About Testing the Hypothesis of Equality of Two Bernoulli Regression Curves

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Abstract

The limiting distribution of an integral square deviation between two kernel type estimators of Bernoulli regression functions is established in the case of two independent samples. The criterion of testing is constructed for both simple and composite hypotheses of equality of two Bernoulli regression functions. The question of consistency is studied. The asymptotics of behavior of the power of test is investigated for some close alternatives.

Keywords: Bernoulli Regression Function, Power of Test, Consistency, Composite Hypothesis

1. Introduction

Let random variables $Y^{(i)}$, $i=1,2$, take two values 1 and 0 with probabilities p_i (succes) and $1-p_i$, $i=1,2$ (failure), respectively. Assume that the probability of success p_i is the function of an independent variable $x\in[0,1]$, i.e. $p_i = p_i(x) = \mathsf{P}\{Y^{(i)}=1 \mid x\}$ $(i=1,2)$ (see [1]-[3]). Let t_j, $j=1,\dots,n$, be the devision points of the interval $[0,1]$:

$$t_j=\frac{2j-1}{2n},\qquad j=1,\dots,n.$$

Let further $Y_i^{(1)}$ and $Y_i^{(2)}$, $i=1,\dots,n$, be mutually independent random Bernoulli variables with $\mathsf{P}\{Y_i^{(k)}=1 \mid t_i\}=p_k(t_i)$, $\mathsf{P}\{Y_i^{(k)}=0 \mid t_i\}=1-p_k(t_i)$, $i=1,\dots,n$, $k=1,2$. Using the samples $Y_1^{(1)},\dots,Y_n^{(1)}$ and $Y_1^{(2)},\dots,Y_n^{(2)}$ we want to chek the hypothesis

$$H_0: p_1(x)=p_2(x)=p(x),\quad x\in[0,1],$$

against the sequence of "close" alternatives of the form

$$H_{1n}: p_k(x)=p(x)+\alpha_n u_k(x)+o(\alpha_n),\quad k=1,2.$$

where $\alpha_n \to 0$ relevantly, $u_1(x) \neq u_2(x)$, $x \in [0,1]$ and $o(\alpha_n)$ uniformly in $x \in [0,1]$.

The problem of comparing two Bernoulli regression functions arises in some applications, for example in quantal biossays in pharmacology. There x denotes the dose of a drung and $p(x)$ the probability of response to the dose x.

We consider the crietrion of testing the hypothesis H_0 based on the statistic

$$T_n = \frac{1}{2}\, nb_n \int_{\Omega_n(\tau)} \left[\hat{p}_{1n}(x) - p_{2n}(x)\right]^2 p_n^2(x)\,dx = \frac{1}{2}\, nb_n \int_{\Omega_n(\tau)} \left[p_{1n}(x) - p_{2n}(x)\right]^2 dx,$$

$$\Omega_n(\tau) = \left[\tau b_n, (1-\tau) b_n\right], \quad \tau > 0,$$

where

$$\hat{p}_{in}(x) = p_{in}(x)\, p_n^{-1}(x),$$

$$p_{in}(x) = \frac{1}{nb_n} \sum_{j=1}^{n} K\left(\frac{x - t_j}{b_n}\right) Y_j^{(i)}, \quad i = 1, 2,$$

$$p_n(x) = \frac{1}{nb_n} \sum_{i=1}^{n} K\left(\frac{x - t_i}{b_n}\right),$$

$K(x)$ is some distribution density and $b_n \to 0$ is a sequence of positive numbers, $\hat{p}_{in}(x)$ is the kernel estimator of the regression function (see [4], [5]).

2. Assumptions and Notation

We assume that a kernel $K(x) \geq 0$ is chosen so that it is a function of bounded variation and satisfies the conditions: $K(x) = K(-x)$, $K(x) = 0$ for $|x| \geq \tau > 0$, $\int K(x)dx = 1$. The class of such functions is denoted by $H(\tau)$.

We also introduce the notation:

$$T_n^{(1)} = \frac{1}{2}\, nb_n \int_{\Omega_n(\tau)} \left[\tilde{p}_{1n}(x) - \tilde{p}_{2n}(x)\right]^2 dx, \qquad \tilde{p}_{in}(x) = p_{in}(x) - Ep_{in}(x), \qquad i = 1, \dots, n,$$

$$Q_{ij} = \psi_n(t_i, t_j), \quad \psi_n(u, v) = \int_{\Omega_n(\tau)} K\left(\frac{x-u}{b_n}\right) K\left(\frac{x-v}{b_n}\right) dx,$$

$$\sigma_n^2 = \frac{1}{(nb_n)^2} \sum_{k=2}^{n} d_k \sum_{i=1}^{k-1} d_i Q_{ik}^2, \qquad d_i = d(t_i) = \sum_{k=1}^{2} p_k(t_i)\left(1 - p_k(t_i)\right).$$

3. Auxiliary Assertions

Lemma 1 ([6]). *Let* $K(x) \in H(\tau)$ *and* $p(x)$, $x \in [0,1]$, *be a function of bounded variation. If* $nb_n \to \infty$, *then*

$$\frac{1}{nb_n} \sum_{i=1}^{n} K^{\nu_1}\left(\frac{x - t_i}{b_n}\right) K^{\nu_2}\left(\frac{y - t_i}{b_n}\right) p^{\nu_3}(t_i) = \frac{1}{b_n} \int_0^1 K^{\nu_1}\left(\frac{x-u}{b_n}\right) K^{\nu_2}\left(\frac{y-u}{b_n}\right) p^{\nu_3}(u)\,du + O\left(\frac{1}{nb_n}\right)$$

uniformly in $x, y \in [0,1]$, *where* $\nu_i \in N \cup \{0\}$, $i = 1,2,3$.

Lemma 2. *Let* $K(x) \in H(\tau)$, $p(x) \in C^1[0,1]$ *and* $u_1(x)$, $u_2(x)$ *be continuous functions on* $[0,1]$. *If*

$nb_n^2 \to \infty$ *and* $\alpha_n b_n^{-1/2} \to 0$, *then for the hypothesis* H_{1n}

$$b_n^{-1}\sigma_n^2 \to \sigma^2(p) = 2\int_0^1 p^2(x)\left(1-p(x)\right)^2 dx \int_{|x|\le 2\tau} K_0^2(x)\,dx \quad (1)$$

and

$$b_n^{-1/2}\left(\Delta_n - \Delta(p)\right) = O\left(b_n^{1/2}\right) + O\left(\alpha_n b_n^{-1/2}\right) + O\left(\frac{1}{nb_n^{3/2}}\right), \quad (2)$$

where

$$\Delta_n = ET_n^{(1)}, \quad \Delta(p) = \int_0^1 p(x)\left(1-p(x)\right)dx \int_{|x|\le\tau} K^2(u)\,du,$$
$$K_0 = K * K, \quad * \text{ is the convolution operator}.$$

Proof. We have

$$\sigma_n^2 = \frac{1}{(nb_n)^2}\left(\frac{1}{2}\sum_{i,k=1}^n d_i d_k Q_{ik}^2 - \frac{1}{2}\sum_{i=1}^n d_i^2 Q_{ii}^2\right) = A_1(n) + A_2(n) \quad (3)$$

where

$$d_k = d(t_k) = p_1(t_k)\left(1-p_1(t_k)\right) + p_2(t_k)\left(1-p_2(t_k)\right), \quad k = 1,\ldots,n,$$

$$d_k = 2p(t_k)\left(1-p(t_k)\right) + O(\alpha_n), \; k = 1,2, \quad (4)$$

uniformly in $t_k \in [0,1]$.

It can be easily established that

$$b_n^{-1}\left|A_2(n)\right| = \frac{1}{2}n^{-2}b_n^{-3}\sum_{i=1}^n d_i^2 \left(\int_{\Omega_n(\tau)} K^2\left(\frac{x-t_i}{b_n}\right)dx\right)^2 \le c_1\frac{1}{nb_n} + c_2\frac{\alpha_n}{nb_n}. \quad (5)$$

From the definition of Q_{ik} and (4) follows

$$A_1(n) = \frac{1}{2}(nb_n)^{-2}\int_{\bar{\Omega}_n(\tau)}\left[\sum_{i=1}^n\left(2p(t_i)\left(1-p(t_i)\right) + O(\alpha_n)\right)K\left(\frac{x-t_i}{b_n}\right)K\left(\frac{y-t_i}{b_n}\right)\right]^2 dx\,dy,$$
$$\bar{\Omega}_n(\tau) = \Omega_n(\tau)\times\Omega_n(\tau).$$

Further, using Lemma 1 and also taking into account that $p(x)\in C^1[0,1]$ and $\left[\frac{x-1}{b_n}, \frac{x}{b_n}\right] \supset [-\tau, \tau]$ for all $x \in \Omega_n(\tau)$, it is easy to show that

$$b_n^{-1}A_1(n) = 2\int_{\Omega_n(\tau)} p^2(x)\left(1-p(x)\right)^2 \int_{\frac{x-1}{b_n}+\tau}^{\frac{x}{b_n}-\tau} K_0^2\left(\frac{x-y}{b_n}\right)dx\,dy + O(b_n) + O\left(\left(\alpha_n b_n^{-1/2}\right)^2\right) + O(\alpha_n) + O\left(\frac{1}{nb_n^2}\right).$$

Thus

$$b_n^{-1}A_1(n)\to 2\int_0^1 p^2(x)(1-p(x))^2\,dx\int_{|x|\le 2\tau}K_0^2(x)\,dx\,. \tag{6}$$

From (5) and (6) follows statement (1).

Further, using the above-mentioned method, we can write

$$\Delta_n = ET_n^{(1)} = \frac{1}{2}(nb_n)^{-1}\int_{\Omega_n(\tau)}\sum_{i=1}^{n}K^2\left(\frac{x-t_i}{b_n}\right)d(t_i)\,dx =$$

$$=\int_{\Omega_n(\tau)}\left[\int_{\frac{x-1}{b_n}}^{\frac{x}{b_n}}K^2(u)\,p(x-b_nu)(1-p(x-b_nu))\,du\right]dx+O\left(\frac{1}{nb_n}\right)+O(\alpha_n)=$$

$$=\int_0^1 p(x)(1-p(x))\,dx\int_{|x|\le\tau}K^2(x)\,dx+O(b_n)+O(\alpha_n)+O\left(\frac{1}{nb_n}\right).$$

Thus

$$b_n^{-1/2}\left(\Delta_n-\Delta(p)\right)=O\left(b_n^{1/2}\right)+O\left(\alpha_n b_n^{-1/2}\right)+O\left(\frac{1}{nb_n^{3/2}}\right).$$

The lemma is proved.

Asymptotical Normality of the Statistic T_n

We have the following assertion.

Theorem 1. *Let* $K(x)\in H(\tau)$ *and*

$$p(x), u_1(x), u_2(x)\in C^1[0,1].$$

If $nb_n^2\to\infty$, $\alpha_n b_n^{-1/2}\to 0$ *and* $nb_n^{1/2}\alpha_n^2\to c_0$, $0<c_0<\infty$, *then for the hypothesis* H_{1n}

$$b_n^{-1/2}\left(T_n-\Delta(p)\right)\sigma^{-1}(p)\xrightarrow{d}N(a,1),$$

where $\Delta(p)$ *and* $\sigma^2(p)$ *are defined in Lemma 2 and* $\xrightarrow{d}$ *denotes convergence in distribution and* $N(a,1)$ *is a random variable having the standard normal distribution with parameters* $(a,1)$,

$$a=\frac{c_0}{2\sigma(p)}\int_0^1\left(u_1(x)-u_2(x)\right)^2dx\,.$$

Proof. We have

$$T_n = T_n^{(1)}+L_n^{(1)}+L_n^{(2)},$$

where

$$L_n^{(1)} = nb_n\int_{\Omega_n(\tau)}\left[\tilde p_{1n}(x)-\tilde p_{2n}(x)\right]\left[Ep_{1n}(x)-Ep_{2n}(x)\right]dx,$$

$$L_n^{(2)} = \frac{1}{2}nb_n\int_{\Omega_n(\tau)}\left[Ep_{1n}(x)-Ep_{2n}(x)\right]^2dx.$$

By the Lemma 1, it is clear

$$b_n^{-1/2}L_n^{(2)}=\frac{1}{2}nb_n^{1/2}\alpha_n^2\int_{\Omega_n(\tau)}\left\{\frac{1}{b_n}\int_0^1 K\left(\frac{x-t}{b_n}\right)\left[u_1(t)-u_2(t)\right]dt+O\left(\frac{1}{nb_n}\right)\right\}^2 dx\,. \tag{7}$$

Since $\left[\frac{x-1}{b_n},\frac{x}{b_n}\right]\supset[-\tau,\tau]$ for all $x\in\Omega_n(\tau)$, then from (7) we find

$$b_n^{-1/2}L_n^{(2)}=\frac{1}{2}nb_n^{1/2}\alpha_n^2\int_{\Omega_n(\tau)}\left[\int_{-\tau}^{\tau} K(t)\left(u_1(x-b_nt)-u_2(x-b_nt)\right)dt+O\left(\frac{1}{nb_n}\right)\right]^2 dx\,. \tag{8}$$

Further, since $u_1(x),u_2(x)\in C^1[0,1]$, then from (8) we have

$$b_n^{-1/2}L_n^{(2)}\longrightarrow\frac{c_0}{2}\int_0^1\left(u_1(t)-u_2(t)\right)^2 dt\,. \tag{9}$$

Now, we show that $b_n^{-1/2}L_n^{(1)}\xrightarrow{\mathbb{P}}0$. We have

$$b_n^{-1/2}L_n^{(1)}=\frac{1}{2}nb_n^{1/2}\int_{\Omega_n(\tau)}\tilde{p}_{1n}(x)\left(Ep_{1n}(x)-Ep_{2n}(x)\right)dx- -\frac{1}{2}nb_n^{1/2}\int_{\Omega_n(\tau)}\tilde{p}_{2n}(x)\left(Ep_{1n}(x)-Ep_{2n}(x)\right)dx=$$

$$=I_n^{(1)}+I_n^{(2)}\,. \tag{10}$$

It is clear that

$$E\left|I_n^{(1)}\right|\le\left(E\left(I_n^{(1)}\right)^2\right)^{1/2}=\frac{1}{2}nb_n^{1/2}\left[E\left(\int_{\Omega_n(\tau)}\tilde{p}_{1n}(x)\left(Ep_{1n}(x)-Ep_{2n}(x)\right)dx\right)^2\right]^{1/2}=$$

$$=\frac{1}{2}nb_n^{1/2}\left[\int_{\bar{\Omega}_n(\tau)}\operatorname{cov}\left(p_{1n}(x_1),p_{1n}(x_2)\right)\left(Ep_{1n}(x_1)-Ep_{2n}(x_1)\right)\left(Ep_{1n}(x_2)-Ep_{2n}(x_2)\right)dx_1\,dx_2\right]^{1/2},$$

$$\bar{\Omega}_n(\tau)=\Omega_n(t)\times\Omega_n(\tau).$$

Easily verify, that

$$\operatorname{cov}\left(p_{1n}(x_1),p_{1n}(x_2)\right)=\frac{1}{(nb_n)^2}\sum_{i=1}^n K\left(\frac{x_1-t_i}{b_n}\right)K\left(\frac{x_2-t_i}{b_n}\right)p_1(t_i)\left(1-p_1(t_i)\right)$$

and by Lemma 2 we can now write

$$\operatorname{cov}\left(p_{1n}(x_1),p_{1n}(x_2)\right)=n^{-1}b_n^{-2}\times$$
$$\times\int_0^1 K\left(\frac{x_1-u}{b_n}\right)K\left(\frac{x_2-u}{b_n}\right)p_1(u)\left(1-p_1(u)\right)du+O\left(\frac{1}{(nb_n)^2}\right).$$

Thus

$$E\left|I_n^{(1)}\right| \le \frac{1}{2} n b_n^{1/2} \left\{ \int_{\bar{\Omega}_n(\tau)} \left[\frac{1}{nb_n^2} \int_0^1 K\left(\frac{x_1-u}{b_n}\right) K\left(\frac{x_2-u}{b_n}\right) p_1(u)(1-p_1(u))\,du + \frac{1}{(nb_n)^2} \right] \times \right.$$

$$\left. \times \left(Ep_{1n}(x_1) - Ep_{2n}(x_1)\right)\left(Ep_{1n}(x_2) - Ep_{2n}(x_2)\right) dx_1 dx_2 \ \times \right\}^{1/2} \le c_3 \sqrt{n}\, b_n^{1/2} \alpha_n = c_3 \frac{n b_n^{1/2} \alpha_n^2}{\sqrt{n}\, \alpha_n} \longrightarrow 0,$$

since, by condition, $nb_n^{1/2}\alpha_n^2 \to c_0$, $0 < c_0 < \infty$ and

$$\sqrt{n}\,\alpha_n = \frac{\sqrt{nb_n^{1/2}\alpha_n^2}}{b_n^{1/4}} \to \infty .$$

So, $I_n^{(1)} \xrightarrow{\mathbb{P}} 0$. Analogously we can show that $I_n^{(2)} \xrightarrow{\mathbb{P}} 0$.

Hence

$$L_n^{(1)} \xrightarrow{\mathbb{P}} 0 . \tag{11}$$

Further, to prove the theorem it remains to show

$$\frac{T_n^{(1)} - \Delta_n}{\sigma_n} \xrightarrow{d} N(0,1) . \tag{12}$$

Since the proof of (12) is similar to that of Theorem 1 from [7], we omit it.

Using the representation $T_n = T_n^{(1)} + L_n^{(1)} + L_n^{(2)}$, Lemma 2, (9), (11) and (12), we find that

$$b_n^{-1/2}\left(\frac{T_n - \Delta(p)}{\sigma(p)}\right) \xrightarrow{d} N\left(\frac{c_0}{2\sigma(p)} \int_0^1 \left(u_1(x) - u_2(x)\right)^2 dx, 1\right).$$

The theorem is proved.

The conditions of Theorem 1 for b_n and α_n are fulfilled if we assume $b_n = b_0 n^{-\delta}$ and $\alpha_n = \alpha_0 n^{-1/2+\delta/4}$ for $0 < \delta < 1/2$.

Corollary. *Let* $K(u) \in H(\tau)$ *and* $p(x) \in C^1[0,1]$. *If* $nb_n^2 \to \infty$, *then for the hypothesis* H_0

$$b_n^{-1/2}\left(T_n - \Delta(p)\right)\sigma^{-1}(p) \xrightarrow{d} N(0,1) . \tag{13}$$

4. Application of the Statistic T_n for the Hypothesis Testing

As an important application of the result of the corollary let us construct the criterion of testing the simple hypothesis H_0: $p_1(x) = p_2(x) = p(x)$ (this is the case with given $p(x)$); the critical domain is defined by the inequality

$$T_n \ge d_n(\alpha) = \Delta(p) + b_n^{1/2}\sigma(p)\lambda_\alpha ,$$

And from Theorem 1 we establish that the local behavior of the power $\mathbb{P}_{H_{1n}}\left(T_n \ge d_n(\alpha)\right)$ is as follows

$$\mathbb{P}_{H_{1n}}\left(T_n \ge d_n(\alpha)\right) \longrightarrow 1-\Phi\left(\lambda_\alpha - \frac{A(u)}{\sigma(p)}\right),$$

where

$$A(u) = \frac{c_0}{2}\int_0^1 \left(u_1(x)-u_2(x)\right)^2 dx, \quad u=(u_1,u_2),$$

$\Phi(\lambda_\alpha)=1-\alpha$, $\Phi(\lambda)$ is a standard normal distribution.

Note note that in (13) the statistic function T_n is normalized by the values $\Delta(p)$ and $\sigma^2(p)$ which depend on $p(x)$. If $p(x)$ is not defined by a hypothesis, then the parameters $\Delta(p)$ and $\sigma^2(p)$ should be replaced respectively by

$$\tilde{\Delta}_n = \int_{\Omega_n(\tau)} \lambda_n(x)dx \int_{|x|\le\tau} K^2(x)dx,$$
$$\tilde{\sigma}_n^2 = 2\int_{\Omega_n(\tau)} \lambda_n^2(x)dx \int_{|x|\le 2\tau} K_0^2(x)dx,$$

$$\lambda_n(x) = p_{1n}(x)\left(p_n(x)-p_{1n}(x)\right)+p_{2n}(x)\left(p_n(x)-p_{2n}(x)\right)$$

and we show that

$$b_n^{-1/2}\left(\tilde{\Delta}_n - \Delta(p)\right)\xrightarrow{\mathbb{P}} 0, \quad \tilde{\sigma}_n^2 \xrightarrow{\mathbb{P}} \sigma^2(p). \tag{14}$$

Let us prove (14). Since $p_n(x)=1+O\left(\frac{1}{nb_n}\right)$ uniformly in $x\in\Omega_n(\tau)$ and $|p_{in}(x)|\le c_4$, $x\in[0,1]$, $i=1,2$, we obtain

$$b_n^{-1/2}E\left|\tilde{\Delta}_n-\Delta(p)\right| \le$$
$$\le c_5 b_n^{-1/2}\left\{\int_{\Omega_n(\tau)}\left(E\left(p_{1n}(x)-Ep_{1n}(x)\right)^2\right)^{1/2}dx + \int_{\Omega_n(\tau)}\left(E\left(p_{2n}(x)-Ep_{2n}(x)\right)^2\right)^{1/2}dx\right\}+$$
$$+b_n^{-1/2}\int_{\Omega_n(\tau)}\left|Ep_{1n}(x)-p(x)\right|dx + b_n^{-1/2}\int_{\Omega_n(\tau)}\left|Ep_{2n}(x)-p(x)\right|dx.$$

Further, using Lemma 1 and and also taking into account that $p(x)\in C^1[0,1]$ and $\left[\frac{x-1}{b_n},\frac{x}{b_n}\right]\supset[-\tau,\tau]$ for all $x\in\Omega_n(\tau)$, it is easy to see that

$$b_n^{-1/2}E\left|\tilde{\Delta}_n-\Delta(p)\right| = O\left(\frac{1}{\sqrt{n}\,b_n}\right)+O\left(b_n^{1/2}\right)+O\left(\frac{1}{nb^{3/2}}\right).$$

Hence $b_n^{-1/2}\left(\tilde{\Delta}_n-\Delta(p)\right)\xrightarrow{\mathbb{P}}0$. Analogously, it can be shown that $\tilde{\sigma}_n^2\xrightarrow{\mathbb{P}}\sigma^2(p)$.

Theorem 2. *Let* $K(x)\in H(\tau)$ *and*

$$p_1(x)=p_2(x)\in C^1[0,1].$$

If $nb_n^2\to\infty$, *then for* $n\to\infty$

$$b_n^{-1/2}\left(T_n-\tilde{\Delta}_n\right)\tilde{\sigma}_n^{-1}\xrightarrow{d}N(0,1).$$

Proof. Follows from (13) and (14).

Theorem 2 enables us to construct an asymptotical criterion of testing the composite hypothesis $H_0:\ p_1(x)=p_2(x),\ \ x\in[0,1]$. The critical domain for testing this hypothesis is defined by the inequality

$$T_n\ge\tilde{d}_n(\alpha)=\tilde{\Delta}_n+b_n^{-1/2}\tilde{\sigma}_n\lambda_\alpha,\quad \Phi(\lambda_\alpha)=1-\alpha\,. \tag{15}$$

Theorem 3. *Let* $K(x)\in H(\tau)$, $p_1(x),p_2(x)\in C^1[0,1]$. *If* $nb_n^2\to\infty$, *then for* $n\to\infty$

$$\mathbb{P}_{H_1}\left(T_n\ge\tilde{d}_n(\alpha)\right)\to 1$$

Here the alternative hypothesis H_1 *is any pair* $\left(p_1(x),p_2(x)\right)$, $p_1(x),p_2(x)\in C^1[0,1]$, $0\le p(x)\le 1$, $i=1,2$, *such that* $p_1(x)\ne p_2(x)$ *on the set of positive measure.*

Proof. Is similar to the proof of Theorem 3 from [7].

Remark. Let t_i be the division points of the interval $[0,1]$, which are chosen so that

$$H(t_j)=\frac{2j-1}{2n}\,,\qquad j=1,\dots,n,$$

where $H(x)=\int_0^x h(u)\,du$, $h(u)$ is some known continuous distribution density o $[0,1]$. In this case, by a similar reasoning to the above one we can be generalize the results obtained in this paper.

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