Generalised Conditional Expectation on Extended Positive Part of Crossed Product of a Von Neumann Algebra.

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Abstract

Extension of generalised conditional expectation onto the set of generalised positive operators (the extended positive part) of crossed product of von Neumann algebra is considered.

Keywords: von Neumann algebra, crossed product, generalised conditional expectation.

1.0 Introduction:

Conditional expectation in von Neumann algebra which is the non-commutative extension of the usual conditional expectation in probability theory was first studied by Umegaki [11]. Tomiyama [10] showed that each projection of norm one from a von Neumann algebra onto its von Neumann sub algebra enjoys most of the properties of conditional expectation. In Takesaki [7], the conditional expectation from a von Neumann algebra M onto its von Neumann sub algebra N exist only when N is globally invariant under the modular automorphism group σ_t^{φ} associated with φ.To recover this drawback, Accardi and Cechni [1]gave the concept of generalised conditional expectation on von Neumann algebras using Tomita-Takesaki theory. Vandaele [13] gave the construction of crossed product of a von Neumann algebra. The motivation for this paper is as a result of Lance [8] and Goldstein [2] where the possibility of extending conditional expectation to extended positive part of a von Neumann algebra and also to the crossed product of von Neumann algebras was shown. In this paper, we extend the notion of generalised conditional expectation onto the extended positive part of crossed product of von Neumann algebra.

2.0 Preliminaries:

We recall the notions and results on conditional expectation and generalised positive operators. We also recall the construction of crossed product of a von Neumann algebra as given by Vandaele [13].

Definition:

A projection of norm one from a von Neumann algebra M onto a von Neumann sub algebra N is a mapping $\pi : M \to N$ such that:

i. π is order preserving
ii. $\pi(axb) = a\pi(x)b$, $\forall a,b \in N$, $x \in M$
iii. π is $*$ preserving
iv. $\pi(x)^* \pi(x) \le \pi(x^* x)$, $x \in M$

Definition:

Let φ be a faithful normal trace such that $\varphi(I) = 1$ in a von Neumann algebra M , then a projection of norm one such that:

i. $\pi(I)=1$, $I\in M$
ii. $\pi(\alpha x+\beta y)=\alpha\pi(x)+\beta\pi(y)$, $x,y\in M$, $\alpha,\beta\in\mathbb{C}$
iii. π is $*$ preserving
iv. π is positive preserving
v. π is faithful
vi. $\pi(z)=z$, $z\in N$
vii. π is norm one
viii. $\pi(x)^*\pi(x)\le\pi(x^*x)$, $x\in M$
ix. $x_\alpha\nearrow x\Rightarrow\pi(x_\alpha)\nearrow\pi(x)$, $x_\alpha,x\in M$
x. $\varphi\pi(x)\le\varphi(x)$, $x\in M$, $\varphi\in M_*^+$

Is called a conditional expectation from M onto N .

Definition:

A von Neumann sub algebra N of M is said to be expected if it is invariant withrespect to the modular automorphism group associated with the pair $(M,\varphi_\circ)$, i.e. a σ weakly continuous $\varphi_\circ$ invariant projection of norm one from N onto M .

Definition:

A weight φ on a von Neumann algebra M is a function $\varphi:M_+\to[0,\infty]$, such that

i. $\varphi(x+y)=\varphi(x)+\varphi(y)$, $x,y\in M_+$
ii. $\varphi(\lambda x)=\lambda\varphi(x)$, $x\in M_+$, $\lambda\ge 0$.

We say that φ is faithful if $\varphi(x^*x)=0\Rightarrow x=0$. φ is normal if $\varphi(x)=\sup\varphi(x_i)$, with x as the limit of a bounded increasing net of operators $\{x_i\}_{i\in I}$ in M_+ and φ is semifinite if n_φ is σ - weakly dense in M .
To any weight φ is associated a σ - weakly continuous one parameter group of $*$ automorphisms $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ on the von Neumann algebra M , called the modular automorphism group defined by
$\sigma_t(x)=\pi^{-1}(\Delta_\varphi^{it}\pi(x)\Delta_\varphi^{-it})$.

Definition:

A generalised positive operator affiliated with a von Neumann algebra M is the set of maps $\hat{x}:M_\bullet^+\to[0,\infty]$ which is positively homogeneous, additive and lower semi continuous, where $M_\bullet^+$ is the positive part of the predual $M_\bullet$ of M . The set of all such maps is called the extended positive part of M and is denoted by $M_\bullet^\wedge$. They are "weights " on the predual of the von Neumann algebra M .Each element $x\in M_+$ defines an element in $M_+^\wedge$ by $\varphi\to\varphi(x)$, $\varphi\in M_\bullet^+$, hence we can regard $M_+\subset M_+^\wedge$ [3].

Definition:

Let $x,y\in M_+,a\in M$ and $\lambda\ge 0$ we define $x+y,\lambda x$ and a^*xa by

(1) $(\lambda x)\phi=\lambda x(\phi)$, $\phi\in M_+^*$

(2) $(x+y)\phi = x(\phi)+y(\phi)$, $\phi \in M_{+}^{*}$

(3) $(a^{*}xa)(\phi) = x(a\phi a^{*})$, $\phi \in M_{+}^{*}$

Remark: $a\phi a^{*}(x) = \phi(a^{*}xa)$, $x \in M$,

hence we have $a\phi a^{*}(1) = \phi(a^{*}1a) = \phi(aa^{*})$, $1 \in M$ (**)

Definition:

If $(x_i)_{i\in I}$ is increasing net in M_{+} then $x(\varphi) = \sup_i x_i(\varphi), \varphi \in M_{*}^{+}$ defines an element in M_{+}. In particular if $(x_j)_{j\in J}$ is a family of elements in M_{+}, then $x(\varphi) = \sum_{j\in J} x_j(\varphi) \in M_{+}$. The relationship between operators in von Neumann algebra and its positive part is given in corollary 1.6 of Haagerup paper [4].

Corollary:

Any $x \in M_{+}$ is the pointwise limit of an increasing sequence of bounded operators in M_{+}.

Definition:

If $x \in M_{+}$, $\varphi \in M_{*}^{+}$ and $(x_n)_{n\in\mathbb{N}}$ in M , with $x_n \nearrow x$ then $x(\varphi) = \lim_n \varphi(x_n)$.

Definition:

Let $\hat{x} \in M_{+}^{\wedge}$, a weight on $M_{+}^{\wedge}$ is given by ;

$$\varphi(\hat{a}) = \lim_m \varphi(a_m) = \lim_{n,m} \tau(x_n \bullet a_m) = \lim_{n,m} \tau(x_n^{\frac{1}{2}} a_m x_n^{\frac{1}{2}}) , \quad a \in M_{+} \quad []$$

We have a theorem from Haagerup [4] which states the form of the spectral resolution for the generalised positive operator.

Theorem:

L et M be a von Neumann algebra. Each element $x \in M_{+}^{\wedge}$ has a spectral resolution of the form

$\hat{x}(\varphi) = \int_{0}^{\infty} \lambda d\varphi(e_\lambda) + \infty(p)$, $\varphi \in M_{\bullet}^{+}$. Where $(e_\lambda)_{\lambda\in[0,\infty]}$ is an increasing family of projections in M such that $\lambda \to e_\lambda$ is strongly continuous from right, and $\lim_{\lambda\to\infty} = 1-p$.

2.0 Generalised Conditional Expectation:

Let M be a semfinite von Neumann algebra and N its von Neumann subalgebra. Then there exists a conditional expectation E from M onto N which is a projection of norm one (Tomiyama [11]). The conditional expectation $\in(x) = \pi_N^{-1}(E\pi_M(x)E)$ in Takesaki [ ] exist only when N is globally invariant under the modular automorphism group $(\sigma_t^{\varphi})_{t\in\mathbb{R}}$ associated with a faithful normal weight φ. Generalised conditional expectation of Accardi and Checchni [1] defined by $\varepsilon(a)=\pi^{-1}(j_{u_0}pj_u\pi(a)j_uj_{u_0}p)$ always exist but it is not a projection of norm one neither does it enjoy the useful property $\varepsilon(axb)=a\varepsilon(x)b$, $a,b\in N$ and $x\in M$. In Goldstein [2] he extends conditional expectation E to the extended positive part of a von Neumann Algebra and also to the crossed product of a von Neumann algebra (which he denoted by E and $\overline{E}$ respectively). Here we extend the generalised conditional expectation ε to the generalised positive operators of crossed product of a von Neumann algebra. We denote our extended generalised conditional expectation by $\overset{\wedge}{\varepsilon}$.

We showed that $\overset{\wedge}{\varepsilon}$ is invariant with respect to a given normal weight on $\overline{\hat{M}}_+$. To show the possibility of extending the generalised conditional expectation onto the generalised positive operators of crossed product of a von Neumann algebra, we follow the same argument of the proof in [2] which of course is the same even for a generalised conditional expectation $\overset{\wedge}{\varepsilon}$.

We now give the operators of our extended positive part of crossed product of von Neumann algebra using [13] and also give the generalised positive operators of the crossed product using the same argument of [4].

Definition 1:

A generalised positive operator $\overline{x}^{\wedge}$ affiliated with a crossed product of a von Neumann algebra $\overline{M}$ is the set of maps $\hat{x}:\overline{M}_*^+\to[0,\infty]$ which is positively homogeneous, additive and lower semi continuous, where $\overline{M}_*^+$ is the positive part of the predual $\overline{M}_*$ of $\overline{M}$. The set of all such maps is called the extended positive of $\overline{M}$ and is denoted by $\overline{\hat{M}}_+$.

Definition 2:

Let $\hat{x}\in\overline{\hat{M}}_+$ we define a weight on $\overline{\hat{M}}_+$ by

$$\overline{\varphi}_{\hat{x}}(\overline{y})=\lim_{n,m}\overline{\tau}^{\wedge}(\overline{x}_m\bullet\overline{y}_n)=\lim_{n,m}\overline{\tau}^{\wedge}(\overline{x}_m^{\frac{1}{2}}\overline{y}_n\overline{x}_m^{\frac{1}{2}})\ ,\ \overline{y}\in\overline{M}_+$$

Remark: The generalised positive operators are added multiplied by scalars in a natural way.

Definition 3:

Let H be a Hilbert space. Denote by $C_c(R)$ the space of continuous functions with compact support on R. The tensor product $\hat{L}_2(R.ds)\otimes\hat{H}$ is spanned by elements of the form $f\otimes\hat{\xi}$ where $C_c(R)$, $\hat{\xi}\in\hat{H}$. The Hilbert space completion of the linear combination of such function is denoted by $\overline{\hat{H}}$. Let $\hat{M}$ be a von Neumann algebra acting in $\hat{H}$ and $\overset{\wedge}{\varphi}_0$ a fixed faithful normal state on $\hat{M}$. Denote by $\overset{\wedge}{\sigma}$ the modular automorphism group associated with $\hat{M}$ and $\hat{\varphi}_0$. The crossed product of $\hat{M}$ by $\hat{\sigma}$ is the von Neumann algebra $\overline{\hat{M}}$ acting on $\overline{\hat{H}}$ generated by $\pi(x)$, $x \in \hat{M}$ and $\hat{\lambda}(s), s \in R$ where

$$(\pi(x)\hat{\xi})(t) = \sigma_{-t}(x)\hat{\xi}(t)\ ,\ (\hat{\lambda}(s)\hat{\xi})(t) = \hat{\xi}(t-s).$$

Let $\overline{\hat{M}_\circ}$ be the * algebra generated algebraically by operators $\pi(x)$, $x \in \hat{M}$ and $\hat{\lambda}(s), s \in R$. Then $\overline{\hat{M}}$ is the $\hat{\sigma}$ weak closure of $\overline{\hat{M}_\circ}$. Every element $\bar{x} \in \overline{\hat{M}}$ may be represented as $\hat{x} = \sum_k \hat{\lambda}(s_k)\hat{\pi}(\hat{x}_k)$ for some $s_1,.,.,s_k \in R, \hat{x}_1,.,.,\hat{x}_k \in \hat{M}$.

Theorem 4:

The $\bar{\varepsilon}$ restricted to M_+ extends uniquely to a map of $\overline{\hat{M}}$ onto $\hat{N}$ which is positive, additive, order preserving , normal and satisfies $(\hat{\varepsilon}\hat{x})(\phi) = \hat{x}(\phi \circ \bar{\varepsilon})$.

Proof : Using Goldstein [2]

Let $\bar{x} \in \overline{\hat{M}}$, $\bar{x}_n \in \overline{M}_+$ and $\bar{x}_n \nearrow \bar{x}$ since $\bar{\varepsilon}$ is positive, $\overline{\varepsilon x}_n \nearrow \bar{y}$ for some $\bar{y} \in \overline{N}_+$

Put $\hat{\varepsilon}\hat{x} = \hat{y}$, if $\bar{z}_n \in \overline{M}_+$, $\bar{z}_n \nearrow \hat{x}$, then for each $\bar{\phi} \in \overline{M}_*$

$\lim_n \bar{\phi}(\bar{x}_n) = \lim_n \bar{\phi}(\bar{z}_n)$ i.e. $\bar{x}_n - \bar{z}_n \to 0$, σ - weakly

Where $\overline{\varepsilon x}_n - \overline{\varepsilon z}_n \to 0$, σ - weakly, and thus $\lim_n \bar{\phi}(\bar{x}_n) - \lim_n \bar{\phi}(\overline{\varepsilon z}_n - \overline{\varepsilon z}_n) \to 0$, σ - weakly

Implies $\overline{\varepsilon z}_n \nearrow \hat{\varepsilon}\hat{x}$

We have $(\hat{\varepsilon}\hat{x})(\bar{\phi}) = \lim_n \bar{\phi}(\overline{\varepsilon x}_n) = \hat{x}(\bar{\phi} \circ \bar{\varepsilon})$

Hence, $(\hat{\varepsilon}\hat{x})(\bar{\phi}) = \hat{x}(\bar{\phi} \circ \bar{\varepsilon})$. It is obvious that $\hat{\varepsilon}$ is positive, additive and also normal.

To show that $\hat{\varepsilon}$ is invariant with respect to a faithful normal weight we refer to the theorem

Theorem 4:

Let $\overline{M}$ be semifinite and $\overline{N}$ its subalgebra and $\overline{M}_+$ and $\overline{N}_+$ be their respective extended positive part such that $\overline{N}_+ \subset \overline{M}_+$, then $\bar{\varphi}_{\hat{x}} = \bar{\varphi}_{\hat{x}} \circ \hat{\varepsilon}$

Proof:

Let $\hat{x} \in \overset{\wedge}{\overline{M}}_{+}$, we define a weight on $\overset{\wedge}{\overline{M}}_{+}$ by

$$\overline{\varphi}_{\hat{x}}(\hat{a}) = \lim_n \overline{\varphi}_{\hat{x}}(\overline{a}_n) = \lim_{m,n} \overline{\tau}(\overline{x}_m \cdot \overline{a}_n) = \lim_{m,n} \overline{\tau}(\overline{x}_m^{-1/2}\overline{a}_n \overline{x}_m^{-1/2}) \ , \ \hat{a} \in \overset{\wedge}{\overline{M}}_{+}$$

Let $\overset{\wedge}{\varepsilon} : \overset{\wedge}{\overline{M}}_{+} \to \overset{\wedge}{\overline{N}}_{+}$ be our extended generalised conditional expectation on $\overset{\wedge}{\overline{M}}_{+}$ onto $\overset{\wedge}{\overline{N}}_{+}$,

If $\hat{a} \in \overset{\wedge}{\overline{M}}_{+}$, $\hat{x} \in \overset{\wedge}{\overline{N}}_{+}$ then, $\overline{x} \nearrow \hat{x}$ and $\overline{a}_n \nearrow \hat{a}$ with $\overline{x}_m \in \overline{N}_{+}$ and $\overline{a}_n \in \overline{M}_{+}$

$$\overline{\varphi}_{\hat{x}}(\hat{a}) = \overset{\wedge}{\tau}(\hat{x} \cdot \hat{a}) = \lim_{m,n} \overline{\tau}(\overline{x}_m \cdot \overline{\varepsilon}(\overline{a}_n)) = \lim_{m,n} \overline{\tau}(\overline{x}_m^{-1/2}\overline{\varepsilon}(\overline{a}_n)\overline{x}_m^{-1/2})$$

$$= \lim_m \overline{\tau}(\overline{x}_m^{-1/2} \lim_n \overline{\phi}(\overline{\varepsilon}\overline{a}_n)\overline{x}_m^{-1/2}) = \lim_m \overline{\tau}(\overline{x}_m^{-1/2}\overset{\wedge}{\varepsilon}(\hat{a})\overline{x}_m^{-1/2})$$

$$= \lim_m \overline{\tau}(\overset{\wedge}{\varepsilon}(\hat{a})(\overline{x}_m^{-1/2}\overline{\phi}\overline{x}_m^{-1/2}) = \lim_m \overline{\tau}(\overset{\wedge}{\varepsilon}(\hat{a})(\overline{\phi}(\overline{x}_m^{-1/2} 1 \overline{x}_m^{-1/2})))$$

$$= \lim_m \overline{\tau}(\overset{\wedge}{\varepsilon}(\hat{a})\overline{\phi}(\overline{x}_m)) = \overline{\tau}(\overset{\wedge}{\varepsilon}(\hat{a}) \lim_m \overline{\phi}(\overline{x}_m)) = \overset{\wedge}{\tau}(\overset{\wedge}{\varepsilon}(\hat{a})(\hat{x}))$$

$$\overline{\varphi}_{\hat{x}}(\hat{a}) = \overset{\wedge}{\tau}(\overset{\wedge}{\varepsilon}(\hat{a})(\hat{x})) = \overline{\varphi}_{\hat{x}}(\overset{\wedge}{\varepsilon}\hat{a})$$

Hence $\overline{\varphi}_{\hat{x}} = \overline{\varphi}_{\hat{x}} \circ \overset{\wedge}{\varepsilon}$

We used the relation in remark (**) and the assumption that the increasing sequences are densely defined on $\overline{M}_{+}$ and $\overline{N}_{+}$ to prove the above theorem.

Theorem 5:

If $\overline{\varphi}_{\hat{x}}$ is a weight on $\overset{\wedge}{\overline{M}}_{+}$ and $\overset{\wedge}{\varepsilon}$, $\overline{\varepsilon}$ are the conditional expectations on $\overset{\wedge}{\overline{M}}_{+}$ and $\overline{M}_{+}$ respectively then

$$\overline{\varphi}_{\hat{x}}(\hat{E}\hat{a}) = \overline{\varphi}_{\hat{x}}(\hat{a}(\overline{\phi} \circ \overline{E}))$$

Proof

$$\overline{\varphi}_{\hat{x}}(\overset{\wedge}{\varepsilon}\hat{a}) = \overline{\varphi}_{\hat{x}}(\lim_n \overline{\phi}(\overline{\varepsilon}\overline{a}_n)) \quad \text{where } \overline{a}_n \in \overline{M}_{+} \text{ and } \overline{a}_n \nearrow \hat{a}$$

$$= \overline{\tau}(\overline{x}_m^{-1/2} \lim_n \overline{\phi}(\overline{\varepsilon}\overline{a}_n)\overline{x}_m^{-1/2}) = \lim_n \overline{\tau}((\overline{x}_m^{-1/2}\overline{\phi}(\overline{\varepsilon}\overline{a}_n)\overline{x}_m^{-1/2})$$

$$= \overline{\tau}(\overline{x}_m^{-1/2} \lim_n \overline{\phi}(\overline{\varepsilon}\overline{a}_n)\overline{x}_m^{-1/2}) = \lim_n \overline{\tau}((\overline{x}_m^{-1/2}(\overline{\phi} \circ \overline{\varepsilon})\overline{a}_n)\overline{x}_m^{-1/2})$$

$$= \bar{\tau}(\bar{x}_m^{-1/2}\hat{a}(\bar{\phi}\circ\bar{\varepsilon})\bar{x}_m^{-1/2}) = \hat{\tau}(\hat{a}(\bar{x}_m^{-1/2}(\bar{\phi}\circ\bar{\varepsilon})\bar{x}_m^{-1/2}))$$

$$= \hat{\tau}(\hat{a}(\bar{\phi}\circ\bar{\varepsilon})\bar{x}_m) = \bar{\varphi}_{\hat{x}}(\hat{a}(\bar{\phi}\circ\bar{\varepsilon}))$$

Hence $\bar{\varphi}_{\hat{x}}(\hat{\varepsilon}\hat{a}) = \bar{\varphi}_{\hat{x}}(\hat{a}(\bar{\phi}\circ\bar{\varepsilon}))$.

REFERENCES

[1] Accardi, L. and Cecchini, C.(1982),Conditional Expectation in von Neumann algebra and a theorem of Takesaki, Journal Functional Analysis, 45. Pp245-273.

[2] Goldstein, S. (1991), Conditional Expectations and Stochastic Integrals in non commutative Lp spaces , Math.Proc.Camb.Phil.Soc.110,365-383.

[3] Haagerup, U.(1979), Normal Weights on W*-algebra, Journal functional Analysis, 19. 302-317.

[4] Haagerup, U.(1979), Operator Valued Weights in von Neumann Algebras I , J. Func. Anal.32.175-206.

[5] Haagerup, U.(1979), Operator Valued Weights in von Neumann Algebras II , J. Func. Anal.33. 339-361.

[6] Ibrahim, Y. and Yusuf, A. (2010), Martingales of the Generalised Conditional Expectations on the Generalised Positive Operators of a von Neumann Algebra, J. NAMP. 17, 1-6.

[7] Lance, E.C.(1978), Martingales Convergence in von Neumann Algebras, Math.Proc.Camb.Phil.Soc.84,47-56.

[8] Takesaki, M.(1972) Conditional Expectation in von Neumann Algebras, J. Func. Aal.9, 306-321.

[9] Takesaki, M.(1972) Theory of Operator AlgebrasI, Springer–Verlag, New York.

[10] Tomiyama, J.(1957), On the Projection of norm one in W* algebras II, Tohoku Math. J.33,608-611.

[11] Umegaki, H. (1954), Conditional Expectation in Operator Algebras, Tohoku Math, J. 6 177-181.

[12] Umegaki, H. (1956), Conditional Expectation in Operator Algebras, Tohoku Math, J. 8,86--100.

[13] Van Daele, A.(1978), Continuous Crossed Products and type III von Neumann Algebras, Cambridge University Press, Cambridge.