = \frac{1}{3} h\left(c_y - g_e\right)^2.$$

Upon substituting for h, we get

$$D_{t_1}{}^2 = \frac{\lambda r^2}{3}\left(r + g_o\right).$$

Thus,

$$D_{t_1}{}^2 = \frac{\lambda c_y{}^3\left(1 - \frac{g_e}{c_y}\right)}{3\left(1 - \frac{g_e}{c_y}\rho\right)}.$$

To obtain $E\left[D_{t_1}{}^2\right]$ we divide the above result by the total number of vehicles, λc_y

$$E\left[D_{t_1}{}^2\right] = \frac{c_y{}^2\left(1 - \frac{g_e}{c_y}\right)^3}{3\left(1 - \frac{g_e}{c_y}\rho\right)}. \tag{12}$$

Equation (12) is the second moment of the deterministic delay component. Thus, utilizing (10) and (12), we have

$$Var\left[D_{t_1}\right] = \frac{c_y{}^2\left(1 - \frac{g_e}{c_y}\right)^3\left(1 + 3\frac{g_e}{c_y} - 4\frac{g_e}{c_y}\rho\right)}{12\left(1 - \frac{g_e}{c_y}\rho\right)^2}, \tag{13}$$

as the variance of the deterministic component, D_{t_1}.

4.2 Stochastic Delay Component

The component is established through a coordinate transformation technique based on the queueing system $M_{+\Delta}/G_{+\Delta}/1$ with the usage of compressed queueing processes. Under this system, the vehicles arrive at the intersection in a Poisson process. The inter-arrival times follow a shifted exponential distribution given by

$$A(t) = 1 - \lambda e^{-\lambda t}.$$

The service times are iid random variables following a general distribution characterized by its Probability density function determined by $f_X(x)$ or $F_X(x)$. Suppose N_t vehicles are on the queue at time t and, R_t being the residual service time of vehicle j. Residual service time herein, is the time until the vehicle found by vehicle j being served by the traffic light completes the service. Then for us to describe the state of the queueing system at time t, we need to compute the value of N_t, the probability that j vehicles are on the queue by

$$P_r\left[N_t = j\right] = \pi_j. \tag{14}$$

We shall use the generating function technique to compute (14) as follows

$$\mathrm{P}(s) = \sum_{j\geq 0} s^j P_r\left(N_t = j\right),$$

where $\mathrm{P}(s)$ is the transform of the system size distribution.

$$\mathrm{P}(s) = \sum_{i\geq 0}\sum_{j\geq 0}\pi_i P_{ij} s^j$$

where $\pi_j = \sum_{i\geq 0}\mathrm{P}_{ij}\pi_i$ for $j \geq 0$ and P_{ij} is a transition probability

Note that the transition $0 \rightarrow j$ occurs if and only if j arrivals occur in the service time following an idle period, whereas the transition $i \rightarrow j$ (with $i > 0$) occurs if and only if $j-i+1$ arrivals occur during a service time. If q_j is the probability of j arrivals in a service time and $Q(s)$ is the generating function of $\{q_j\}$, we have

$$P(s) = \pi_0 \frac{Q(s)(1-s)}{Q(s)-s}. \tag{15}$$

The matrix of (15) takes the form

$$\mathrm{P}(s) = \begin{bmatrix} q_0 & q_1 & q_2 & \cdots \\ 0 & q_0 & q_1 & \cdots \\ 0 & 0 & q_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

To compute q_j, first note that N_t follow a Poisson distribution with parameter λt at time t. Thus,

$$q_j = \int_{t=0}^{\infty} \frac{(\lambda t)^j}{j!} e^{-\lambda t} f_X(t)\,dt,$$

where $f_X(t)$ is the service time distribution. The generating function of q_j herein denoted by $Q(s)$ is given by

$$Q(s) = \sum_{i \geq 0} q_j s^j.$$

Thus,

$$Q(s) = \int_{t=0}^{\infty} \sum_{j=0}^{\infty} e^{-\lambda t(1-s)} f_X(t)\,dt. \tag{16}$$

Notice that in (16), the Laplace transform of the service time distribution is

$$Q(s) = X^* \lambda(1-s), \tag{17}$$

by definition, X^* in (17) is referred to as the service time transform. Equation (17) is also referred to as the Laplace-Stieltjes transform (LST) or Pollaczek-Khintchine (P-K) transform of the service time distribution with first and second moments denoted by $E[X]$ and $E[X^2]$, respectively.

Next, we compute π_0 in (17) by employing L'Hospital's rule with the assumption that $P(1) = Q(1) = 1$

$$P'(s) = \frac{d}{ds}[\cdot]$$

$$P'(s) = \pi_0 \left(\frac{Q'(s)[1-s] - Q(s)}{Q'(s)-1} \right).$$

Upon taking the limit $s \rightarrow 1$, we get

$$\lim_{s \to 1} P'(s) = \lim_{s \to 1} [\cdot]$$

$$\lim_{s \to 1} P'(s) = \frac{-\pi_0}{Q'(1)-1}.$$

Applying Little's theorem, defined by $\rho = \lambda E[X]$, we have

$$\pi_0 = 1 - \rho .$$

Thus,

$$\mathrm{P}(s) = \frac{(1-\rho) X^* \lambda (1-s)(1-s)}{X^* \lambda (1-s) - s} . \tag{18}$$

4.2.1 Mean

First, we break D_{t_2} of (3) into W_t and X_t, where W_t is the waiting time for vehicle j and X_t is the service time for vehicle j. Therefore, D_{t_2} is given by

$$D_{t_2} = W_t + X_t .$$

Thus expectation of D_{t_2} is

$$E\left[D_{t_2}\right] = E[W_t] + E[X_t].$$

To obtain $E\left[D_{t_2}\right]$, first we compute $E[W_t]$. Assuming First Come First Served (FCFS) discipline, we have

$$W_t = \rho R_t + X_{t-1} + X_{t-2} + ... + X_{t-Q_t}$$

$$W_t = \rho R_t + \sum_{i=1}^{Q_t} X_{t-i}$$

where W_t, ρ, R_t, X_t and Q_t are as provided in the list of symbols. Therefore $E[W_t]$ is

$$E[W_t] = \rho E[R_t] + E\left[\sum_{i=1}^{Q_t} X_{t-i}\right].$$

The Q_t as defined in the list of symbols is a random variable hence,

$$E[W_t] = \rho E[R_t] + EE\left[X_t \middle| Q_t\right].$$

Since X_t is independent of Q_t, we have

$$E[W_t] = \rho E[R_t] + E[X_t] \cdot E[Q_t].$$

Upon taking the limit $t \to \infty$, we get

$$\underset{t\to\infty}{Limit}\, E[W_t] = \underset{t\to\infty}{Limit} \left\{\rho E[R_t] + E[X_t] \cdot E[Q_t]\right\}.$$

Hence,

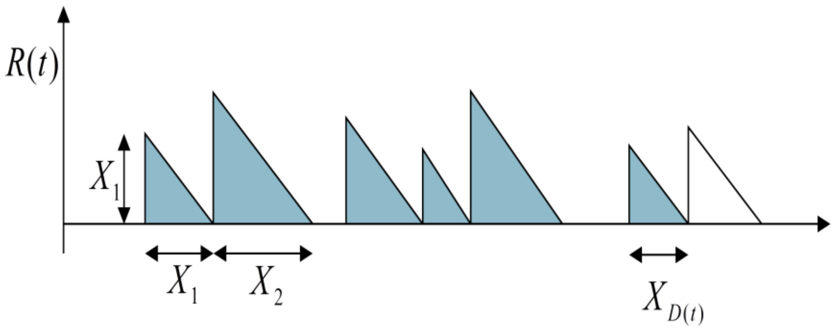
$$E[W] = \rho E[R] + E[X] E[Q]. \tag{19}$$

The expectations $E[R]$ and $E[Q]$ in (19) are those observed by arriving vehicle at the intersection. From Poisson Arrivals See Time Averages (PASTA) property, the statistical measures (mean, variance and distribution) of the number of vehicles in the queueing system observed by an arrival is the same as those observed by an independent Poisson inspector. If we assume that vehicles arrive at the intersection in a Poisson process, then the expected number of vehicles in the queue excluding the one being served is given by

$$E[Q] = \lambda E[W],$$

utilizing this relation in (19), we get

$$E[W] = \frac{\rho E[R]}{1-\rho}, \tag{20}$$



where, ρ is the traffic intensity defined by $\rho=\lambda E[X]$ (Little's law). To compute $E[R]$ in (20), consider Figure (2) below.

Figure 2: Diagram representing long-term residual service time.

In Figure (2), we present a diagrammatic description of a long-term expected residual time.

To compute the (unconditional) mean residual service time $E[R]$, consider the process $\{R(t), t \geq 0\}$ where $R(t)$ is the residual service time of the vehicle in service at time t. And consider a very long time interval $[0, T]$. Then

$$E[R] = \frac{1}{T}\int_0^T R(t)\, d(t).$$

Let $X(T)$ be the number of service completions by time T and X_i the i^{th} service time. Notice that the function $R(t)$ takes the value zero when there is no vehicle in service and jumps to the value of X_i at the time the i^{th} service time commences. During a service time it linearly decreases with rate of one and reaches zero at the end of a service time. Therefore, $E[R(t)]$ is equal to the sum of the areas of $X(T)$ isosceles right triangles where the side of the i^{th} triangle is X_i. For large T, we can ignore the last possibly incomplete triangle to obtain

$$E[R] = \frac{1}{T}\sum_{i=1}^{X(T)} \frac{1}{2} X_i^2$$

$$E[R] = \frac{1}{2}\cdot\frac{X(T)}{T}\cdot\frac{1}{X(T)}\cdot\sum_{i=1}^{X(T)} X_i^2 .$$

Letting T approach infinity and employing the law of large numbers, the latter gives

$$E[R] = \frac{1}{2}\lambda E\left[X^2\right], \tag{21}$$

where $E\left[X^2\right]$ is the second moment of the service time.

Utilizing (21) in (20), we obtain

$$E[W] = \frac{\rho}{2(1-\rho)}\lambda E\left[X^2\right]. \tag{22}$$

Thus, we establish the expected time a vehicle spends in the queue, $E\left[D_{t_2}\right]$ as

$$E\left[D_{t_2}\right]=\frac{\rho}{2(1-\rho)}\lambda E\left[X^2\right]+E\left[X\right]. \tag{23}$$

Upon employing the compressed queueing processes, Equation (23) reduces to

$$E\left[D_{t_2}\right]=\frac{\rho\lambda E\left[X^2\right]}{2(1-\rho)(1-\lambda\Delta)}+E\left[X\right]. \tag{24}$$

4.2.2 Variance

In this section, we are interested in the computation of $Var\left[D_{t_2}\right]$. First note that

$$D_{t_2}=W_t+X_t.$$

Therefore,

$$Var\left[D_{t_2}\right]=Var\left[W_t\right]+Var\left[X_t\right]-2Cov\left(W_t\cdot X_t\right),$$

but we know that W_t and X_t are independent random variables, thus

$$Var\left[D_{t_2}\right]=Var\left[W_t\right]+Var\left[X_t\right]. \tag{25}$$

Note that D_{t_2} is a sum of two independent random variables, that is, W_t and X_t. If the generating function of X_t is $X_t^*(s)$ and that of W_t is $W_t^*(s)$, the joint transformed probability generating function of D_{t_2} is

$$P_t^*(s)=W_t^*(s)+X_t^*(s),$$

where $P^*(s)$ and $X^*(s)$ are P-K transforms of the queueing system size and service time distributions respectively. Taking limits as $t\to\infty$, we have

$$\lim_{t\to\infty}P_t^*(s)=\lim_{t\to\infty}\left\{W_t^*(s)+X_t^*(s)\right\}.$$

Thus,

$$P^*(s)=W^*(s)+X^*(s).$$

Since the transform of the sum of two independent random variables is equivalent to the product of their transforms for instance see Ivo and Jacques, 2002, Section 2.3, then

$$P^*(s)=W^*(s)\times X^*(s).$$

Upon utilizing (17) and (18), $W^*(s)$ is given by

$$W^*(s)=\frac{(1-\rho)s}{\lambda X^*(s)+s-\lambda}.$$

Applying Little's law, $\rho=\lambda E\left[X\right]$, we have

$$W^*(s)=\frac{1-\rho}{1-\rho R^*(s)}. \tag{26}$$

Equation (26) is the P-K transform of waiting time distribution, hence to get the first and second moments of waiting time, we differentiate with respect to s and set $s=0$ to get

$$W'^*(s)=\frac{d}{ds}[\cdot]\Big|_{s=0}.$$

Hence,

$$E[W]=\frac{\rho E[R]}{1-\rho}, \tag{27}$$

Again, differentiating (26) twice with respect to s and set $s=0$, we get

$$W''^{*}(s)=\frac{d^2}{ds^2}[\cdot]\Big|_{s=0}.$$

Hence,

$$E\left[W^2\right]=2\left(E[W]\right)^2+\frac{\rho E\left[R^2\right]}{1-\rho}. \tag{28}$$

To compute $E\left[R^2\right]$ again, we consider Figure (2) and deduce that

$$E\left[R^2\right]=\frac{1}{T^2}\int_0^T R(t)\ dt,$$

which simplifies to

$$E\left[R^2\right]=\frac{\lambda E\left[X^3\right]}{3}.$$

Hence,

$$E[W]=\frac{\rho\lambda E\left[X^2\right]}{2(1-\rho)} \tag{29}$$

and

$$E\left[W^2\right]=\rho\lambda\left(\rho\lambda\left(\frac{E\left[X^2\right]}{1-\rho}\right)^2+\frac{E\left[X^3\right]}{3(1-\rho)}\right). \tag{30}$$

Therefore, $E\left[D_{t_2}{}^2\right]$ and $\left(E\left[D_{t_2}\right]\right)^2$ are obtained as

$$E\left[D_{t_2}{}^2\right]=\rho\lambda\left(\rho\lambda\left(\frac{E\left[X^2\right]}{1-\rho}\right)^2+\frac{E\left[X^3\right]}{3(1-\rho)}\right)+E\left[X^2\right] \tag{31}$$

and

$$\left(E\left[D_{t_2}\right]\right)^2=\left(\frac{\rho\lambda E\left[X^2\right]}{2(1-\rho)}+E[X]\right)^2, \tag{32}$$

respectively. Having obtained (31) and (32), the variance becomes

$$Var\left[D_{t_2}\right]=\frac{3}{4}\left(\frac{\rho\lambda E\left[X^2\right]}{1-\rho}\right)^2+\frac{\rho\lambda E\left[X^3\right]}{3(1-\rho)}+\left(1-\frac{\rho\lambda E[X]}{1-\rho}\right)E\left[X^2\right]-\left(E[X]\right)^2. \tag{33}$$

Employing the use of compressed queueing processes, (33) becomes

$$Var\left[D_{t_2}\right]=\frac{3}{4}\left(\frac{\rho\lambda E\left[X^2\right]}{(1-\rho)\cdot(1-\lambda\Delta)}\right)^2+\frac{\rho\lambda E\left[X^3\right]}{3(1-\rho)\cdot(1-\lambda\Delta)}+\left(1-\frac{\rho\lambda E[X]}{(1-\rho)\cdot(1-\lambda\Delta)}\right)E\left[X^2\right]-\left(E[X]\right)^2. \quad (34)$$

4.3 The Moments of Overall Traffic Delay

Notice from (3) that D_t can be split into two independent components, that is, D_{t_1} and D_{t_2}. In the previous sections, we have confined ourselves in the computation of mean and variance of D_{t_1} and D_{t_2}. In this section, we amalgamate the two sections to obtain $E[D_t]$ and $Var[D_t]$. To obtain $E[D_t]$, we have

$$E[D_t]=E\left[D_{t_1}\right]+E\left[D_{t_2}\right].$$

Hence,

$$E[D_t]=\frac{c_y\left(1-\frac{g_e}{c_y}\right)^2}{2\left(1-\frac{g_e}{c_y}\rho\right)}+\frac{\rho\lambda E\left[X^2\right]}{2(1-\rho)(1-\lambda\Delta)}+E[X]. \quad (35)$$

Similarly, $Var[D_t]$ is given by

$$Var[D_t]=Var\left[D_{t_1}\right]+Var\left[D_{t_2}\right]-2Cov\left(D_{t_1}.D_{t_2}\right),$$

and since D_{t_1} and D_{t_2} are independent components, we get $Var[D_t]$ as

$$Var[D_t]=\frac{c_y\left(1-\frac{g_e}{c_y}\right)^3\left(1+3\frac{g_e}{c_y}-4\frac{g_e}{c_y}\rho\right)}{12\left(1-\frac{g_e}{c_y}\rho\right)^2}+\frac{3}{4}\left(\frac{\rho\lambda E\left[X^2\right]}{(1-\rho)\cdot(1-\lambda\Delta)}\right)^2+\frac{\rho\lambda E\left[X^3\right]}{3(1-\rho)\cdot(1-\lambda\Delta)}$$

$$+\left(1-\frac{\rho\lambda E[X]}{(1-\rho)\cdot(1-\lambda\Delta)}\right)E\left[X^2\right]-\left(E[X]\right)^2. \quad (36)$$

5.0 Results and Discussion

In this section, we apply the developed overall traffic delay model on real traffic data collected at Kenyatta Avenue-Kimathi Street signalized intersection between 20th and 22nd February, 2013. The intermediate results from the data are given and simulation on the developed models using MATLAB software is performed for traffic intensities ranging from 0.1 to 1.9.

5.1 Computation of Parameters

For the simplicity of sampling and measurement, we assumed that the data collected on 20th, 21st and 22nd February, 2013 from 5:13 PM to 6:10 PM daily represented the traffic data on general weekdays. Considering a single lane controlled by a fixed-time traffic signal, we recorded the duration of the green lights that allow the vehicles to go through Kenyatta Avenue-Kimathi Street intersection and the number of vehicles passing during the effective green lights after every cycle time of 180 seconds. The data collected is provided in Tables 1 – 3.

To compute Δ, we also recorded the speed of vehicles on the queue and the distance between them on Friday, 22nd February, 2013, from 5:13 PM – 6:10 PM. The data is given in Table 4. Assuming this data to be a representative for all weekdays, we compute the average weekday speed of vehicles in the queue and distance in between them.

The average effective green time is

$$g_e = \frac{1}{3}\times\left(\frac{1396}{20}+\frac{1356}{20}+\frac{1342}{20}\right)$$
$$= 68.23 \text{ sec.}$$

Average service time during the green light is

$$\overline{x_{g_e}} = \frac{1}{3}\times\left(\frac{1396}{1405}+\frac{1356}{1383}+\frac{1342}{1302}\right)$$
$$= 1.002 \text{ sec.}$$

Average effective red time is

$$\overline{x_{g_e}} = \frac{1}{3}\times\left(\frac{2204}{20}+\frac{2240}{20}+\frac{2256}{20}\right)$$
$$= 111.67 \text{ sec.}$$

In our model, we assume that the traffic light is always running. Thus, service time of the first vehicle passing through the intersection when a green light turns on is considered to be equal to the red light duration. We denote the average service time for that vehicle as $\overline{x_r}$ given by

$$\overline{x_r} = t_r = 111.67 \text{ sec.}$$

For each green light during 5:13 PM – 6:10 PM, there is only one vehicle which has the service time $\overline{x_r}$. All the other vehicles have the service time $\overline{x_{g_e}}$. The green lights turn on 20 times, so the number of vehicles with service time $\overline{x_r}$ is equal to 20. The probability that a vehicle has a service time $\overline{x_r}$ is given by

$$\Pr\left[X = \overline{x_r}\right] = \frac{1}{3}\times\left(\frac{20}{1405}+\frac{20}{1383}+\frac{20}{1302}\right)$$
$$= 0.015.$$

And the probability that a vehicle has service time $\overline{x_{g_e}}$ is

$$\Pr\left[X = \overline{x_{g_e}}\right] = 1 - \Pr\left[X = \overline{x_r}\right]$$
$$= 0.985.$$

Thus, the average service time becomes

$$\bar{x} = \Pr\left[X = \overline{x_{g_e}}\right].\overline{x_{g_e}} + \Pr\left[X = \overline{x_r}\right].\overline{x_r}$$
$$= 2.66 \text{ sec.}$$

The average service rate is

$$\mu = \frac{1}{2.66} = 0.38 \text{ sec.}$$

Based on the data (Table 4), we can get the average speed of a vehicle in the queue during the time period (5:13 PM - 6:10 PM) as

$$\frac{256}{20} = 12.8 \text{ Km/h.}$$

Converting the above result to M/s, we have

$$12.8\times\frac{1000}{3600}=3.56 \text{ M/s.}$$

The average distance between the vehicles in the queue is

$$\overline{d}=\frac{25.7}{20}=1.285 \text{ M.}$$

Hence, Δ is obtained as

$$\Delta=\frac{1.285}{3.56}$$
$$=0.36 \text{ sec.}$$

5.2 Simulation of $\mathbf{E}\left[\mathbf{D}_t\right]$

Using Equation (35) and the collected data, we split $E\left[D_t\right]$ into $E\left[D_{t_1}\right]$ and $E\left[D_{t_2}\right]$ as described in Figure (3) by MATLAB software when we assumed that service times follow Exponential distribution with parameter $1/\lambda$.

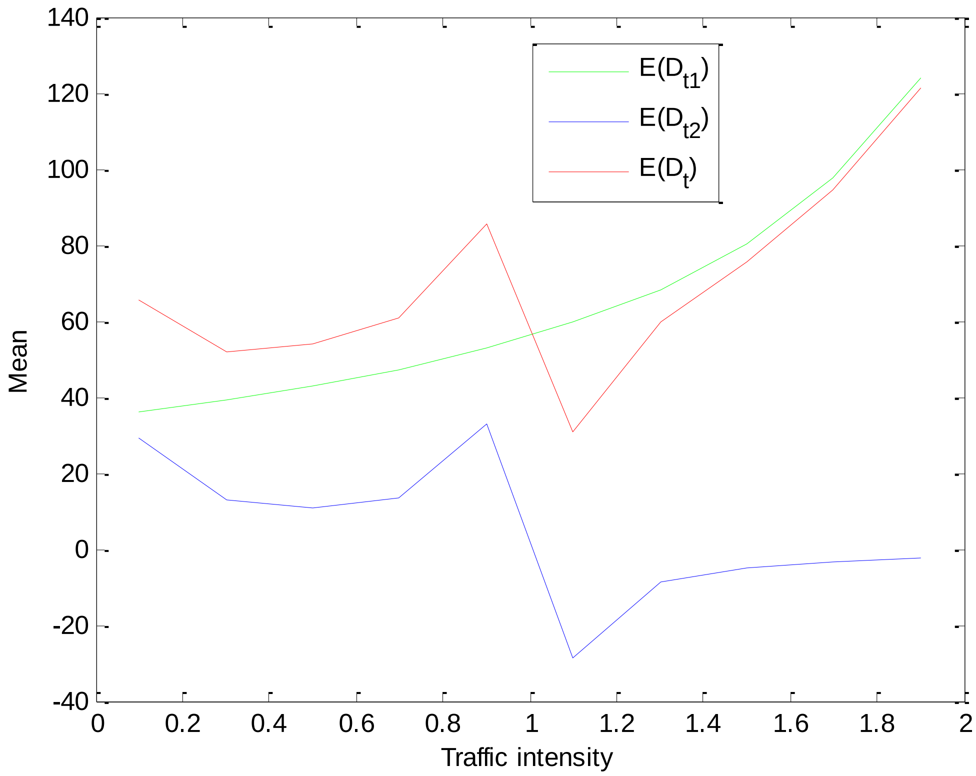


Figure 3: $\mathbf{E}\left[\mathbf{D}_{t_1}\right], \mathbf{E}\left[\mathbf{D}_{t_2}\right]$ and $\mathbf{E}\left[\mathbf{D}_t\right]$ versus ρ using Exponential distribution of service times.

From Figure (3), it is clear to note that the stochastic delay model is only applicable to undersaturated conditions ($\rho<1$) and estimate infinite delay when arrival flow approaches capacity. However, when arrival flow exceeds capacity oversaturated queues exist and continuous delay occurs. It is also evident that the deterministic delay model estimates continuous delay, but it does not completely deals with the effect of randomness when the arrival flows are close to capacity, and also fail when the traffic intensity is between 1.0 and 1.1. The figure shows that both components of our overall traffic delay model are incompatible when the traffic intensity is equal to 1.0. Therefore, our overall traffic delay model is used to fill the gap between the two models and also give more realistic results in the estimation of delay at signalized intersections. It predicts the delay for both undersaturated and oversaturated traffic conditions without having any discontinuity at the traffic intensity of

1.0. Similarly, with the assumption of service times following Gamma distribution, we obtained Figure (4) below by MATLAB.

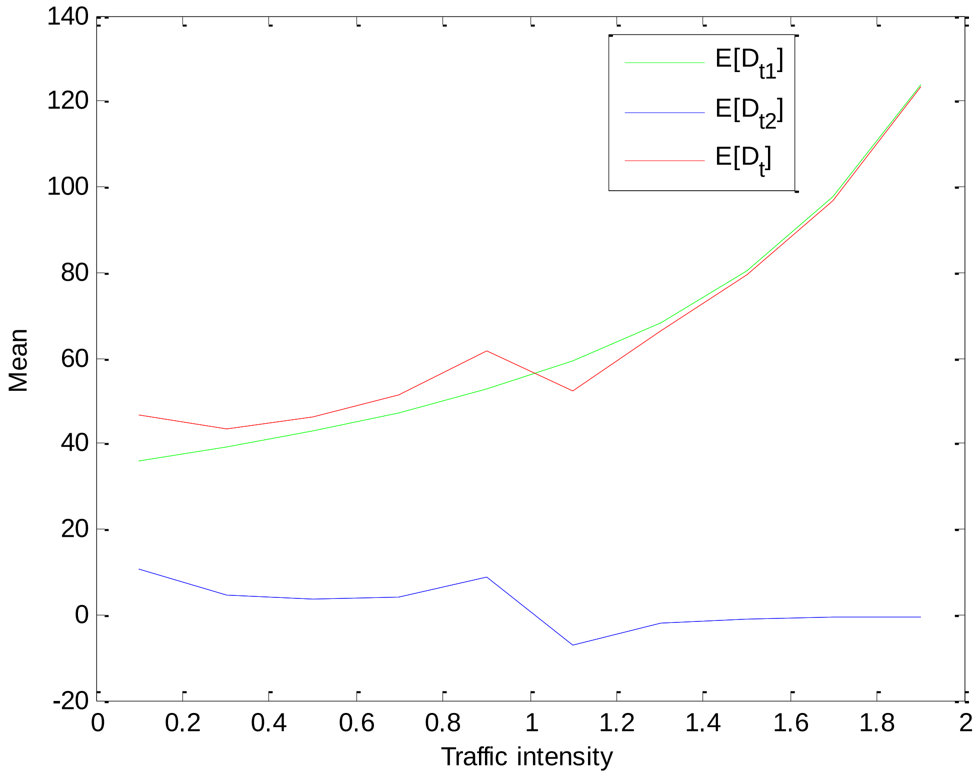


Figure 4: $\mathbf{E\left[D_{t_1}\right], E\left[D_{t_2}\right] \text{and} \, E\left[D_t\right]}$ **versus** ρ **using Gamma distribution of service times.**

We depict that under this assumption, $E\left[D_t\right]$ increases rapidly with ρ than in the Exponential assumption under oversaturated traffic conditions ($\rho \geq 1.15$), although the general behaviour is similar to the Exponential assumption. From the figure, $E\left[D_{t_1}\right]$ remains the same as that of exponential distribution of service times. Comparing Figure (3) and Figure (4), Figure (3) estimates a lower value of $E\left[D_t\right]$ than Figure (4), that is, Figure (4) estimates $E\left[D_t\right]$ to be 43.12 seconds while Figure (3) estimates $E\left[D_t\right]$ to be 30.93 seconds. Also, Figure (4) estimates higher values of $E\left[D_t\right]$ as $\rho \geq 1.5$. This is contrary to what $E\left[D_t\right]$ with exponential distribution of service times estimates. Therefore, exponential distribution of service times is far much preferred since we are interested in a reduced mean of overall traffic delay at the intersection.

5.3 Simulation of $\mathbf{Var\left[D_t\right]}$

To investigate the major contributor to $Var\left[D_t\right]$ by D_{t_1} and D_{t_2}, we plot the graphs of $Var\left[D_t\right]$, $Var\left[D_{t_1}\right]$ and $Var\left[D_{t_2}\right]$ versus ρ as shown in Figure (5) below.

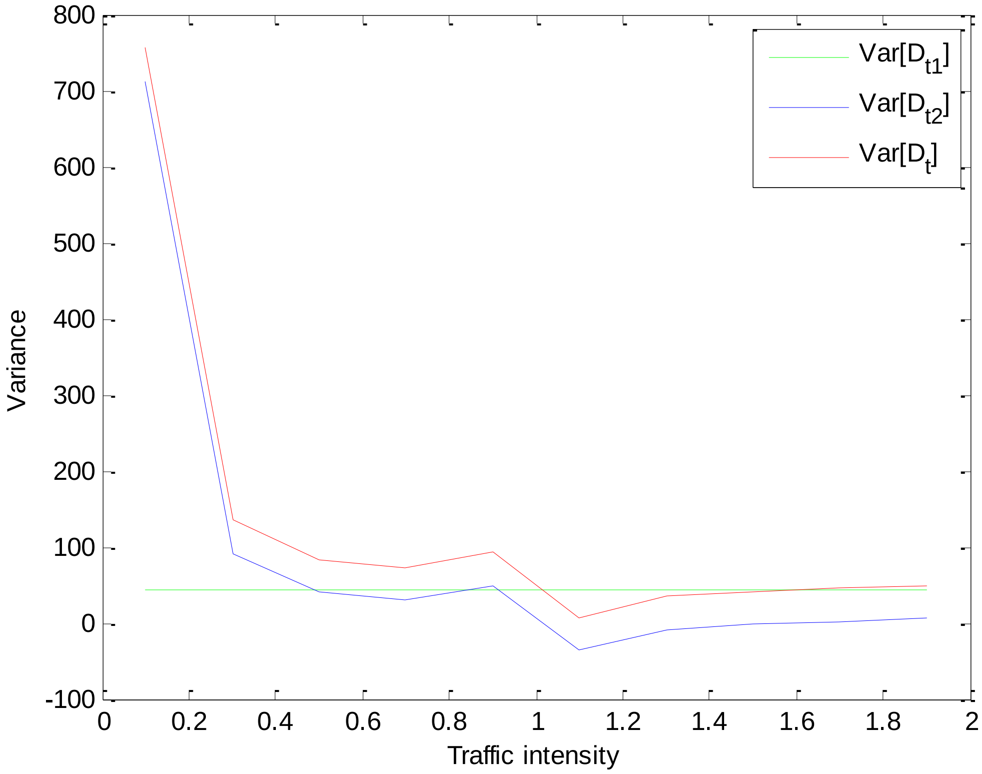


Figure 5: $\mathbf{Var}\left[\mathbf{D}_{\mathbf{t}_1}\right]$, $\mathbf{Var}\left[\mathbf{D}_{\mathbf{t}_2}\right]$ and $\mathbf{Var}\left[\mathbf{D}_{\mathbf{t}}\right]$ versus ρ using Exponential distribution of service times.

From Figure (5), the deterministic model shows no variation because of its constant service times while stochastic model provides a reasonable estimate of variance only under light traffic conditions ($\rho << 1.0$), that is, the variance is time-independent and infinite variance is estimated as ρ approaches 1.0. Therefore, the contributing factor in the estimation of $Var\left[D_t\right]$ is D_{t_2} since $Var\left[D_{t_1}\right]$ is zero. A similar scenario is depicted when we assume Gamma distribution for service times as shown in Figure (6) below.

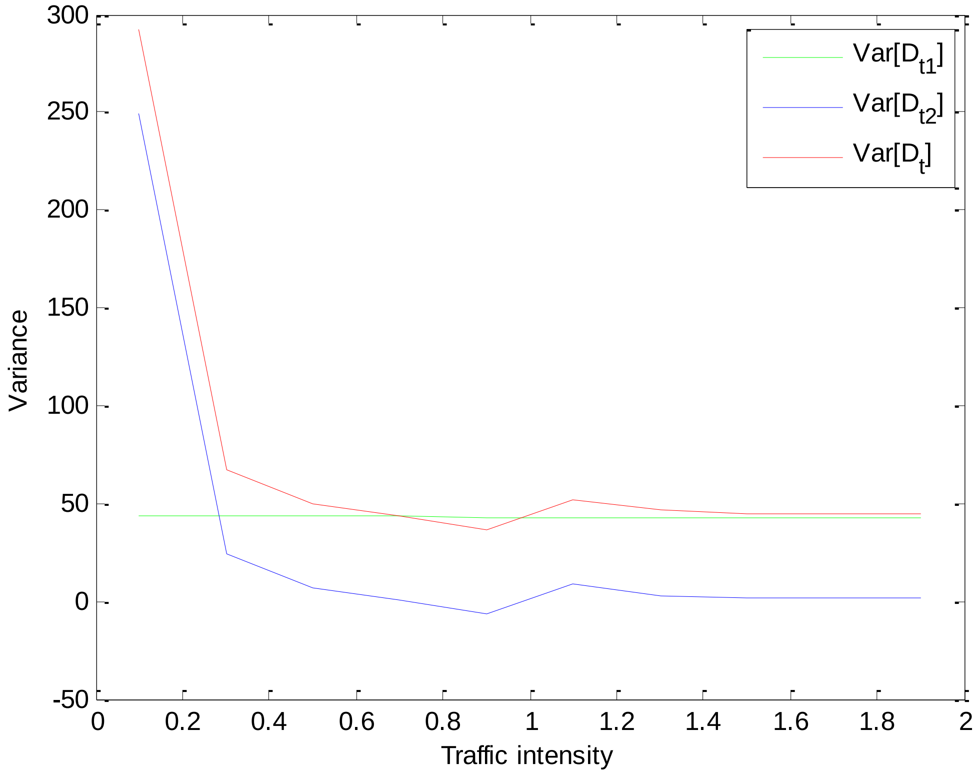


Figure 7: $\mathbf{Var}\left[\mathbf{D}_{\mathbf{t}_1}\right]$**,** $\mathbf{Var}\left[\mathbf{D}_{\mathbf{t}_2}\right]$ **and** $\mathbf{Var}\left[\mathbf{D}_{\mathbf{t}}\right]$ **versus** ρ **using Gamma distribution of service times.**

Again, D_{t_2} remains constant as that of exponential distribution of service times due to its deterministic nature of arrivals and service. The stochastic delay component estimates infinite variance when $0.7 \le \rho \ge 0.9$ contrary to its assumption of steady-state (Hurdle, 1984). This disregards our assumption that D_{t_2} is a steady-state model. Also, Figure (7) estimates higher values of $Var\left[D_t\right]$ as compared to Figure (6), that is, Figure (7) estimates 36.74 seconds while Figure (6) estimates 7.731 seconds as the lowest values of $Var\left[D_t\right]$. Therefore, exponential distribution of service times is far much preferred since a lower variance results to a reduced overall traffic delay at the intersection.

5.4 Application of $\mathbf{Var}\left[\mathbf{D}_{\mathbf{t}}\right]$

Variability of Level of Service

The possible use of delay variability in quantifying level of service for a signalized intersection is illustrated in this section. In this study, the level of service at the intersection was defined in terms of expected overall traffic delay. With the ability to estimate the variance of overall traffic delay, it is feasible to integrate the concept of reliability into design and analysis of a signalized intersection. For example, delay of a certain percentile, instead of expected value, can be used to define the level of service. A 95th-percentile delay means that 95 percent of the vehicles would encounter a traffic delay less than or equal to this delay. The percentile value can be approximately estimated using $E\left[D_t\right]+z_\alpha\sqrt{Var\left[D_t\right]}$ where, z_α is a statistic for the normal distribution and can be determined on the basis of the pre-specified reliability level. Figure (8) below shows expected overall traffic delay and 90th-percentile delay (with $z_\alpha \approx 1.3$) under different traffic intensities. It is assumed that the ranges of traffic delay values used in defining each level of service in the HCM are also applicable to vehicles, as shown in Figure (8). It can be observed that for the given case with a traffic intensity of 0.9, the expected overall traffic delay is 85.6 seconds, which would yield LoS C (point a). However, if the 90th percentile delay is used, the LoS would be D (point b). On the other hand, in order to guarantee that 90 percent of the vehicles going through the intersection encounter LoS C or higher, the traffic intensity needs to be reduced to 0.7 (point c) by either increasing the capacity or reducing the number of arrivals per unit time.

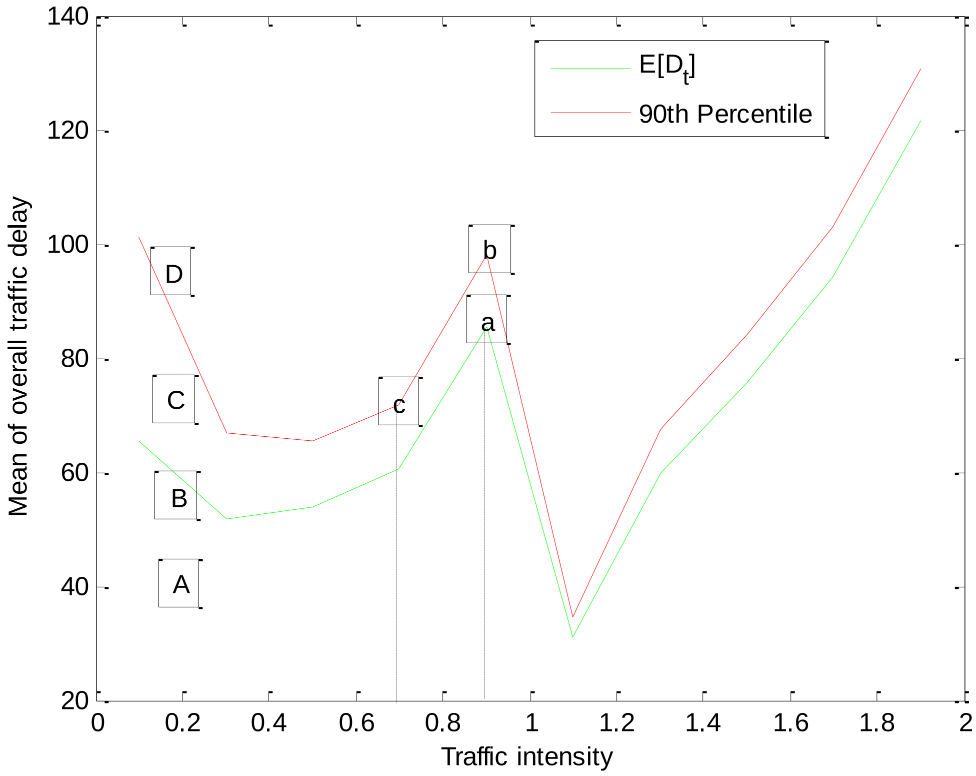


Figure 8: $E[D_t]$ **and 90th – percentile delay (with** $z_\alpha \approx 1.3$ **) versus** ρ **.**

6.0 Conclusion and Recommendation

6.1 Conclusion

Considering the uniform and random properties of traffic flows, the models for estimating deterministic and stochastic delay components of traffic delay were successfully developed in this study. With the application of compressed queueing processes in order to better describe the variation in traffic flows, the developed models indeed estimate the mean and variance of traffic delay at the signalized intersection.

From the developed moments of the deterministic and stochastic delay components of traffic delay, the central moments of the overall traffic delay model were developed. These moments estimate the mean and variance of the overall traffic delay at the signalized intersection.

To validate the developed model, the model was applied to real traffic data collected at Kenyatta Avenue - Kimathi Street intersection and a simulation was performed for traffic intensities ranging from 0.1 to 1.9 using MATLAB software. The simulation results confirmed the result that exists in literature that oversaturated conditions and random delay renders the stochastic model unrealistic. Furthermore, the results preferred exponential distribution of service times to gamma distribution since it resulted to a lower variance hence led to a reduced overall traffic delay.

6.2 Recommendation

In the study presented herein, the overall traffic delay model was developed for a fixed-time traffic light, and further studies should be conducted for vehicle-actuated type of traffic lights.

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TABLES

Table 1: Traffic data collected on Wednesday, 20th February, 2013

Time (PM)	t (sec)	R (sec)	AR (sec)	G (sec)	Y (sec)	l_1 (sec)	l_2 (sec)	g_e (sec)	r (sec)	No. of vehicles passed
5:13	3600	103	10	52	15	3	4	70	110	73
5:16	3600	104	10	51	15	3	3	70	110	71
5:19	3600	102	10	53	15	4	2	72	108	72
5:22	3600	103	10	52	15	3	4	70	110	75
5:25	3600	101	10	54	15	2	3	74	106	76
5:28	3600	104	10	51	15	3	3	70	110	70
5:31	3600	106	10	49	15	3	4	67	113	70
5:34	3600	107	10	48	15	3	3	67	113	69
5:37	3600	102	10	53	15	4	3	71	109	73
5:40	3600	104	10	51	15	3	4	69	111	71
5:43	3600	97	10	52	15	3	3	77	103	73
5:46	3600	103	10	52	15	3	3	71	109	73
5:49	3600	104	10	51	15	3	3	70	110	70
5:52	3600	103	10	52	15	4	3	70	110	68
5:55	3600	105	10	50	15	3	4	68	112	67
5:58	3600	106	10	49	15	3	2	69	111	69
6:01	3600	106	10	49	15	3	3	68	112	65
6:04	3600	108	10	47	15	2	3	67	113	69
6:07	3600	105	10	50	15	3	4	68	112	66
6:10	3600	107	10	48	15	2	3	68	112	65
	Total							**1396**	**2204**	**1405**

Table 2: Traffic data collected on Thursday, 21st February, 2013

Time (PM)	t (sec)	R (sec)	AR (sec)	G (sec)	Y (sec)	l_1 (sec)	l_2 (sec)	g_e (sec)	r (sec)	No. of vehicles passed
5:13	3600	105	10	50	15	3	4	68	112	71
5:16	3600	106	10	49	15	3	3	68	112	69
5:19	3600	104	10	51	15	4	2	70	110	70
5:22	3600	105	10	49	15	3	4	68	112	73
5:25	3600	101	10	52	15	2	3	72	106	74
5:28	3600	104	10	49	15	3	3	68	110	68
5:31	3600	108	10	47	15	3	4	65	115	68
5:34	3600	109	10	46	15	3	3	65	115	67
5:37	3600	104	10	51	15	4	3	69	111	72
5:40	3600	106	10	49	15	3	4	67	113	69
5:43	3600	106	10	49	15	3	3	68	112	71
5:46	3600	105	10	50	15	3	3	69	111	71
5:49	3600	106	10	49	15	3	3	68	112	69
5:52	3600	105	10	50	15	4	3	68	112	71
5:55	3600	107	10	48	15	3	4	66	114	70
5:58	3600	108	10	47	15	3	2	67	113	67
6:01	3600	106	10	47	15	3	3	66	114	68
6:04	3600	110	10	45	15	2	3	65	115	65
6:07	3600	107	10	48	15	3	4	66	114	66
6:10	3600	109	10	46	15	2	3	66	114	64
	Total							**1356**	**2240**	**1383**

Table 3: Traffic data collected on Friday, 22nd February, 2013

Time (PM)	t (sec)	R (sec)	AR (sec)	G (sec)	Y (sec)	l_1 (sec)	l_2 (sec)	g_e (sec)	r (sec)	No. of vehicles passed
5:13	3600	106	10	49	15	3	4	67	113	65
5:16	3600	105	10	50	15	3	3	69	111	67
5:19	3600	104	10	51	15	4	2	70	110	67
5:22	3600	107	10	48	15	3	4	66	114	64
5:25	3600	104	10	51	15	2	3	71	109	68
5:28	3600	104	10	49	15	3	3	68	110	67
5:31	3600	106	10	48	15	3	4	67	113	65
5:34	3600	109	10	46	15	3	3	65	115	64
5:37	3600	105	10	50	15	4	3	68	112	64
5:40	3600	107	10	48	15	3	4	66	114	64
5:43	3600	107	10	48	15	3	3	67	113	65
5:46	3600	104	10	51	15	3	3	70	110	67
5:49	3600	107	10	48	15	3	3	67	113	65
5:52	3600	106	10	49	15	4	3	67	113	65
5:55	3600	107	10	47	15	3	4	66	114	64
5:58	3600	108	10	47	15	3	2	67	113	65
6:01	3600	109	10	46	15	3	3	65	115	64
6:04	3600	110	10	45	15	2	3	65	115	64
6:07	3600	109	10	46	15	3	4	64	116	63
6:10	3600	108	10	47	15	2	3	67	113	65
	Total							**1342**	**2256**	**1302**

Table 4: Average speed per vehicle and distance between the vehicles on the queue

Time (PM)	No. of vehicles passed	Average speed per vehicle (Km/h)	Distance between the vehicles on the queue (Meters)
5:13	65	13	1.2
5:16	67	14	1.4
5:19	67	14	1.3
5:22	64	12	1.3
5:25	68	15	1.2
5:28	67	14	1.4
5:31	65	13	1.3
5:34	64	12	1.4
5:37	64	12	1.4
5:40	64	12	1.3
5:43	65	13	1.2
5:46	67	14	1.2
5:49	65	13	1.1
5:52	65	13	1.3
5:55	64	12	1.3
5:58	65	13	1.3
6:01	64	12	1.4
6:04	64	12	1.4
6:07	63	10	1.1
6:10	65	13	1.2
Total	**1302**	**256**	**25.7**