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Bounded Solutions of Certain (Differential and first order systems of Differential Equations)

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Abstract : In this paper we prove the of solutions, of some kinds of differential equations, and system of first order differential equations are bounded.

Key words: Boundedness of solutions, first order ordinary Differential equations, systems of differential equations and limit cycles ..

1- Introduction

A solution x(t) is called bounded on $[0, \infty)$ if there exist a positive constant M such that $|x(t)| \leq M$ for all $t \in [0, \infty)$ [6] .

Boundedness is one of the mathematical properties that is needed in applications such as population growth, trophic function, capacity, voltage, differential and integral equations, chemical reactions etc. [1], [2]. It can be determined by finding the functions that make the solution bounded, or by using the stability of limit cycles, or determining the bounded region in which if the solution inter the region then it stay their for all t .

Differential equations of first order describe many real life applications such as population growth, nuclear decoy, Newton law of cooling, so the concept of boundedness of the solutions, is important in this sense.

Mawhin [3] studied the boundedness of solutions of the differential equation.

$$x' + \lambda x = f(t)$$

and that of its corresponding difference equation

$$\Delta x_m + \lambda x_m = f_m \quad , \ m \in \mathbb{Z}$$

Jose L. and Bravo and his colleagues, [4] studied the first order differential equation

$$x' = f(x, t)$$

and they defined a subset D of all continuous functions

$$f: \mathbb{R} x \mathbb{R} \to \mathbb{R}$$

such that f satisfies four conditions and defining subsets of D called D_+ and D_- of all points f such that

$$x' = f(x, t)$$

has bounded solutions.

Tineo [7] studied

$$x' = f(x, t) + \lambda$$

and showed that there exists $\lambda_o \in \mathbb{R}$ such that the equation has at least one or two separated bounded solutions . Also he showed that when f satisfies the concavity condition with respect to x, then the equation has exactly one or two separated bounded solutions.

Rao [6] studied the bounded of solutions of differential equations

$$x' = Ax \ , \ x' = \big(A + B(t)\big)x \ , \ x' = B(t)x,$$

$$x' = Ax + f(t, x) \text{ and } x' = B(t)x + f(t, x).$$

Where A is nxn constant matrix and B(t) is nxn variable matrix.

2- Preliminaries

We can use the following concepts and theorems for studying boundedness.

Theorem (1) [5]:- Let $\lim_{x\to\infty} x^p f(x) = A$ then

(1) $\int_a^\infty f(x)dx$ is convergent if $p > 1$ and A is finite.

(2) $\int_a^\infty f(x)dx$ is divergent if $p \leq 1$ and $A \neq 0$ and A may be infinite.

Theorem (2) [5]:- Let $\lim_{x\to a^+} (x-a)^p f(x) = A$ then

(1) $\int_a^b f(x)dx$ is convergent if $p < 1$ and A is finite.

(2) $\int_a^b f(x)dx$ is divergent if $p \geq 1$ and $A \neq 0$, A may be infinite.

Remark [6]:-

(1) Every solution inside the limit cycles is bounded.

(2) If there exist one stable limit cycle then every solution is bounded.

(3) If there exist one semi-stable limit cycle enclosing sink point then every solution is bounded.

(4) If there exist one semi-stable limit cycle enclosing source point then the solution outside is unbounded.

3- Main Results

In this section we find the conditions that ensure the boundedness of solutions.

3.1 Boundedness of solutions of certain differential equations.

The following lemmas concern the boundedness of solutions of specific third and first order ordinary differential equations.

Lemma (1):- Every solution of

$$u''' + pu' = w(t) \qquad (1)$$

where p is a positive real number, is bounded if $\int_0^\infty |w(t)|\,dt < \infty$.

Proof:-

The general solution of the corresponding homogeneous part is

$$u_c = c_1 + c_2 \cos\ pt + c_3 \sin pt \qquad (2)$$

It is clear that the solution (2) is bounded. Now to find the particular of solution of equation (1), we apply the variation of parameters method. For this purpose, let

$u_p = v_1 + v_2 \cos pt + v_3 \sin pt$, where v_1, v_2 and v_3 are functions should be determined. To do so, are can solve system of equations,

$$v_1' + v_2' \cos pt + v_3' \sin pt = 0$$

$$-pv_2' \sin pt + v_3' \cos pt = 0 \qquad (3)$$

$$-p^2 v_2' \cos pt - v_3' \sin pt = w(t)$$

It is a system of three parameters v_1', v_2' and v_3' to find them we use Grammar method, are can get the following :

$$v_1' = w(t)\frac{1}{p^2}$$

$$v_2' = \frac{1}{p^2} w(t) \cos pt$$

$$v_3' = -\frac{1}{p^2} w(t) \sin pt$$

To find v_1, v_2 and v_3 , we integrate v_1', $v_2'$and v_3' from 0 to t, we get

$$v_1 = \frac{1}{p^2}\int_0^t w(s)\, ds$$

$$v_2 = \frac{1}{p^2}\int_0^t w(s) \cos ps\, ds$$

$$v_3 = -\frac{1}{p^2}\int_0^t w(s) \sin ps\, ds$$

Thus, the particular solution is,

$$u_p = \frac{1}{p^2}\int_0^t w(s)\, ds + \frac{1}{p^2}\cos pt \int_0^t \cos ps\, w(s) ds - \frac{1}{p^2}\sin pt \int_0^t \sin ps\, w(s) ds$$

$$= \frac{1}{p^2}\int_0^t [1 + \cos p(t+s)]\, w(s) ds$$

$$|u_p| \le \frac{2}{p^2}\int_0^t |w(s)|\, ds$$

is bounded if $\int_0^\infty |w(s)|\, ds < \infty$

Thus, the solution of equation (1) is bounded.

<u>Lemma (2)</u>:- Every solution of the initial value problem

$$u' - 2u - f(t)u^2 = 0, \quad u(0) = u_0 \neq 0 \qquad (4)$$

approaches to zero as $t \to \infty$, when $f(t) < e^{-2t}$

<u>Proof</u>:-

The solution of this Bernoulli equation is

$$u^{-2}u' - 2u^{-1} = f(t) \qquad (5)$$

Let $z=u^{-1}$, $dz = -u^{-2}\, du$

$\frac{-dz}{dt}$ -2z = f(t)

$\frac{dz}{dt}$ +2z =- f(t)

I.F. = e^{2t}

$\frac{d}{dt}$ [z e^{2t}] = - e^{2t} f(t)

$z\, e^{2t} = -\int_0^t e^{2t}\, f(t)dt$

z(t)= $-e^{-2t}\int_0^t e^{2t}\, f(t)dt$

z(t) approaches zero as t→ ∞ $[f(t) < e^{-2t}]$.

Lemma (3):- Every solution of

$$u' + \alpha u = f(t)\,, \qquad \alpha > 0 \tag{6}$$

approach to zero as $|f(t)|$ approaches 0 as $t \to \infty$.

Proof:- The Equation (6) is first order linear equation, its general solution is

$u = e^{-\alpha t}[\int_0^t e^{\alpha s}\, f(s)ds + c]$

$|u(t)| \leq e^{-\alpha t}\int_0^t |f(s)|e^{\alpha s}ds + |c|e^{-\alpha t}$

since, $|f(t)| \to 0$, as $t \to \infty$, then $\lim_{t\to\infty}|u(t)| = 0$, thus

$u(t)$ approaches 0 as $t \to \infty$.

Lemma (4):- Every solution of ;

$$u' + (\alpha + f(t))u(t) = 0\,, \qquad \alpha > 0, \qquad u(0) \neq 0 \tag{7}$$

Converges to zero if f(t) is continuous on [0,∞) and $\int_0^\infty f(s)\, ds > 0$.

Proof:-

Rewrite equation (7) as

$\frac{du}{u(t)} = -(\alpha + f(t))dt$

Now by integrating both sides from zero to t, we get

u(t)= $e^{-\alpha t - \int_0^t f(s)ds + c}$

Now

$|u(t)| \leq \left|e^{-\alpha t - \int_0^t f(s)ds + c}\right| = e^{-\alpha t - \int_0^t f(s)ds + c}$

Which is bounded and approaching zero as $t \to \infty$.

3.2 Boundedness of solution of systems of first order differential equation.

Lemma (1):- Let $\phi(t)$be fundamental matrix of $\mathbf{x}' = \mathbf{Ax}$ such that $\|\phi(t)\| \leq K$, $t \geq 0$ and

$$\liminf_{t\to\infty} \int_0^t \mathbf{Tr\ A(s)ds} > -\infty \qquad (8)$$

If B(t) is a real nxn continuous matrix for $t \geq 0$ with

$$\int_0^\infty \|\mathbf{B(s)} - \mathbf{A(s)}\| < \infty \qquad (9)$$

then every solution of $\mathbf{y}' = \mathbf{B(t)y}$, is bounded on $[0,\infty)$.

Proof:-

The solution 0f $\mathbf{y}' = \mathbf{B(t)y}$ can be written as:

$$\mathbf{y(t)} = \mathbf{x(t)} - \int_t^\infty \phi(\mathbf{t})\, \phi^{-1}(\mathbf{s})\, [\mathbf{B(s)} - \mathbf{A(s)}]\mathbf{y(s)ds} \qquad (\mathbf{10})$$

Where $\phi(t)$ is the fundemental matrix of the system $\mathbf{x'=A(t)x}$, such that $\phi(0) = I$.

It should be noted that $x(t) = \phi(t)x_0$, since$\|\phi(t)\| \leq k$, $t \geq 0$, and

Let $\phi^{-1}(t) = \frac{\text{adj}\,\phi(t)}{\det\phi(t)}$, then $\phi(t)^{-1}$ is bounded in view of the condition(8)

$c_1 = \max(\sup\|\phi^{-1}(t)\|, \sup\|\phi(t)\|)$.

By taking the norm to equation (10), we get

$$\|y(t)\| \leq k_1 + c_1 \int_t^\infty \|B(s) - A(s)\| \|y(s)\|\, ds$$

Now by using Gronwall-Reid-Bellman inequality, then

$$\|y(t)\| \leq k_1 \exp[c_1 \int_t^\infty \|B(s) - A(s)\| ds$$

the result follows, by using the inequality (9).

Lemma (2):- If all the solutions of

$$\mathbf{x'=A(t)x} \qquad (11)$$

are bounded, then all the solutions of (11)

$$\mathbf{x'=A(t)x+f(t,x)} \qquad (12)$$

are also bounded provided that the following conditions are satisfied:

(1) $\|f(t; x)\| \leq |\alpha(t)| \|x\|$
(2) $\int_0^\infty \alpha(t)dt < \infty$
(3) $\lim_{t\to\infty} \int_0^t \text{Tr}\, A(s)ds > -\infty$
(4) $\text{Tr}\, A(t) = 0$.

Proof:-

Expressing the solution z(t) of **x'=A(t)x+f(t,x)** in terms of the solution x(t) of x'=A(t)x, we have

$$z(t)= x(t)+ \int_0^t \phi(t-s)f\big(s,z(s)\big)ds \qquad (13)$$

Where $\phi(t)$ is the fundamental matrix of (11) with $\phi(0) = I$.

It should be noted that x(t) = $\phi(t)\ z_0$. Since all the solutions of (11) are bounded, then $\|\phi(t)\|$ is bounded, and since

$\phi^{-1}(t) = \frac{\text{adj}\ \phi(t)}{\det \phi(t)} = \frac{\text{adj}\ \phi(t)}{\exp[\int_0^t \text{Tr}\ A(s)ds]}$, then $\|\phi^{-1}(t)\|$ is bounded also,

that $\|\phi^{-1}(t)\|$ is bounded. Now, let

$c_1 = \max(\sup\|\phi^{-1}(t)\|, \sup\|\phi(t)\|)$

then,

$$\|z(t)\| \le c_1 + c_1 \int_0^t \alpha(s)\|z(s)\|ds$$

and by using Gronwall-Reid-Bellman inequality, and from condition (2) we have the result.

Lemma (3):- If all the solutions of **x'=A(t)x** ,

approach zero as t→ ∞, then the same is true for the solutions of system **x'=A(t)x+f(t,x)** provided that conditions (1) and (2) in lemma(2) together with condition (3) or (4), hold.

Proof:-

Expressing the solution z(t) of **x'=A(t)x+f(t,x)** in terms of the solution x(t) of **x'=A(t)x**, we have

$$z(t)= x(t)+ \int_0^t \phi(t-s)f\big(s,z(s)\big)ds \qquad (14)$$

Where $\phi(t)$ is the fundamental matrix of **x'=A(t)x** with $\phi(0) = I$.

It should be noted that z(0) = $x(0) = z_0$, since all the solutions of (11) are approaching zero as t → ∞, we have $\|\phi(t)\|$ is approaching 0 as $t \to \infty$.

Thus, in the view of condition(3) and the fact that $\|\phi(t)\|$ is approaching 0 as $t \to \infty$, we have $\|\phi^{-1}(t)\|$ is approaching 0 as $t \to \infty$. Since both $\|\phi(t)\|$ and $\|\phi^{-1}(t)\|$ are approaching 0 as $t \to \infty$ and are continuous , then they are bounded for all t≥ $0. Now, let$

$c_1 = \max(\sup\|\phi^{-1}(t)\|, \sup\|\phi(t)\|)$

Therefore, from equation (14) we obtain:

$$\|z(t)\| \le \|\phi(t)\|\|z_0\| + \int_0^t \|\phi(t-s)\|\|f(s)z(s)\|ds$$

and from the condition (1) of lemma (2), we have

$$\|z(t)\| \le c_1 + c_1 \int_0^t |\alpha(s)|\|z(s)\|ds$$

Now by applying Gronwall-Reid-Bellman inequality we obtain, for all $t \geq 0$

$\|z(t)\| \leq c_1 \exp[c_1 \int_0^t |\alpha(s)|\, ds]$

therefore $\|z(t)\| \leq kc_1$, since all the solutions of (11) approach zero as $t \to \infty$ then, the same is true for (12) .

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