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Numerical Solution of Linear Volterra-Fredholm Integro-Differential Equations Using Lagrange Polynomials

Muna M. Mustafa[1] and Adhra'a M. Muhammad[2*]
Department of Mathematics, College of Science for Women, University of Baghdad, Baghdad-Iraq
1 E-mail: munamm05@yahoo.com 2 E-mail:Athr30@yahoo.com

Abstract:
In this paper, we introduce a numerical method for solving linear Volterra-Fredholm integro-differential Equations (LVFIDE's) of the first order. To solve these equations, we consider the polynomial approximation from original Lagrange polynomial approximation, barycentric Lagrange polynomial approximation, and modified Lagrange polynomial approximation. Finally, some examples are included to improve the validity and applicability of the techniques.
Keywords: Linear Volterra-Fredholm integro-differential equation, Original Lagrange polynomial, Barycentric Lagrange Polynomial, Modified Lagrange polynomial.

1. Introduction:

In recent year, there has been a growing interest in the integro-differential equation which are a combination of differential and Volterra-Fredholm integral equation. Integro-differential equation play an important role in many branches of linear and non-linear functional analysis and their applications in the theory of engineering, mechanics, physics, chemistry, biology, economics, and elctrostations. The mentioned integro- differential equations are usually difficult to solve analytically, so approximation methods is required to obtain the solution of the linear and non-linear integro-differential equation [7].

Many researchers studied and discuss the linear Volterra-Fredholm integro-differential equations, E. Boabolian, Z. Masouri and S. Hatamazadeh-Varmazyar [5] in 2008 construct new direct method to solve non-linear Volterra-Fredholm integral and integro-differential equation using operational matrix block-pulse functions. AL-Jubory A. [2] in 2010 introduced some approximation method for solving Volterra-Fredholm integral and integro-differential equation. M. Dadkah, Kajanj. M. Tavassoli and S. Mahdavi [13] in 2010 used numerical solution of non-linear Volterra-Fredholm integro-differential equations using Legendre wavelets. R. Mohesn and S. H. Kiasoltani [14] in 2011 study the solving of non-linear system of Volterra-Fredholm integro-differential equation by using discrete collocation method. Gherjalar H. D. and M. Hossein [6] in 2012 solved integral and integro-differential equation by using B-splines function.

Also, Lagrange interpolation polynomial used to solve integral equations. Adibi, H. and Rismani, A. M. [1] in 2010 applied Legendre-spectral method to solve functional integral equations where the Legendre Gauss points are used as collocation nodes and Lagrange scheme is employed to interpolate the quantities needed. Shahsavaran, A. [15] in 2011 presented a numerical method for solving nonlinear VFIE's based upon Lagrange functions approximations together with the Gaussian quadrature rule and then utilized to reduce the VFIE to the solution of algebraic equations. Muna M. Mustafa and Iman N. Ghanim [11] in 2014 used Lagrange polynomials to solve linear Volterra-Fredholm integral equations.

Barycentric Lagrange and modified Lagrange polynomials presented in: Berrut, J.-P. And Trefethen, L. N. [3] were they discuss Lagrange polynomial interpolation and barycentric Lagrange polynomial interpolation. Muthumalai, R. K. [12] derived an interpolation formula that generalizes both Newton interpolation formula and barycentric Lagrange interpolation formula, and Higham, N. J. [8] give an error analysis of the evaluation of barycentric Lagrange formula and modified formula.

In this work, Lagrange polynomial, and Barycentric Lagrange Polynomial are used to solve LVFIDE's numerically. The remainder of the paper is organized as follows: the methods of the solution (Lagrange polynomial, and Barycentric Lagrange Polynomial), test examples are investigated and the corresponding tables are presented. Finally, the report ends with a brief conclusion.

2. Methods of Solution:

The Linear Volterra-Fredholm integro-differential equation (LVFIDE) of the first order is:

$$u'(x) + p(x)u(x) = f(x) + \int_a^x k_1(x,t)u(t)dt + \int_a^b k_2(x,t)u(t)dt \qquad (1)$$

Where $a \le x \le b$; $f(x), k_1(x,t)$, and $k_2(x,t)$ are continuous functions and $u(x)$ is the unknown function to be determined.

Now, to solve Eq. (1) using original Lagrange polynomial method, barycentric Lagrange Polynomial method, and modified Lagrange polynomial method; the derivation of the methods are showing as follows:

2.1 Original Lagrange Polynomial Method:

Lagrange interpolation is praised for analytic utility and beauty. The Lagrange approach is in most cases the method of choice for dealing with polynomial interpolants [4]. This method has the advantage that the values $x_0, x_1, \ldots, x_n$ need not be equally spaced or taken in consecutive order [9].

Now, define Lagrange formula for a set of n+1 data points $\{(x_0,t_0), (x_1,t_1), \ldots, (x_n,t_n)\}$as [4]:

$$P_n(x) = \sum_{m=0}^{n} u_m L_{n,m}(x) \qquad (2)$$

Where

$$L_{n,m}(x) = \prod_{\substack{k=0 \\ k \ne m}}^{n} \frac{(x - x_k)}{(x_m - x_k)} \qquad (3)$$

Now, for $x = x_m$ then [16]:

$$P_n(x_m) = u_m \forall m = 0,1,\ldots,n \qquad (4)$$

Which that is mean:

$$L_{n,m}(x) = \begin{cases} 1 & if x = x_m \\ 0 & if x = x_k \ ,(k \ne m) \end{cases}$$

The error of Lagrange polynomial is [10]:

$$E_n(x) = \frac{W_n(x)}{(n+1)!} u^{(n+1)}(\xi(x)) \qquad (5)$$

Where $W_n(x) = \prod_{k=0}^{n}(x - x_k)$ and $\xi(x)$ lies in the interval [a, b].

Now, by differentiated (2), we obtain:

$$P'_n\,(x) = \sum_{j=0}^{n} u_j\, L'_{n,j}(x) \qquad (6)$$

In order to solve LVFIDE using Lagrange polynomial, we substituting Eq. (6) in Eq. (1), to get:

$$\sum_{j=0}^{n} u_j\, L'_{n,j}(x) + p(x)\sum_{j=0}^{n} u_j\, L_{n,j}(x) = f(x) + \int_a^x k_1(x,t)\,(\sum_{j=0}^{n} u_j\, L_{n,j}(t))dt$$
$$+ \int_a^b k_2(x,t)\,(\sum_{j=0}^{n} u_j\, L_{n,j}(t))\,dt$$

Therefore:

$$u_0\left(L'_{n,0}(x) + p(x)l_{n,0}(x) - \int_a^x k_1\,(x,t)L_{n,0}(t)dt - \int_a^b k_2\,(x,t)L_{n,0}(t)dt\right)$$
$$+u_1\left(L'_{n,1}(x) + p(x)l_{n,1}(x) - \int_a^x k_1\,(x,t)L_{n,1}(t)dt - \int_a^b k_2\,(x,t)L_{n,1}(t)dt\right)$$
$$+u_2\left(L'_{n,2}(x) + p(x)l_{n,2}(x) - \int_a^x k_1\,(x,t)L_{n,2}(t)dt - \int_a^b k_2\,(x,t)L_{n,2}(t)dt\right)+\ldots$$

$$+ u_n \left(L'_{n,n}(x) + p(x) l_{n,n}(x) - \int_a^x k_1(x,t) L_{n,n}(t) dt - \int_a^b k_2(x,t) L_{n,n}(t) dt \right) = f(x)$$

Putting x=x_i, for i=1, 2,…, n , to get a system of n equations, which is:

$$D\vec{u} = \vec{C} \qquad (7)$$

When $D = d_{ij}$, $\vec{C} = c_i$.

$$c_i = f(x_i) - u_0 \left(L'_{n,1}(x_i) + p(x_i) l_{n,1}(x_i) - \int_a^{x_i} k_1(x_i,t) L_{n,1}(t) dt - \int_a^b k_2(x_i,t) L_{n,1}(t) dt \right) \qquad (8)$$

And

$$d_{ij} = L'_{n,j}(x_i) + p(x_i) l_{n,j}(x_i) - \int_a^{x_i} k_1(x_i,t) L_{n,j}(t) dt - \int_a^b k_2(x_i,t) L_{n,j}(t) dt) \qquad (9)$$

For all i, j =1, 2,…, n.

The Algorithm

The numerical solution of LVFIDE's, by using original Lagrange polynomial, is obtained as follows:

Step1:

Put $h = \frac{b-a}{n}$, $n \epsilon \mathbb{N}$, u (a) =u_0 (initial condition is given)

Step2:

Set $x_i = a + ih$, with $x_0 = a$ and $x_n = b$, i=0, 1,…, n.

Step3:

Use step 1 and step 2 and Eq. (9) to find d_{ij} (Note that for derivative and integral in Eq. (9), we compute the exact value).

Step4:

Compute c_i using Eq. (8) (Note that for integral in Eq. (8), we compute the exact value).

Step5:

Solve the system Eq. (7) using step3 and step4 and Gaussian Elimination Method.

2.2 Barycentric Lagrange Polynomial Method:

Barycentric interpolation is not new, but most students, most mathematical scientists, and even many numerical analysis do not know about it [4].

Now, introduce Barycentric Lagrange formula for a set of n+1 data points{(x_0,t_0),(x_1,t_1) ,…, (x_n,t_n)}as [3]:

$$P_n(x) = \frac{\sum_{j=0}^{n} \frac{w_j}{x-x_j} u_j}{\sum_{j=0}^{n} \frac{w_j}{x-x_j}} \qquad (10)$$

Where

$$w_j = \frac{1}{\prod_{k \neq j}(x_j - x_k)} \qquad , k, j = 0,1, \dots, n \qquad (11)$$

And, the error for Barycentric Lagrange polynomial is [12]:

$$E_n(x) = l(x) \frac{u^{(n+1)}(\xi(x))}{(n+1)!} \qquad (12)$$

Where

$$l(x) = \prod_{j=0}^{n} (x - x_j) \qquad (13)$$

Now, putting:

$$P_n(x)=\sum_{j=0}^{n} u_j B_{n,j}(x) \quad (14)$$

And, the derivative for P_n(x)

$$P'_n(x)=\sum_{j=0}^{n} u_j B'_{n,j}(x) \quad (15)$$

In order to solve LVFIDE using barycentric Lagrange polynomial, we substituting Eq's. (14) and (15) in Eq. (1), to get:

$$\sum_{j=0}^{n} u_j B'_{n,j}(x) + p(x)\sum_{j=0}^{n} u_j B_{n,j}(x) = f(x) + \int_a^x k_1(x,t)\,(\sum_{j=0}^{n} u_j B_{n,j}(t))dt + \int_a^b k_2(x,t)\,(\sum_{j=0}^{n} u_j B_{n,j}(t))\,dt$$

Therefore:

$$u_0\left(B'_{n,0}(x)+p(x)B_{n,0}(x)-\int_a^x k_1(x,t)B_{n,0}(t)dt-\int_a^b k_2(x,t)B_{n,0}(t)dt\right)$$
$$+u_1\left(B'_{n,1}(x)+p(x)B_{n,1}(x)-\int_a^x k_1(x,t)B_{n,1}(t)dt-\int_a^b k_2(x,t)B_{n,1}(t)dt\right)$$
$$+u_2\left(B'_{n,2}(x)+p(x)B_{n,2}(x)-\int_a^x k_1(x,t)B_{n,2}(t)dt-\int_a^b k_2(x,t)B_{n,2}(t)dt\right)+\ldots$$
$$+u_n\left(B'_{n,n}(x)+p(x)B_{n,n}(x)-\int_a^x k_1(x,t)B_{n,n}(t)dt-\int_a^b k_2(x,t)B_{n,n}(t)dt\right)=f(x)$$

Putting x=x_i, for i=1, 2,…, n, to get a system of n equations, which is:

$$D\vec{u}=\vec{C} \quad (16)$$

Where $D = d_{ij}$, $\vec{C} = c_i$.

$$c_i = f(x_i) - u_0\,(B'_{n,1}(x_i)+p(x_i)B_{n,1}(x_i)-\int_a^{x_i} k_1(x_i,t)B_{n,1}(t)dt-\int_a^b k_2(x_i,t)B_{n,1}(t)dt) \quad (17)$$

And

$$d_{ij} = B'_{n,j}(x_i)+p(x_i)B_{n,j}(x_i)-\int_a^{x_i} k_1(x_i,t)B_{n,j}(t)dt-\int_a^b k_2(x_i,t)B_{n,j}(t)dt \quad (18)$$

For all i, j =1, 2,…, n.

The Algorithm

The numerical solution of LVFIDE's, by using barycentric Lagrange polynomial, is obtained as follows:

Step1:

Put $h=\frac{b-a}{n}$, $n\epsilon\mathbb{N}$, u (a) =u_0 (initial condition is given)

Step2:

Set $x_i = a + ih$, with $x_0 = a$ and $x_n = b$, i=0, 1,…, n.

Step3:

Use step 1 and step 2 and Eq. (18) to find d_{ij} (Note that for derivative and integral in Eq. (18), we compute The exact value).

Step4:

Compute c_i using Eq. (17) (Note that for integral in Eq. (17), we compute the exact value).

Step5:

Solve the system Eq. (16) using step3 and step4 and Gaussian Elimination Method.

2.3 Modified Lagrange Polynomial Method:

First, define modified Lagrange formula for a set of n+1 data points $\{(x_0,t_0),(x_1,t_1),\ldots,(x_n,t_n)\}$ as [8]:

$$P_n(x) = l(x)\sum_{j=0}^{n}\frac{w_j}{x-x_j}u_j \qquad (19)$$

Where

$$l(x) = \prod_{j=0}^{n}(x-x_j) \qquad (20)$$

And

$$w_j = \frac{1}{\prod_{k\neq j}(x_j-x_k)} \qquad ,k,j = 0,1,\ldots,n \qquad (21)$$

Now putting

$$P_n(x)=\sum_{j=0}^{n} w_j\, u_j M_{n,j}(x) \qquad (22)$$

And, the derivative for $P_n(x)$

$$P'_n(x)=\sum_{j=0}^{n} w_j\, u_j\, M'_{n,j}(x) \qquad (23)$$

In order to solve LVFIDE using Modified Lagrange polynomial, we substitute Eq's. (22) and (23) in Eq. (1), to get:

$$\sum_{j=0}^{n} w_j\, u_j\, M'_{n,j}(x) + p(x)\sum_{j=0}^{n} w_j u_j\, M_{n,j}(x) = f(x) + \int_a^x k_1(x,t)\,\Big(\sum_{j=0}^{n} w_j u_j\, M_{n,j}(t)\Big)dt$$

$$+\int_a^b k_2(x,t)\,\Big(\sum_{j=0}^{n} w_j u_j M_{n,j}(t)\Big)\,dt$$

Therefore:

$$w_0\,u_0\left(M'_{n,0}(x) + p(x)M_{n,0}(x) - \int_a^x k_1(x,t)M_{n,0}(t)dt - \int_a^b k_2(x,t)M_{n,0}(t)dt\right)$$
$$+w_1\,u_1\left(M'_{n,1}(x) + p(x)M_{n,1}(x) - \int_a^x k_1(x,t)M_{n,1}(t)dt - \int_a^b k_2(x,t)M_{n,1}(t)dt\right)$$
$$+w_2\,u_2\left(M'_{n,2}(x) + p(x)M_{n,2}(x) - \int_a^x k_1(x,t)M_{n,2}(t)dt - \int_a^b k_2(x,t)M_{n,2}(t)dt\right)+\ldots$$
$$+w_n\,u_n\left(M'_{n,n}(x) + p(x)M_{n,n}(x) - \int_a^x k_1(x,t)M_{n,n}(t)dt - \int_a^b k_2(x,t)M_{n,n}(t)dt\right) = f(x)$$

Now, putting $x=x_i$, for i =1, 2,…, n, yields a system of n equations, which is:

$$D\vec{u} = \vec{C} \qquad (24)$$

Where $D = d_{ij}$, $\vec{C} = c_i$.

$$c_i = f(x_i) - w_0\,u_0\left(M'_{n,1}(x_i) + p(x_i)M_{n,1}(x_i) - \int_a^{x_i} k_1(x_i,t)M_{n,1}(t)dt - \int_a^b k_2(x_i,t)M_{n,1}(t)dt\right) \qquad (25)$$

And

$$d_{ij} = w_j\,(M'_{n,j}(x_i) + p(x_i)M_{n,j}(x_i) - \int_a^{x_i} k_1(x_i,t)M_{n,j}(t)dt - \int_a^b k_2(x_i,t)M_{n,j}(t)dt) \qquad (26)$$

For all i, j =1, 2,…, n.

The Algorithm

The numerical solution of LVFIDE's, by using barycentric Lagrange polynomial, is obtained as follows:

Step1:

Put $h = \frac{b-a}{n}$, $n \epsilon \mathbb{N}$, u (a) $= u_0$ (initial condition is given)

Step2:

Set $x_i = a + ih$, with $x_0 = a$ and $x_n = b$, i=0,1,…,n.

Step3:

Use step1, step2 and Eq. (26) to find d_{ij} (Note that for derivative and integral in Eq. (26), we Compute the exact value).

Step4:

Compute c_i using Eq. (25) (Note that for integral in Eq. (25), we compute the exact value).

Step5:

Solve the system Eq. (24) using step3 and step4 and Gaussian Elimination Method.

3. Test Examples:

In this section, we give some of the numerical examples to illustrate the above methods for solving the LVFIDE's of the first order.

The exact solution is known and used to show that the numerical solution obtained with our methods is correct. We used MATLAB v 7.6 to solve the examples.

Example 1: Consider the LVFIDE:

$$u'(x) + xu(x) = f(x) + \int_0^x (x-t)u(t)dt + \int_0^1 (1+xt)u(t)dt \qquad (27)$$

Where $f(x) = -\sin(x) + x\cos(x) - 1 + \cos(x) + x - \sin(1) - x\cos(1) - x\sin(1)$

With the exact solution$u(x) = \cos(x)$.

Tables 1 and 2 represent the absolute error by using Lagrange polynomial, Barycentric Lagrange polynomial, and Modified Lagrange polynomial with n=5 and n=8 respectively, where $||err||_\infty$is the maximum absolute error, and R.T. represent the running time.

Table (1)

The Absolute Error of Example 1 by using Lagrange polynomial, Barycentric Lagrange polynomial, and Modified Lagrange polynomial with n=5

x	*Lagrange approximation Error*	*Barycentric Lagrange approximation Error*	*Modified Lagrange approximation Error*
0.2	2.391765344234020e-005	2.391765344289532e-005	2.391765344167407e-005
0.4	2.857260188393607e-005	2.857260188426913e-005	2.857260188426913e-005
0.6	3.408742998700642e-005	3.408742998722847e-005	3.408742998711745e-005
0.8	3.818385863973983e-005	3.818385864007290e-005	3.818385863973983e-005
1	4.299947868480203e-005	4.299947868502407e-005	4.299947868469101e-005
$\|\|err\|\|_\infty$	**4.299947868480203e-005**	**4.299947868502407e-005**	**4.299947868469101e-005**
R.T.	**4.123351239158373e000**	**1.250613505479534e+001**	**3.464698443517950e+000**

Table (2)
The Absolute Error of Example 1 by using Lagrange polynomial, Barycentric Lagrange polynomial, and Modified Lagrange polynomial with n=8

x	*Lagrange approximation Error*	*Barycentric Lagrange approximation Error*	*Modified Lagrange approximation Error*
0.125	1.265974325370678e-009	1.265996973920380e-009	1.265974325370678e-009
0.25	1.452257314404903e-009	1.452277964553161e-009	1.452257314404903e-009
0.375	1.671212834786218e-009	1.671230709376914e-009	1.671212834786218e-009
0.5	1.867390464482810e-009	1.867408894185019e-009	1.867390464482810e-009
0.625	2.055702386094538e-009	2.055722814198191e-009	2.055702386094538e-009
0.75	2.222863337841829e-009	2.222885875369229e-009	2.222863337841829e-009
0.875	2.387074760079599e-009	2.387098407830024e-009	2.387074760079599e-009
1	2.489904948888011e-009	2.489930595039880e-009	2.489904948888011e-009
$\|\|err\|\|_\infty$	**2.489904948888011e-009**	**2.489930595039880e-009**	**2.489904948888011e-009**
R.T.	**1.313352769194746e+001**	**7.551065389257385e+002**	**1.115002813213844e+001**

Example 2: Consider the LVFIDE:

$$u'(x) + (x+1)u(x) = f(x) + \int_0^x (x^2 t - t)u(t)dt + \int_0^1 (xt)u(t)dt \qquad (28)$$

Where $f(x) = (x+1)e^x - x^2 + 1 + e^x x^2 - e^x x^3 + e^x x - x$

With the exact solution $u(x) = e^x$.

Tables 3 and 4 represent the absolute error by using Lagrange polynomial, Barycentric Lagrange polynomial, and Modified Lagrange polynomial with n=5 and n=8 respectively, where $||err||_\infty$ is the maximum absolute error, and R.T. represent the running time.

Table (3)
The Absolute Error of Example 1 by using Lagrange polynomial, Barycentric Lagrange polynomial, and Modified Lagrange polynomial with n=5

X	*Lagrange approximation Error*	*Barycentric Lagrange approximation Error*	*Modified Lagrange approximation Error*
0.2	2.680112904229759e-005	2.680112904007714e-005	2.680112904229759e-005
0.4	1.921056893694484e-005	1.921056893516848e-005	1.921056893694484e-005
0.6	1.523593756602715e-005	1.523593756402875e-005	1.523593756602715e-005
0.8	1.053798391836835e-005	1.053798391659200e-005	1.053798391836835e-005
1	1.025547619182277e-005	1.025547619093459e-005	1.025547619182277e-005
$\|\|err\|\|_\infty$	**2.680112904229759e-005**	**2.680112904007714e-005**	**2.680112904229759e-005**
R.T.	**3.882641321834108e+000**	**1.276567664025521e+001**	**3.484865105051906e+000**

Table (4)
The Absolute Error of Example 1 by using Lagrange polynomial, Barycentric Lagrange polynomial, and Modified Lagrange polynomial with n=8

X	*Lagrange approximation Error*	*Barycentric Lagrange approximation Error*	*Modified Lagrange approximation Error*
0.125	3.071339849114452e-009	3.071380039187943e-009	3.071339849114452e-009
0.25	2.557183576001876e-009	2.557209777265257e-009	2.557183576001876e-009
0.375	2.206390625758559e-009	2.206405280702484e-009	2.206390625758559e-009
0.5	1.849937758535702e-009	1.849947306453714e-009	1.849937758535702e-009
0.625	1.551605732785788e-009	1.551612616168541e-009	1.551605732785788e-009
0.75	1.277102423813403e-009	1.277106864705502e-009	1.277102423813403e-009
0.875	1.091139179010270e-009	1.091142287634739e-009	1.091139179010270e-009
1	8.018856689773202e-010	8.018892216909990e-010	8.018856689773202e-010
$\|\|err\|\|_\infty$	**3.071339849114452e-009**	**3.071380039187943e-009**	**3.071339849114452e-009**
R.T.	**1.266073276521567e+001**	**7.406448682922203e+002**	**1.151814706240349e+001**

4. Conclusion:

In this work, we applied original Lagrange polynomial, barycentric Lagrange polynomial, and modified Lagrange polynomial for solving the LVFIDE of the first order. According to the numerical results which obtain from the illustrative examples, we conclude that:

- The approximate solutions obtained by MATLAB software show the validity and efficiency of the proposed method.
- The barycentric Lagrange polynomial gives better accuracy than other polynomials.
- As n (the number of knots) increase, the error term is decreased in all the used polynomials.
- The method can be extended and applied to non-linear VFIDE.
- The method can be extended also for solving LVFIDE's of nth order.

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