A Predator-prey Model With Transition Two Infectious Diseases in Prey and Harvesting of The Predator

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Abstract

In this paper, we proposed and study prey-predator model involving effect of transition different infectious diseases in prey population with the optimal harvesting for metapopulation, especially predator population, using dynamics programming. Sufficient conditions on the system parameters are derived which guarantee that the equilibrium points of the system are globally asymptotically stable while the harvesting which has an effect on the stability of this system satisfies certain conditions. Also, we discuss the existence of local bifurcation at each equilibrium point. Finally the effects both of the disease and harvest on the dynamical behavior the model are discussed numerically.

Keywords: SIS epidemics disease, Predator-prey model, Harvest management; Global stability; Lyaponov function; local bifurcation.

1. Introduction

It is well known that in nature there is no species lived alone rather than that there are hundreds or thousands of species interact with each other in any given environment. On the other hand densely populated areas are a good incubator for the spread of infectious diseases. Therefore, there is increasing opportunity for the spread of diseases among the communities interacting with each other. Many diseases are transmitted in the species not only through contact, but also directly from environment, such as, influenza, bird flu and others. Anderson and May [1] formulated a prey–predator model involving disease in prey species. Later on many researchers, especially in the last two decades, have proposed and studied different predator–prey models in presence of disease in one of the species see for example [2-11] and the references there in. The effect of constant-rate harvesting on the dynamics of predator-prey systems has been investigated by many authors, see, for example, Brauer and Soudack [12,13], Dai and Tang [14], Myerscough et al. [15], Xiao and Ruan [16], very rich and interesting dynamical behaviors have been observed see for example [17-23] and the references there in, such as the stability of the equilibria, existence of Hopf bifurcation, limit cycles, homoclinic loops, Bogdanov-Takens bifurcations, and even catastrophe. It is also observed that in some cases, before a catastrophic harvest rate is reached the effect of harvesting is to stabilize the equilibrium of the population system. On the other hand, many researchers proposed and study eco-epidemic model containing two disease strains in the same population. see for example [24,25] and the references there in.

On contrast to all the above studies, in this paper a prey-predator model involving, in addition, harvest in predator species the two different SIS infectious diseases in prey species is proposed and analyzed. It is assumed that the both diseases spread within prey population by contact, between susceptible individuals and infected individuals. Furthermore, in this model, Holling type II as a functional response and linear disease incidence for describing the transition both of diseases are used.

2. Mathematical Model

To describe the model for a prey-predator system, we consider the following notation:

- Let $N(t)$ and $p(t)$ be the population densities of the prey species and predator species at time t, respectively.
- The prey grows logistically with intrinsic growth rate $h_1 > 0$ and carrying capacity $h_2 > 0$.
- There is an two different SIS epidemic diseases spread among the prey population and it transmitted between the prey individuals (but not the predator) by contact, according to linear incidence rate with first and second infection rate constants $h_3 > 0$ and $h_4 > 0$, respectively. Therefore, the total prey population is divided into three classes: susceptible that is denoted by $x(t)$, infected by first disease that is denoted by $y(t)$ infected by second disease that is denoted by $z(t)$. Hence at any time t the total prey population is $N(t) = x(t) + y(t) + z(t)$.
- The predator preys upon only the susceptible prey according to Holling type-II functional response with maximum attack rate $h_5 > 0$. Furthermore it is assumed that $e > 0$ represent the conversion rate constant
- Both of the infected prey can be recovered and become susceptible again with recovery rate constant $h_6 > 0$ and $h_7 > 0$, respectively.
- Furthermore it is assumed that there is disease induced mortality rate represented by $h_8 > 0$ and $h_9 > 0$, respectively.
- The predator grows logistically with intrinsic growth rate $h_{10} > 0$ and carrying capacity $h_{11} > 0$.
- Finally, $q > 0$ is the catch ability co-efficient of the predator, $E > 0$ is the harvesting effort and qEp is the catch-rate function based on the CPUE (catch-per-unit-effort) hypothesis.

Consequently, the model with the above assumptions can be written in the following form:

$$\begin{aligned}
\frac{dx}{dt} &= x\left[h_1\left(1-\frac{x+y+z}{h_2}\right) - h_3 y - h_4 z - \frac{h_5 p}{1+x}\right] + h_6 y + h_7 z = \mathrm{N}_1(x, y, z, p) \\
\frac{dy}{dt} &= y(h_3 x - h_6 - h_8) = \mathrm{N}_2(x, y, z, p) \\
\frac{dz}{dt} &= z(h_4 x - h_7 - h_9) = \mathrm{N}_3(x, y, z, p) \\
\frac{dp}{dt} &= p\left[h_{10}\left(1-\frac{p}{h_{11}}\right) + \frac{e h_5 x}{1+x} - qE\right] = \mathrm{N}_4(x, y, z, p)
\end{aligned} \tag{1}$$

The system (1) has the following domain $\Re_+^4 = \{(x, y, z, p),\ x \geq 0,\ y \geq 0,\ z \geq 0 \ and\ p \geq 0\}$. Moreover, the above four nonlinear differential equations are continuously differentiable on int. $\Re_+^4$ and hence they are Lipschizian on $\Re_+^4$. Thus for each set of initial conditions, say $x(0) \geq 0$, $y(0) \geq 0$, $z \geq 0$ and $p(0) \geq 0$, system (1) has a unique solution. Therefore, the domain $\Re_+^4$ is an invariant for the system (1). Further in the following theorem the sufficient condition for uniformly bounded of the solution of the system (1) is established.

For later purposes, it is necessary to have the Jacobian of system (1) at hand, it is reported below.

$$J_k = \left(\beta_{ij}^{[k]}\right) = \begin{bmatrix} \beta_{11}^{[k]} & \beta_{12}^{[k]} & \beta_{13}^{[k]} & \beta_{14}^{[k]} \\ \beta_{21}^{[k]} & \beta_{22}^{[k]} & 0 & 0 \\ \beta_{31}^{[k]} & 0 & \beta_{33}^{[k]} & 0 \\ \beta_{41}^{[k]} & 0 & 0 & \beta_{44}^{[k]} \end{bmatrix}$$

Where: $i = 1,2,3,4$; $j = 1,2,3,4$; $k = 0,1,\ldots,9$ and

$$\beta_{11}^{[k]} = h_1\left(1 - \frac{2}{h_2}x\right) - \left(h_3 + \frac{h_1}{h_2}\right)y - \left(h_4 + \frac{h_1}{h_2}\right)z - \frac{h_5 p}{(1+x)^2}\,;\ \beta_{12}^{[k]} = \frac{-h_1}{h_2}x - h_3 x + h_6\,;$$

$$\beta_{13}^{[k]} = \frac{-h_1}{h_2}x - h_4 x + h_7\,;\ \beta_{14}^{[k]} = \frac{-h_5 x}{1+x}\,;\ \beta_{21}^{[k]} = h_3 y\,;\ \beta_{22}^{[k]} = h_3 x - h_6 - h_8\,;\ \beta_{23}^{[k]} = \beta_{24}^{[k]} = 0\,;$$

$$\beta_{31}^{[k]} = h_4 z\,;\ \beta_{33}^{[k]} = h_4 x - h_7 - h_9\,;\ \beta_{32}^{[k]} = \beta_{34}^{[k]} = 0\,;\ \beta_{41}^{[k]} = \frac{e h_5 p}{(1+x)^2}\,;\ \beta_{42}^{[k]} = \beta_{43}^{[k]} = 0\,;$$

$$\beta_{44}^{[k]} = h_{10}\left(1 - \frac{2}{h_{11}}p\right) + \frac{e h_5 x}{1+x} - qE$$

Theorem (1): All the trajectories of system (1), which initiate in $\Re_+^4$ are uniformly bounded.

Proof: From the first and four equations of system (1) we obtain that;

$$\frac{dx}{dt} \le h_1 x\left(1 - \frac{N}{h_2}\right) \text{ and } \frac{dp}{dt} \le h_{10} p\left(1 - \frac{p}{h_{11}}\right)$$

Clearly by solving the above differential inequalities we get

$$\limsup_{t\to\infty} N(t) \le h_2 \text{ and } \limsup_{t\to\infty} p(t) \le h_{11}$$

Define the function $\mathrm{M}(t) = x(t) + y(t) + z(t) + \frac{1}{e}p(t)$ and then take its time derivative along the solution of system (1), gives

$$\frac{d\mathrm{M}}{dt} \le h_1 x + \frac{h_{10} - qE}{e}p - \phi y - \phi z \qquad \text{where} \quad \phi = \min\{h_8, h_9\}$$

$$\le (h_1 + \phi)x + \frac{1}{e}((h_{10} - qE) + \phi)p - \phi\mathrm{M}$$

$$\le \pi - \phi\mathrm{M} \qquad \text{where} \quad \pi = (h_1 + \phi)h_2 + \frac{((h_{10} - qE) + \phi)}{e}h_{11}$$

Now, by using Gronwall lemma [26], it obtains that:

$$0 < \mathrm{M}(t) \le \mathrm{M}(0)e^{-\phi t} + \frac{\pi}{\phi}\left(1 - e^{-\phi t}\right)$$

which yields $\limsup_{t\to\infty}\mathrm{M}(t) \le \frac{\pi}{\phi}$ that is independent of the initial conditions. Thus the proof is complete.

3. Equilibrium points

System (1) has the following equilibrium:

- The **vanishing equilibrium point** $E_0 = (0,0,0,0)$ always exists.
- The **axial equilibrium point on the x-axis** $E_1 = (x_1, 0, 0, 0)$ where $x_1 = h_2$, E_1 always exists.

- The **axial equilibrium point on the p-axis** $E_2 = (0,0,0,p_2)$ where: $p_2 = \frac{h_{11}}{h_{10}}[h_{10} - qE]$ exists if and only if $h_{10} > qE$.

- The **second disease and predator free equilibrium point** $E_3 = (x_3, y_3, 0, 0)$ where:

$$x_3 = \frac{h_6 + h_8}{h_3} \text{ and } y_3 = \frac{h_1 x_3 (h_2 - x_3)}{(h_2 h_8 + h_1 x_3)} \quad (2)$$

exists uniquely in the interior of the first quadrant of $xy-$plane under the following necessary and sufficient condition $x_3 < h_2$.

- The **first disease and predator free equilibrium point** $E_4 = (x_4, 0, z_4, 0)$ where:

$$x_4 = \frac{h_7 + h_9}{h_4} \text{ and } z_4 = \frac{h_1 x_4 (h_2 - x_4)}{(h_2 h_9 + h_1 x_4)} \quad (3)$$

exists uniquely in the interior of the first quadrant of $xz-$plane under the following necessary and sufficient condition $x_4 < h_2$

- The **simple prey-predator equilibrium point** $E_5 = (x_5, 0, 0, p_5)$ where:

$p_5 = \frac{h_1}{h_2 h_5}(1 + x_5)(h_2 - x_5)$ while x_5 represents a positive root of the following second order polynomial equation $A_1 x^2 + A_2 x + A_3 = 0$ where:

$$A_1 = \frac{-h_1 h_{10}}{h_2 h_5 h_{11}} < 0; \quad A_2 = \left(qE - h_{10} - e h_5 + \frac{h_1 h_{10}(h_2 - 1)}{h_2 h_5 h_{11}} \right); \quad A_3 = \left(qE - h_{10} + \frac{h_1 h_{10}}{h_5 h_{11}} \right)$$

Therefore, straight forward computation shows that E_5 exists uniquely in the interior of the first quadrant of $xp-$plane if and only if the following conditions are hold.

$$x_5 < h_2, \quad \frac{h_{10}}{h_5 h_{11}}(h_5 h_{11} - h_1) < qE \text{ and } h_1 < h_5 h_{11}$$

- The **first disease free equilibrium point** $E_6 = (x_6, 0, z_6, p_6)$ where:

$$x_6 = \frac{h_7 + h_9}{h_4}; \; p_6 = \frac{h_{11}}{h_{10}}\left[h_{10} + \frac{e h_5 x_6}{1 + x_6} - qE \right] \text{ and } z_6 = \frac{x_6 \left[\frac{h_1}{h_2}(h_2 - x_6) - \frac{h_5 p_6}{1 + x_6} \right]}{h_9 + \frac{h_1 x_6}{h_2}} \quad (4)$$

exists uniquely in the interior of the first octant of $xzp-$plane under the following necessary and sufficient conditions:

$$x_6 < h_2; \; qE < h_{10} + \frac{e h_5 x_6}{1 + x_6} \text{ and } \frac{h_5 p_6}{1 + x_6} < \frac{h_1}{h_2}(h_2 - x_6)$$

- The **second disease free equilibrium point** $E_7 = (x_7, y_7, 0, p_7)$ where:

$$x_7 = \frac{h_6 + h_8}{h_3}; \; p_7 = \frac{h_{11}}{h_{10}}\left[h_{10} + \frac{e h_5 x_7}{1 + x_7} - qE \right] \text{ and } y_7 = \frac{x_7 \left[\frac{h_1}{h_2}(h_2 - x_7) - \frac{h_5 p_7}{1 + x_7} \right]}{h_8 + \frac{h_1 x_7}{h_2}} \quad (5)$$

exists uniquely in the interior of the first octant of $xyp-$plane under the following necessary and sufficient conditions:

$$x_7 < h_2 \,;\, qE < h_{10} + \frac{eh_5 x_7}{1+x_7} \quad \text{and} \quad \frac{h_5 p_7}{1+x_7} < \frac{h_1}{h_2}(h_2 - x_7)$$

- The **predator free equilibrium point** $E_8 = (x_8, y_8, z_8, 0)$ where:

$$x_8 = \left\{\frac{h_6+h_8}{h_3}, \frac{h_7+h_9}{h_4}\right\};$$

$$z_8 = \left[\frac{h_1 x_8}{h_2}(h_2 - x_8) - y_8\left(h_8 + \frac{h_1 x_8}{h_2}\right)\right] \Big/ \left(h_9 + \frac{h_1 x_8}{h_2}\right)$$

and y_8 is any positive variable, E_8 exists uniquely in the interior of the first octant of $xyz-$plane under the following necessary and sufficient conditions $x_8 < h_2$; $h_4(h_6+h_8) = h_3(h_7+h_9)$. and

$$y_8\left(h_8 + \frac{h_1 x_8}{h_2}\right) < \frac{h_1 x_8}{h_2}(h_2 - x_8)$$

- The **coexistence equilibrium point** $E_9 = (x_9, y_9, z_9, p_9)$ where

$$\begin{aligned} x_9 &= \left\{\frac{h_6+h_8}{h_3}, \frac{h_7+h_9}{h_4}\right\};\; p_9 = \frac{h_{11}}{h_{10}}\left[h_{10} + \frac{eh_5 x_9}{1+x_9} - qE\right] \\ z_9 &= \left[\left(\frac{h_1 x_9}{h_2}(h_2 - x_9) - \frac{h_5 p_9}{1+x_9}\right) - y_9\left(h_8 + \frac{h_1 x_9}{h_2}\right)\right] \Big/ \left(h_9 + \frac{h_1 x_9}{h_2}\right) \end{aligned} \tag{6}$$

and y_9 is any positive variable, E_9 exists uniquely in the Int. $\Re_+^4$ if and only if the following conditions are hold.

$$x_9 < h_2 \,;\quad h_4(h_6+h_8) = h_3(h_7+h_9);$$

$$qE < h_{10} + \frac{eh_5 x_9}{1+x_9};\; \frac{h_5 p_9}{1+x_9} < \frac{h_1}{h_2}(h_2 - x_9) \text{ and } y_8\left(h_8 + \frac{h_1 x_8}{h_2}\right) < \frac{h_1 x_8}{h_2}(h_2 - x_8)$$

4. Stability of the system

At the vanishing equilibrium point E_0 the eigenvalues are h_1; $-(h_6+h_8)$; $-(h_7+h_9)$; $h_{10} - qE$, showing that it is an unstable saddle .

Theorem (2): The equilibrium point E_1 is globally asymptotically stable in $\Re_+^4$ if and only if:

$$h_2 < \min\left\{\frac{h_6+h_8}{h_3}; \frac{h_7+h_9}{h_4}\right\} \quad \text{and} \quad \left(h_{10} + \frac{eh_2 h_5}{1+h_2}\right) < qE \tag{7}$$

Proof: The Jacobian matrix of the system (1) at E_1 is given by:

$$J_1 = \begin{pmatrix} -h_1 & -h_1 - h_2 h_3 + h_6 & -h_1 - h_2 h_4 + h_7 & \dfrac{-h_2 h_5}{1+h_2} \\ 0 & h_2 h_3 - h_6 - h_8 & 0 & 0 \\ 0 & 0 & h_2 h_4 - h_7 - h_9 & 0 \\ 0 & 0 & 0 & h_{10} + \dfrac{eh_2 h_5}{1+h_2} - qE \end{pmatrix}$$

So, the characteristic equation of J_1 can be written by

$$\left(-h_1-\lambda_x^{[1]}\right)\left((h_2h_3-h_6-h_8)-\lambda_y^{[1]}\right)\left((h_2h_4-h_7-h_9)-\lambda_z^{[1]}\right)\left(\left(h_{10}+\frac{eh_2h_5}{1+h_2}-qE\right)-\lambda_p^{[1]}\right)=0$$

from which, we obtain that:

$$\lambda_x^{[1]}=-h_1 \quad , \ \lambda_y^{[1]}=h_2h_3-h_6-h_8 \ , \ \lambda_z^{[1]}=h_2h_4-h_7-h_9 \ \text{ and } \ \lambda_p^{[1]}=h_{10}+\frac{eh_2h_5}{1+h_2}-qE$$

Here $\lambda_x^{[1]},\lambda_y^{[1]},\lambda_z^{[1]}$ and $\lambda_p^{[1]}$ denote to the eigenvalues in the $x-$direction, $y-$direction, $z-$direction and $p-$direction, respectively. So, it is easy to verify that, all the eigenvalues have negative real parts if and only if the condition (7) holds. Therefore, the equilibrium point E_1 is locally asymptotically stable in $\Re_+^4$. Furthermore, it is a globally asymptotically stable too.

Theorem (3): The equilibrium point E_2 is globally asymptotically stable in $\Re_+^4$ if and only if:

$$\frac{h_5h_{11}}{h_{10}}(h_{10}-qE)<h_1 \tag{8}$$

Proof: The Jacobian matrix of the system (1) at E_2 is given by:

$$J_2=\begin{pmatrix} h_1-\dfrac{h_5h_{11}}{h_{10}}(h_{10}-qE) & h_6 & h_7 & 0 \\ 0 & -(h_6+h_8) & 0 & 0 \\ 0 & 0 & -(h_7+h_9) & 0 \\ \dfrac{eh_5h_{11}}{h_{10}}(h_{10}-qE) & 0 & 0 & -(h_{10}-qE) \end{pmatrix}$$

So, the characteristic equation of J_2 can be written by

$$\begin{aligned}&\left(-(h_7+h_9)-\lambda_z^{[2]}\right)\left(-(h_{10}-qE)-\lambda_p^{[2]}\right)\\ &\times\left(\left(\lambda^{[2]}\right)^2+\left((h_6+h_8)-\left(h_1-\frac{h_5h_{11}}{h_{10}}(h_{10}-qE)\right)\right)\lambda^{[2]}-\left(h_1-\frac{h_5h_{11}}{h_{10}}(h_{10}-qE)\right)(h_6+h_8)\right)=0\end{aligned}$$

From which, we obtain that:

$$\lambda_z^{[2]}=-(h_7+h_9),\ \lambda_p^{[2]}=-(h_{10}-qE),$$

$$\lambda_x^{[2]}\times\lambda_y^{[2]}=-\left(h_1-\frac{h_5h_{11}}{h_{10}}(h_{10}-qE)\right)(h_6+h_8)$$

$$\lambda_x^{[2]}+\lambda_y^{[2]}=\left(h_1-\frac{h_5h_{11}}{h_{10}}(h_{10}-qE)-(h_6+h_8)\right)$$

Here $\lambda_x^{[2]},\lambda_y^{[2]},\lambda_z^{[2]}$ and $\lambda_p^{[2]}$ denote to the eigenvalues in the $x-$direction, $y-$direction, $z-$direction and $p-$direction, respectively. So, it is easy to verify that, in addition the condition of exist E_2 all the eigenvalues have negative real parts if and only if satisfies the condition (8) [27]. Therefore, the E_2 is locally asymptotically stable in $\Re_+^4$. Furthermore, it is a globally asymptotically stable too.

Similarly, The second disease and predator free equilibrium point E_3 is locally and globally asymptotically stable if and only if the following conditions hold.

$$h_{10}+\frac{eh_5x_3}{1+x_3}<qE\,;\,x_3<\min\left\{\frac{h_2}{2},\frac{(h_7+h_9)}{h_4}\right\} \text{ and } h_1\left(1-\frac{2}{h_2}x_3\right)<y_3\left(h_3+\frac{h_1}{h_2}\right) \quad (9)$$

The first disease and predator free equilibrium point E_4 is locally and globally asymptotically stable if and only if the following conditions hold.

$$h_{10}+\frac{eh_5x_4}{1+x_4}<qE\,;\,x_4<\min\left\{\frac{h_2}{2},\frac{(h_6+h_8)}{h_3}\right\} \text{ and } h_1\left(1-\frac{2}{h_2}x_4\right)<z_4\left(h_4+\frac{h_1}{h_2}\right) \quad (10)$$

The prey-predator equilibrium point E_5 is locally and globally asymptotically stable if and only if the following conditions hold.

$$\begin{aligned}&p_5<\frac{h_{11}}{2}\,;\,qE<h_{10}\left(1-\frac{2}{h_{11}}p_5\right)+\frac{eh_5x_5}{1+x_5}\,;\,x_5<\min\left\{\frac{h_2}{2},\frac{(h_6+h_8)}{h_3},\frac{(h_7+h_9)}{h_4}\right\}\\&\frac{h_5p_5}{(1+x_5)^2}<h_1\left(1-\frac{2}{h_2}x_5\right)\end{aligned} \quad (11)$$

Theorem (4): If the following conditions hold

$$x_6<\frac{h_6+h_8}{h_3} \quad (12a)$$

$$h_1\left(1-\frac{2}{h_2}x_6\right)<z_6\left(h_4-\frac{h_1}{h_2}\right)+\frac{h_5p_6}{(1+x_6)^2} \quad (12b)$$

Then, the first disease free equilibrium point E_6 is a locally asymptotically stable.

Proof: The Jacobian matrix of the system (1) at E_6 is given by:

$$J_6=\begin{pmatrix}\beta_{11}^{[6]}&\beta_{12}^{[6]}&\beta_{13}^{[6]}&\beta_{14}^{[6]}\\0&\beta_{22}^{[6]}&0&0\\\beta_{31}^{[6]}&0&0&0\\\beta_{41}^{[6]}&0&0&\beta_{44}^{[6]}\end{pmatrix}$$

Where:

$$\beta_{11}^{[6]}=h_1\left(1-\frac{2}{h_2}x_6\right)-z_6\left(h_4-\frac{h_1}{h_2}\right)-\frac{h_5p_6}{(1+x_6)^2}\,;\,\beta_{12}^{[6]}=\frac{-h_1}{h_2}x_6-h_3x_6+h_6\,;\,\beta_{13}^{[6]}=-\left(\frac{h_1}{h_2}x_6+h_9\right);$$

$$\beta_{14}^{[6]}=\frac{-h_5x_6}{1+x_6}\,;\,\beta_{22}^{[6]}=h_3x_6-h_6-h_8\,;\,\beta_{31}^{[6]}=h_4z_6\,;\,\beta_{41}^{[6]}=\frac{eh_5p_6}{(1+x_6)^2}\,;\,\beta_{44}^{[6]}=-\left(h_{10}+\frac{eh_5x_6}{1+x_6}-qE\right)$$

So, the characteristic equation of J_6 can be written by

$$\left(\lambda_y^{[6]}-\beta_{22}^{[6]}\right)\left[\left(\lambda^{[6]}\right)^3+F_1\left(\lambda^{[6]}\right)^2+F_2\left(\lambda^{[6]}\right)+F_3\right]=0$$

with

$$F_1=-\left(\beta_{11}^{[6]}+\beta_{44}^{[6]}\right);\ F_2=\beta_{11}^{[6]}\beta_{44}^{[6]}-\beta_{14}^{[6]}\beta_{41}^{[6]}-\beta_{13}^{[6]}\beta_{31}^{[6]}\,;\ F_3=\beta_{13}^{[6]}\beta_{31}^{[6]}\beta_{44}^{[6]}$$

Here $\lambda_x^{[6]},\lambda_y^{[6]},\lambda_z^{[6]}$ and $\lambda_p^{[6]}$ denote to the eigenvalues in the $x-$direction, $y-$direction, $z-$direction and $p-$direction, respectively. The Routh-Hurwitz conditions require $F_1>0$ and $\Delta=F_1F_2-F_3>0$, which reduces to condition (12b) and $F_3>0$ is always positive, and in addition the negativity of the other eigenvalues, namely condition (12a). So, according to Routh-Hurwitz criterion E_6 is locally asymptotically stable, it is a globally asymptotically stable too.

Theorem (5): If the following conditions hold

$$x_7 < \frac{h_7 + h_9}{h_4} \tag{13a}$$

$$h_1\left(1 - \frac{2}{h_2}x_7\right) < z_7\left(h_4 - \frac{h_1}{h_2}\right) + \frac{h_5 p_7}{(1+x_7)^2} \tag{13b}$$

Then, the second disease free equilibrium point E_7 is a locally asymptotically stable.

Proof: The Jacobian matrix of the system (1) at E_7 is given by:

$$J_7 = \begin{pmatrix} \beta_{11}^{[7]} & \beta_{12}^{[7]} & \beta_{13}^{[7]} & \beta_{14}^{[7]} \\ \beta_{21}^{[7]} & 0 & 0 & 0 \\ 0 & 0 & \beta_{33}^{[7]} & 0 \\ \beta_{41}^{[7]} & 0 & 0 & \beta_{44}^{[7]} \end{pmatrix}$$

Where:

$$\beta_{11}^{[7]} = h_1\left(1 - \frac{2}{h_2}x_7\right) - y_7\left(h_3 - \frac{h_1}{h_2}\right) - \frac{h_5 p_7}{(1+x_7)^2}\ ;\ \beta_{12}^{[7]} = -\left(\frac{h_1}{h_2}x_7 + h_8\right);\ \beta_{13}^{[7]} = \frac{-h_1}{h_2}x_7 - h_4 x_7 + h_7\ ;$$

$$\beta_{14}^{[7]} = \frac{-h_5 x_7}{1+x_7}\ ;\ \beta_{21}^{[7]} = h_3 y_7\ ;\ \beta_{33}^{[7]} = h_4 x_7 - h_7 - h_9\ ;\ \beta_{41}^{[7]} = \frac{e h_5 p_7}{(1+x_7)^2}\ ;\ \beta_{44}^{[7]} = -\left(h_{10} + \frac{e h_5 x_7}{1+x_7} - qE\right)$$

So, the characteristic equation of J_7 can be written by

$$\left(\lambda_z^{[7]} - \beta_{33}^{[7]}\right)\left[\left(\lambda^{[7]}\right)^3 + F_1\left(\lambda^{[7]}\right)^2 + F_2\left(\lambda^{[7]}\right) + F_3\right] = 0$$

with

$$F_1 = -\left(\beta_{11}^{[7]} + \beta_{44}^{[7]}\right);\ F_2 = \beta_{11}^{[7]}\beta_{44}^{[7]} - \beta_{14}^{[7]}\beta_{41}^{[7]} - \beta_{12}^{[7]}\beta_{21}^{[7]}\ ;\ F_3 = \beta_{12}^{[7]}\beta_{21}^{[7]}\beta_{44}^{[7]}$$

Here $\lambda_x^{[7]}, \lambda_y^{[7]}, \lambda_z^{[7]}$ and $\lambda_p^{[7]}$ denote to the eigenvalues in the $x-$direction, $y-$direction, $z-$direction and $p-$direction, respectively. The Routh-Hurwitz conditions require $F_1 > 0$ and $\Delta = F_1F_2 - F_3 > 0$, which reduces to condition (13b) and $F_3 > 0$ is always positive, and in addition the negativity of the other eigenvalues, namely condition (13a).So, according to Routh-Hurwitz criterion E_7 is locally asymptotically stable, it is a globally asymptotically stable too.

Theorem (6): The predator free equilibrium point E_8 is locally asymptotically stable in $\Re_+^4$ if and only if:

$$x + y + z < h_2\ ;\ x < x_8\ ;\ p < h_{11}; x < \min\left\{\frac{h_6 y}{h_3 y_8}, \frac{h_7 z}{h_4 z_8}\right\} \text{ and } \frac{e h_5 x_8}{1+x_8} + h_{10}\left(1 - \frac{p}{h_{11}}\right) < qE \tag{14}$$

Proof: Since the equilibrium point E_8 is non-hyperbolic equilibrium point, then consider the function

$$V^{[8]} = \left(x - x_8 - x_8 \ln\frac{x}{x_8}\right) + \left(y - y_8 - y_8 \ln\frac{y}{y_8}\right) + \left(z - z_8 - z_8 \ln\frac{z}{z_8}\right) + \frac{p}{e}$$

Clearly, $V^{[8]} : \Re_+^4 \to \Re$ and $V^{[8]}(E_8) = 0$ with $V^{[8]}(E) \neq 0\ \ \forall E \neq E_8, E \in \Re_+^4$. Hence it is positive definite function in $\Re_+^4$. Also, the derivative of $V^{[8]}$ with respect to the time t is given as follows.

$$\frac{dV^{[8]}}{dt} = h_1\left(1-\frac{x+y+z}{h_2}\right)(x-x_8)+\frac{1}{x}(h_6 y-h_3 x y_8)(x-x_8)+\frac{1}{x}(h_7 z-h_4 x z_8)(x-x_8)$$
$$+\frac{p}{e}\left[\frac{eh_5 x_8}{1+x_8}+h_{10}\left(1-\frac{p}{h_{11}}\right)-qE\right]$$

In addition condition (14) guarantee that $\frac{dV^{[8]}}{dt}<0$ on subregion of $\Re_+^4$, then $V^{[8]}$ is a Lyapunov function on that subregion which satisfies condition (14). Therefore E_8 is a locally asymptotically stable but not globally.

Theorem (7): the coexistence equilibrium point E_9 is locally asymptotically stable in $\Re_+^4$ if and only if:

$$x+y+z<h_2\,;\; x<x_9\,;\; p<\min\{h_{11}\,,\,p_9\}\,;\; x<\min\left\{\frac{h_6 y}{h_3 y_9}\,,\,\frac{h_7 z}{h_4 z_9}\right\} \text{ and } \frac{eh_5 x_9}{1+x_9}+h_{10}\left(1-\frac{p}{h_{11}}\right)<qE \qquad (15)$$

Proof: Since the equilibrium point E_9 is non-hyperbolic equilibrium point, then consider the function

$$V^{[8]} = \left(x-x_9-x_9\ln\frac{x}{x_9}\right)+\left(y-y_9-y_9\ln\frac{y}{y_9}\right)+\left(z-z_9-z_9\ln\frac{z}{z_9}\right)+\frac{1}{e}\left(p-p_9-p_9\ln\frac{p}{p_9}\right)$$

Clearly, $V^{[9]}:\Re_+^4\to\Re$ and $V^{[9]}(E_9)=0$ with $V^{[9]}(E)\neq 0\quad \forall\, E\neq E_9\,, E\in\Re_+^4$. Hence it is positive definite function in $\Re_+^4$. Also, the derivative of $V^{[9]}$ with respect to the time t is given as follows.

$$\frac{dV^{[9]}}{dt} = h_1\left(1-\frac{x+y+z}{h_2}\right)(x-x_9)+\frac{1}{x}(h_6 y-h_3 x y_9)(x-x_9)+\frac{1}{x}(h_7 z-h_4 x z_9)(x-x_9)$$
$$+\frac{p}{e}\left[\frac{eh_5 x_9}{1+x_9}+\frac{h_{10}}{e}\left(1-\frac{p}{h_{11}}\right)-\frac{qE}{e}\right]$$

In addition condition (15) guarantee that $\frac{dV^{[9]}}{dt}<0$ on subregion of $\Re_+^4$, then $V^{[9]}$ is a Lyapunov function on that subregion which satisfies condition (15). Therefore E_9 is a locally asymptotically stable but not globally.

5. The local bifurcation analysis

Theorem (8): the system (1) at the axial equilibrium point on the x-axis E_1 with the parameter $h_{10}^{[1]}=qE-\frac{eh_5 h_2}{1+h_2}$ has:

- No saddle-node bifurcation.
- No pitchfork bifurcation.
- Transcritical bifurcation.

Proof: According to the Jacobian matrix J_1 the system (1) at the equilibrium point E_1 has zero eigenvalue (say $\lambda^{[1]}=0$) if and only if $h_{10}=h_{10}^{[1]}$. Now, let $V^{[1]}=\left(v_1^{[1]},v_2^{[1]},v_3^{[1]},v_4^{[1]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda^{[1]}=0$. Thus $\left(\tilde{J}_1-\lambda^{[1]}I\right)V^{[1]}=0$ where $\tilde{J}_1=J_1\left(\lambda^{[1]}=0\right)$, then $V^{[1]}=\left(\frac{-h_5 h_2}{h_1(1+h_2)}v_4^{[1]}\,,0\,,0\,,v_4^{[1]}\right)^T$ where $v_4^{[1]}$ represents any nonzero real number. And let $\Psi^{[1]}=\left(\psi_1^{[1]},\psi_2^{[1]},\psi_3^{[1]},\psi_4^{[1]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda^{[1]}=0$ of the matrix $\tilde{J}_1^T$, hence $\Psi^{[1]}=\left(0,0,0,\psi_4^{[1]}\right)^T$ where $\psi_4^{[1]}$ represents any nonzero real number. Now, we have $\left(\Psi^{[1]}\right)^T\left[\mathrm{N}_{h_{10}}\left(E_1,h_{10}^{[1]}\right)\right]=0$ where $\mathrm{N}_{h_{10}}(X,h_{10})$ represents the derivative of $\mathrm{N}(X,h_{10})$ with respect to parameter h_{10}.

Thus, according to Sotomayor's theorem[33], the saddle-node bifurcation cannot occur. While the first condition of transcritical and pitchfork bifurcation is satisfied. Now, it is easy to observed that

$$\left(\Psi^{[1]}\right)^T\left[D\mathrm{N}_{h_{10}}\left(E_1,h_{10}^{[1]}\right)V^{[1]}\right]=v_4^{[1]}\psi_4^{[1]}\neq 0$$

where $D\mathrm{N}_{h_{10}}(X,h_{10})$ represents the derivative of $\mathrm{N}_{h_{10}}(X,h_{10})$ with respect to $X=(x,y,z,P)^T$. Moreover to the above, if $J=D\mathrm{N}(X,h_{10})$ represents the general Jacobian matrix of system (1) at X, then

$$\left(\Psi^{[1]}\right)^T\left[D^2\mathrm{N}\left(E_1,h_{10}^{[1]}\right)\left(V^{[1]},V^{[1]}\right)\right]=-\left[\frac{2h_{10}^{[1]}}{h_{11}}+\frac{2e\,h_5^2\,h_2}{(1+h_2)^3}\right]\left(v_4^{[1]}\right)^2\psi_4^{[1]}\neq 0$$

Thus, the system (1) has transcritical bifurcation at E_1 with the parameter $h_{10}=h_{10}^{[1]}$ while the pitchfork bifurcation does not occur.

Theorem (9): Assume that the following condition holds

$$(h_2-1)(h_{10}-qE)\neq eh_2h_5 \tag{16}$$

Then the system (1) at the axial equilibrium point on the p-axis E_2 with the parameter

$h_1^{[2]}=\frac{h_5h_{11}}{h_{10}}(h_{10}-qE)$ has:

- No saddle-node bifurcation.
- No pitchfork bifurcation.
- Transcritical bifurcation.

Proof: According to the Jacobian matrix J_2 the system (1) at the equilibrium point E_2 has zero eigenvalue (say $\lambda^{[2]}=0$) if and only if $h_1=h_1^{[2]}$. Now, let $V^{[2]}=\left(v_1^{[2]},v_2^{[2]},v_3^{[2]},v_4^{[2]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda^{[2]}=0$. Thus $\left(\tilde{J}_2-\lambda^{[2]}I\right)V^{[2]}=0$ where $\tilde{J}_2=J_2\left(\lambda^{[2]}=0\right)$, then $V^{[2]}=\left(\frac{(h_{10}-qE)}{eh_1^{[2]}}v_4^{[2]},0,0,v_4^{[2]}\right)^T$ where $v_4^{[2]}$ represents any nonzero real number. And let $\Psi^{[2]}=\left(\psi_1^{[2]},\psi_2^{[2]},\psi_3^{[2]},\psi_4^{[2]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda^{[2]}=0$ of the matrix $\tilde{J}_2^T$, hence $\Psi^{[2]}=\left(\psi_1^{[2]},\frac{h_6}{(h_6+h_8)}\psi_1^{[2]},\frac{h_7}{(h_7+h_9)}\psi_1^{[2]},0\right)^T$

where $\psi_1^{[2]}$ represents any nonzero real number. Now, we have $\left(\Psi^{[2]}\right)^T\left[\mathrm{N}_{h_1}\left(E_2,h_1^{[2]}\right)\right]=0$ where $\mathrm{N}_{h_1}(X,h_1)$ represents the derivative of $\mathrm{N}(X,h_1)$ with respect to parameter h_1 .

Thus, according to Sotomayor's theorem for local bifurcation the saddle-node bifurcation cannot occur. While the first condition of transcritical and pitchfork bifurcation is satisfied. Now, it is easy to observed that

$$\left(\Psi^{[2]}\right)^T\left[D\mathrm{N}_{h_1}\left(E_2,h_1^{[2]}\right)V^{[2]}\right]=\frac{h_{10}-qE}{eh_1^{[2]}}v_4^{[2]}\psi_1^{[2]}\neq 0$$

where $D\mathrm{N}_{h_1}(X,h_1)$ represents the derivative of $\mathrm{N}_{h_1}(X,h_1)$ with respect to $X=(x,y,z,P)^T$. Moreover to the above, if $J=D\mathrm{N}(X,h_1)$ represents the general Jacobian matrix of system (1) at X . Obviously, if the condition (16) holds then

$$\left(\Psi^{[2]}\right)^T\left[D^2\mathrm{N}\left(E_2,h_{10}^{[2]}\right)\left(V^{[2]},V^{[2]}\right)\right]=-2\left[\frac{(h_2-1)(h_{10}-qE)}{eh_2}-h_5\right]\frac{(h_{10}-qE)}{eh_1^{[2]}}\left(v_4^{[2]}\right)^2\psi_1^{[2]}\neq 0$$

Thus, the system (1) has transcritical bifurcation at E_2 with the parameter $h_1=h_1^{[2]}$ while the pitchfork bifurcation does not occur.

Theorem (10): The system (1) at the second disease and predator free equilibrium point E_3 with the parameter $h_{10}^{[3]}=qE-\frac{eh_5x_3}{1+x_3}$ has:

- No pitchfork bifurcation.
- No transcritical bifurcation.
- Saddle-node bifurcation.

Proof: The Jacobian matrix of system (1) can be written as:

$$J_3=\begin{pmatrix} h_1-\frac{h_5h_{11}}{h_{10}}(h_{10}-qE) & -\frac{h_1}{h_2}x_3-h_3x_3+h_6 & -\frac{h_1}{h_2}x_3-h_4x_3+h_7 & \frac{-h_5x_3}{1+x_3} \\ h_3y_3 & 0 & 0 & 0 \\ 0 & 0 & h_4x_3-h_7-h_9 & 0 \\ 0 & 0 & 0 & h_{10}+\frac{eh_5x_3}{1+x_3}-qE \end{pmatrix}$$

Clearly the characteristic equation of J_3 can written as:

$$P\left(\lambda^{[3]}\right)=\left[h_{10}+\frac{eh_5x_3}{1+x_3}-qE-\lambda_p^{[3]}\right]\left[h_4x_3-h_7-h_9-\lambda_z^{[3]}\right]\times$$
$$\times\left[\left(\lambda^{[3]}\right)^2-\lambda^{[3]}\left(h_1\left(1-\frac{2}{h_2}x_3\right)-\left(h_3+\frac{h_1}{h_2}\right)y_3\right)+h_3y_3\left(\frac{h_1}{h_2}x_3+h_8\right)\right]$$

So, the system (1) at the equilibrium point E_3 has zero eigenvalue (say $\lambda^{[3]}=0$) if and only if $h_{10}=h_{10}^{[3]}$. Now, let $V^{[3]}=\left(v_1^{[3]},v_2^{[3]},v_3^{[3]},v_4^{[3]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda^{[3]}=0$. Thus $\left(\tilde{J}_3-\lambda^{[3]}I\right)V^{[3]}=0$ where $\tilde{J}_3=J_3\left(\lambda^{[3]}=0\right)$, then $V^{[3]}=\left(0\,,\frac{-h_2h_5x_3}{(1+x_3)(h_1x_3+h_2h_8)}v_4^{[3]}\,,0\,,v_4^{[3]}\right)^T$ where $v_4^{[3]}$ represents any nonzero real number. And let $\Psi^{[3]}=\left(\psi_1^{[3]},\psi_2^{[3]},\psi_3^{[3]},\psi_4^{[3]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda^{[3]}=0$ of the matrix $\tilde{J}_3^T$, hence $\Psi^{[3]}=\left(0\,,0\,,0,\psi_4^{[3]}\right)^T$ where $\psi_4^{[3]}$ represents any nonzero real number. Now, we have $\left(\Psi^{[3]}\right)^T\left[\mathrm{N}_{h_{10}}\left(E_3,h_{10}^{[3]}\right)\right]=\psi_4^{[3]}\neq 0$ where $\mathrm{N}_{h_{10}}(X,h_{10})$ represents the derivative of $\mathrm{N}(X,h_{10})$ with respect to parameter h_{10}.

Thus, according to Sotomayor's theorem, both Transcritical and pitchfork bifurcation cannot occur. While the first condition of saddle-point is satisfied. Moreover to the above, if $J=D\mathrm{N}(X,h_{10})$ represents the general Jacobian matrix of system (1) at X, then

$$\left(\Psi^{[3]}\right)^T\left[D^2\mathrm{N}\left(E_3,h_{10}^{[3]}\right)\left(V^{[3]},V^{[3]}\right)\right]=\frac{-2h_{10}^{[3]}}{h_{11}}\left(v_4^{[3]}\right)^2\psi_4^{[3]}\neq 0$$

Thus, the system (1) has saddle-node bifurcation at E_3 with the parameter $h_{10}=h_{10}^{[3]}$.

Theorem (11): the system (1) at the first disease and predator free equilibrium point E_4 with the parameter $h_{10}^{[4]}=qE-\frac{eh_5x_4}{1+x_4}$ has:

- No saddle-node bifurcation.
- No pitchfork bifurcation.
- Ttranscritical bifurcation.

Proof: The Jacobian matrix of system (1) can be written as:

$$J_4=\begin{pmatrix} h_1\left(1-\frac{2}{h_2}x_4\right)-\left(h_4+\frac{h_1}{h_2}\right)z_4 & -\frac{h_1}{h_2}x_4-h_3x_4+h_6 & -\frac{h_1}{h_2}x_4-h_9 & \frac{-h_5x_4}{1+x_4} \\ 0 & h_3x_4-h_6-h_8 & 0 & 0 \\ h_4z_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_{10}+\frac{eh_5x_4}{1+x_4}-qE \end{pmatrix}$$

Clearly the characteristic equation of J_4 can written as:

$$P\left(\lambda^{[4]}\right)=\left[h_{10}+\frac{eh_5x_4}{1+x_4}-qE-\lambda_p^{[4]}\right]\left[h_3x_4-h_6-h_8-\lambda_y^{[4]}\right]\times$$
$$\times\left[\left(\lambda^{[4]}\right)^2-\lambda^{[4]}\left(h_1\left(1-\frac{2}{h_2}x_4\right)-\left(h_4+\frac{h_1}{h_2}\right)z_4\right)+h_4z_4\left(\frac{h_1}{h_2}x_4+h_9\right)\right]$$

So, the system (1) at the equilibrium point E_4 has zero eigenvalue (say $\lambda^{[4]}=0$) if and only if $h_{10}=h_{10}^{[4]}$. Now, let $V^{[4]}=\left(v_1^{[4]},v_2^{[4]},v_3^{[4]},v_4^{[4]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda^{[4]}=0$. Thus $\left(\tilde{J}_4-\lambda^{[4]}I\right)V^{[4]}=0$ where $\tilde{J}_4=J_4\left(\lambda^{[4]}=0\right)$, then $V^{[4]}=\left(0\,,0\,,\frac{h_2h_5x_4}{(1+x_4)(h_1x_4+h_2h_9)}v_4^{[4]}\,,v_4^{[4]}\right)^T$ where $v_4^{[4]}$ represents any nonzero real number. And let $\Psi^{[4]}=\left(\psi_1^{[4]},\psi_2^{[4]},\psi_3^{[4]},\psi_4^{[4]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda^{[4]}=0$ of the matrix $\tilde{J}_4^T$, hence $\Psi^{[4]}=\left(0\,,0\,,0,\psi_4^{[4]}\right)^T$ where $\psi_4^{[4]}$ represents any nonzero real number. Now, we have $\left(\Psi^{[4]}\right)^T\left[\mathrm{N}_{h_{10}}\left(E_4,h_{10}^{[4]}\right)\right]=0$ where $\mathrm{N}_{h_{10}}(X,h_{10})$ represents the derivative of $\mathrm{N}(X,h_{10})$ with respect to parameter h_{10}.

Thus, according to Sotomayor's theorem for local bifurcation the saddle-node bifurcation cannot occur. While the first condition of transcritical and pitchfork bifurcation is satisfied. Now, it is easy to observed that

$$\left(\Psi^{[4]}\right)^T\left[D\mathrm{N}_{h_{10}}\left(E_4,h_{10}^{[4]}\right)V^{[4]}\right]=v_4^{[4]}\psi_4^{[4]}\neq 0$$

where $D\mathrm{N}_{h_{10}}(X,h_{10})$ represents the derivative of $\mathrm{N}_{h_{10}}(X,h_{10})$ with respect to $X=(x\,,y\,,z\,,P)^T$. Moreover to the above, if $J=D\mathrm{N}(X,h_{10})$ represents the general Jacobian matrix of system (1) at X, then

$$\left(\Psi^{[4]}\right)^T\left[D^2\mathrm{N}\left(E_4,h_{10}^{[4]}\right)\left(V^{[4]},V^{[4]}\right)\right]=\frac{-2h_{10}^{[4]}}{h_{11}}\left(v_4^{[4]}\right)^2\psi_4^{[4]}\neq 0$$

Thus, the system (1) has transcritical bifurcation at E_4 with the parameter $h_{10}=h_{10}^{[4]}$ while the pitchfork bifurcation does not occur.

Theorem (12): the system (1) at the simple prey-predator equilibrium point E_5 with the parameter

$$h_{10}^{[5]}=\frac{1}{\left(1-\frac{2}{h_{11}}p_5\right)}\left[qE-\frac{eh_5x_5}{(1+x_5)}-\frac{eh_5^2x_5p_5}{(1+x_5)^3\left[h_1\left(1-\frac{2}{h_2}x_5\right)-\frac{h_5x_5}{(1+x_5)^2}\right]}\right]\text{ has:}$$

- No pitchfork bifurcation.

- No transcritical bifurcation.
- Saddle-node bifurcation.

Proof: The Jacobian matrix of system (1) can be written as:

$$J_5=\begin{pmatrix} h_1\left(1-\frac{2x_5}{h_2}\right)-\frac{h_5p_5}{(1+x_5)^2} & -\frac{h_1}{h_2}x_5-h_3x_5+h_6 & -\frac{h_1}{h_2}x_5-h_4x_5+h_7 & \frac{-h_5x_5}{1+x_5} \\ 0 & h_3x_5-h_6-h_8 & 0 & 0 \\ 0 & 0 & h_4x_5-h_7-h_9 & 0 \\ \frac{eh_5p_5}{(1+x_5)^2} & 0 & 0 & h_{10}-\frac{2h_{10}p_5}{h_{11}}+\frac{eh_5x_5}{1+x_5}-qE \end{pmatrix}$$

Clearly the characteristic equation of J_5 can written as:

$$P\left(\lambda^{[5]}\right)=\left[h_3x_5-h_6-h_8-\lambda_y^{[5]}\right]\left[h_4x_5-h_7-h_9-\lambda_z^{[5]}\right]\times$$
$$\times\left[\begin{array}{l}\left(\lambda^{[5]}\right)^2-\lambda^{[5]}\left(h_1\left(1-\frac{2}{h_2}x_5\right)-\frac{h_5p_5}{(1+x_5)^2}+h_{10}\left(1-\frac{2}{h_{11}}p_5\right)+\frac{eh_5x_5}{1+x_5}-qE\right)\\ +\left(\left[h_1\left(1-\frac{2}{h_2}x_5\right)-\frac{h_5p_5}{(1+x_5)^2}\right]\left[h_{10}\left(1-\frac{2}{h_{11}}p_5\right)+\frac{eh_5x_5}{1+x_5}-qE\right]+\frac{e(h_5)^2x_5p_5}{(1+x_5)^3}\right)\end{array}\right]$$

So, the system (1) at the equilibrium point E_5 has zero eigenvalue (say $\lambda^{[5]}=0$) if and only if $h_{10}=h_{10}^{[5]}$. Now, let $V^{[5]}=\left(v_1^{[5]},v_2^{[5]},v_3^{[5]},v_4^{[5]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda^{[5]}=0$. Thus $\left(\tilde{J}_5-\lambda^{[5]}I\right)V^{[5]}=0$ where $\tilde{J}_5=J_5\left(\lambda^{[5]}=0\right)$, then $V^{[5]}=\left(\frac{\left[h_{10}^{[5]}\left(1-\frac{2}{h_{11}}p_5\right)+\frac{eh_5x_5}{1+x_5}-qE\right]}{\frac{eh_5p_5}{(1+x_5)^2}}v_4^{[5]},0,0,v_4^{[5]}\right)^T$

where $v_4^{[5]}$ represents any nonzero real number. And let $\Psi^{[5]}=\left(\psi_1^{[5]},\psi_2^{[5]},\psi_3^{[5]},\psi_4^{[5]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda^{[5]}=0$ of the matrix $\tilde{J}_5^T$, hence

$$\Psi^{[5]}=\left(\psi_1^{[5]},\frac{\left[\frac{h_1}{h_2}x_5+h_3x_5-h_6\right]}{\left[h_3x_5-h_6-h_8\right]}\psi_1^{[5]},\frac{\left[\frac{h_1}{h_2}x_5+h_4x_5-h_7\right]}{\left[h_4x_5-h_7-h_9\right]}\psi_1^{[5]},\frac{-\frac{eh_5p_5}{(1+x_5)^2}}{\left[h_{10}^{[5]}\left(1-\frac{2p_5}{h_{11}}\right)+\frac{eh_5x_5}{1+x_5}-qE\right]}\psi_1^{[5]}\right)^T$$

where $\psi_1^{[5]}$ represents any nonzero real number. Now, we have

$$\left(\Psi^{[5]}\right)^T\left[\mathrm{N}_{h_{10}}\left(E_5,h_{10}^{[5]}\right)\right]=\frac{-\frac{eh_5(p_5)^2}{(1+x_5)^2}\left(1-\frac{p_5}{h_{11}}\right)}{\left[h_{10}^{[5]}\left(1-\frac{2}{h_{11}}p_5\right)+\frac{eh_5x_5}{1+x_5}-qE\right]}\psi_1^{[5]}\neq 0$$

where $\mathrm{N}_{h_{10}}(X,h_{10})$ represents the derivative of $\mathrm{N}(X,h_{10})$ with respect to parameter h_{10}.

Thus, according to Sotomayor's theorem, both Transcritical and pitchfork bifurcation cannot occur. While the first condition of saddle-point is satisfied. Moreover to the above, if $J=D\mathrm{N}(X,h_{10})$ represents the general Jacobian matrix of system (1) at X, then

$$\left(\Psi^{[5]}\right)^T\left[D^2\mathrm{N}\left(E_5,h_{10}^{[5]}\right)\left(V^{[5]},V^{[5]}\right)\right]=\frac{\dfrac{2eh_5p_5}{(1+x_5)^2}}{h_{11}\left[h_{10}^{[5]}\left(1-\dfrac{2}{h_{11}}p_5\right)+\dfrac{eh_5x_5}{1+x_5}-qE\right]}\left(v_4^{[5]}\right)^2\psi_1^{[5]}\neq 0$$

Thus, the system (1) has saddle-node bifurcation at E_5 with the parameter $h_{10}=h_{10}^{[5]}$ while the pitchfork bifurcation does not occur.

Theorem (13): The system (1) does not occur any types of local bifurcation at the first disease free equilibrium point E_6 with the parameter $h_3^{[6]}=\dfrac{h_4(h_6+h_8)}{(h_7+h_9)}$

Proof: According to the Jacobian matrix J_6 the system (1) at the equilibrium point E_6 has zero eigenvalue (say $\lambda^{[6]}=0$) if and only if $h_3=h_3^{[6]}$. Now, let $V^{[6]}=\left(v_1^{[6]},v_2^{[6]},v_3^{[6]},v_4^{[6]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda^{[6]}=0$. Thus $\left(\tilde{J}_6-\lambda^{[6]}I\right)V^{[6]}=0$ where $\tilde{J}_6=J_6\left(\lambda^{[6]}=0\right)$, then

$V^{[6]}=\left(0,\dfrac{-[h_1x_6+h_2h_9]}{[h_1x_6+h_2h_8]}v_3^{[6]},v_3^{[6]},0\right)^T$ where $v_3^{[6]}$ represents any nonzero real number. And let

$\Psi^{[6]}=\left(\psi_1^{[6]},\psi_2^{[6]},\psi_3^{[6]},\psi_4^{[6]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda^{[6]}=0$ of the matrix $\tilde{J}_6^T$, hence $\Psi^{6]}=\left(0,\psi_2^{[6]},0,0\right)^T$ where $\psi_2^{[6]}$ represents any nonzero real number. Now, we have $\left(\Psi^{[6]}\right)^T\left[\mathrm{N}_{h_3}\left(E_6,h_3^{[6]}\right)\right]=0$ where $\mathrm{N}_{h_3}(X,h_3)$ represents the derivative of $\mathrm{N}(X,h_3)$ with respect to parameter h_3.

Thus, according to Sotomayor's theorem, the saddle-node bifurcation cannot occur. While the first condition of transcritical and pitchfork bifurcation is satisfied. Now, it is easy to observed that

$$\left(\Psi^{[6]}\right)^T\left[D\mathrm{N}_{h_3}\left(E_6,h_3^{[6]}\right)V^{[6]}\right]=\frac{-x_6[h_1x_6+h_2h_9]}{[h_1x_6+h_2h_8]}v_3^{[6]}\psi_2^{[6]}\neq 0$$

where $D\mathrm{N}_{h_3}(X,h_3)$ represents the derivative of $\mathrm{N}_{h_3}(X,h_3)$ with respect to $X=(x,y,z,P)^T$. Moreover to the above, if $J=D\mathrm{N}(X,h_3)$ represents the general Jacobian matrix of system (1) at X, then

$$\left(\Psi^{[6]}\right)^T\left[D^2\mathrm{N}\left(E_6,h_3^{[6]}\right)\left(V^{[6]},V^{[6]}\right)\right]=0$$

Thus, according to Sotomayor's theorem for local bifurcation both of Transcritical and pitchfork bifurcation cannot occur.

Theorem (14): The system (1) does not occur any types of local bifurcation at the second disease free equilibrium point E_7 with the parameter $h_4^{[7]}=\dfrac{h_3(h_7+h_9)}{(h_6+h_8)}$

Proof: According to the Jacobian matrix J_7 the system (1) at the equilibrium point E_7 has zero eigenvalue (say $\lambda^{[7]}=0$) if and only if $h_4=h_4^{[7]}$. Now, let $V^{[7]}=\left(v_1^{[7]},v_2^{[7]},v_3^{[7]},v_4^{[7]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda^{[7]}=0$. Thus $\left(\tilde{J}_7-\lambda^{[7]}I\right)V^{[7]}=0$ where $\tilde{J}_7=J_7\left(\lambda^{[7]}=0\right)$, then $V^{[7]}=\left(0\,,v_2^{[7]}\,,\frac{\left(h_1x_7+h_2h_8\right)}{\left(h_1x_7+h_2h_9\right)}v_2^{[7]}\,,0\right)^T$ where $v_2^{[7]}$ represents any nonzero real number. And let $\Psi^{[7]}=\left(\psi_1^{[7]},\psi_2^{[7]},\psi_3^{[7]},\psi_4^{[7]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda^{[7]}=0$ of the matrix $\tilde{J}_7^T$, hence $\Psi^{[7]}=\left(0,0,\psi_3^{[7]},0\right)^T$ where $\psi_3^{[7]}$ represents any nonzero real number. Now, we have $\left(\Psi^{[7]}\right)^T\left[\mathrm{N}_{h_4}\left(E_7,h_4^{[7]}\right)\right]=0$ where $\mathrm{N}_{h_4}(X,h_4)$ represents the derivative of $\mathrm{N}(X,h_4)$ with respect to parameter h_4.

Thus, according to Sotomayor's theorem for local bifurcation the saddle-node bifurcation cannot occur. While the first condition of transcritical and pitchfork bifurcation is satisfied. Now, it is easy to observed that

$$\left(\Psi^{[7]}\right)^T\left[D\mathrm{N}_{h_4}\left(E_7,h_4^{[7]}\right)V^{[7]}\right]=\frac{x_7\left(h_1x_7+h_2h_8\right)}{\left(h_1x_7+h_2h_9\right)}v_2^{[7]}\psi_3^{[7]}\neq 0$$

where $D\mathrm{N}_{h_4}(X,h_4)$ represents the derivative of $\mathrm{N}_{h_4}(X,h_4)$ with respect to $X=(x,y,z,P)^T$. Moreover to the above, if $J=D\mathrm{N}(X,h_4)$ represents the general Jacobian matrix of system (1) at X. Obviously, if the condition (16) holds then $\left(\Psi^{[7]}\right)^T\left[D^2\mathrm{N}\left(E_7,h_4^{[7]}\right)\left(V^{[7]},V^{[7]}\right)\right]=0$

Thus, according to Sotomayor's theorem for local bifurcation both of Transcritical and pitchfork bifurcation cannot occur.

Theorem (15): The system (1) at the predator free equilibrium point E_8 with the parameter $h_5^{[8]}=\frac{\left(h_4+h_7+h_9\right)\left(qE-h_{10}\right)}{e\left(h_7+h_9\right)}$ has:

- No saddle-node bifurcation.
- No pitchfork bifurcation.
- Transcritical bifurcation.

Proof: According to the Jacobian matrix J_8 the system (1) at the equilibrium point E_8 has zero eigenvalue (say $\lambda^{[8]}=0$) if and only if $h_5=h_5^{[8]}$. Now, let $V^{[8]}=\left(v_1^{[8]},v_2^{[8]},v_3^{[8]},v_4^{[8]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda^{[8]}=0$. Thus $\left(\tilde{J}_8-\lambda^{[8]}I\right)V^{[8]}=0$ where $\tilde{J}_8=J_8\left(\lambda^{[8]}=0\right)$, then

$V^{[8]} = \left(0 \, , \frac{h_2}{(h_1 x_8 + h_2 h_8)} \left(\frac{(h_1 x_8 + h_2 h_8)}{h_2} v_3^{[7]} + \frac{h_5 x_8}{(1+x_8)} v_4^{[7]} \right), v_3^{[7]} \, , v_4^{[7]} \right)^T$ where $v_3^{[8]}$ *and* $v_4^{[8]}$ represent any nonzero real number. And let $\Psi^{[8]} = \left(\psi_1^{[8]}, \psi_2^{[8]}, \psi_3^{[8]}, \psi_4^{[8]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda^{[8]} = 0$ of the matrix $\tilde{J}_8^T$, hence $\Psi^{[8]} = \left(0 \, , \frac{-(h_1 x_8 + h_2 h_9)}{(h_1 x_8 + h_2 h_8)} \psi_3^{[8]} \, , \psi_3^{[8]}, \psi_4^{[8]} \right)^T$ where $\psi_3^{[8]}$ *and* $\psi_4^{[8]}$ represents any nonzero real number. Now, we have $\left(\Psi^{[8]}\right)^T \left[\mathrm{N}_{h_5}\left(E_8, h_5^{[8]}\right) \right] = 0$ where $\mathrm{N}_{h_5}(X, h_5)$ represents the derivative of $\mathrm{N}(X, h_5)$ with respect to parameter h_5 .

Thus, according to Sotomayor's theorem for local bifurcation the saddle-node bifurcation cannot occur. While the first condition of transcritical and pitchfork bifurcation is satisfied. Now, it is easy to observed that

$$\left(\Psi^{[8]}\right)^T \left[D\mathrm{N}_{h_5}\left(E_8, h_5^{[8]}\right) V^{[8]} \right] = \frac{e x_8}{(1+x_8)} v_4^{[8]} \psi_4^{[8]} \neq 0$$

where $D\mathrm{N}_{h_5}(X, h_5)$ represents the derivative of $\mathrm{N}_{h_5}(X, h_5)$ with respect to $X = (x, y, z, P)^T$. Moreover to the above, if $J = D\mathrm{N}(X, h_5)$ represents the general Jacobian matrix of system (1) at X . Obviously, if the condition (16) holds then $\left(\Psi^{[8]}\right)^T \left[D^2\mathrm{N}\left(E_8, h_5^{[8]}\right)\left(V^{[8]}, V^{[8]}\right) \right] = \frac{-2h_{10}}{h_{11}} \left(v_4^{[8]}\right)^2 \psi_4^{[8]} \neq 0$

Thus, the system (1) has transcritical bifurcation at E_8 with the parameter $h_5 = h_5^{[8]}$ while the pitchfork bifurcation does not occur.

Theorem (16): The system (1) does not occur any types of local bifurcation at the coexistence equilibrium point E_9 with the parameter $h_5^{[9]} = \frac{(1+x_9)^2}{p_9} \left[h_1 \left(1 + \frac{2}{h_2} x_9 \right) - \left(h_3 + \frac{h_1}{h_2} \right) y_9 - \left(h_4 + \frac{h_1}{h_2} \right) z_9 \right]$

Proof: According to the Jacobian matrix J_9 the system (1) at the equilibrium point E_9 has zero eigenvalue (say $\lambda^{[9]} = 0$) if and only if $h_5 = h_5^{[9]}$. Now, let $V^{[9]} = \left(v_1^{[9]}, v_2^{[9]}, v_3^{[9]}, v_4^{[9]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda^{[9]} = 0$. Thus $\left(\tilde{J}_9 - \lambda^{[9]} I\right) V^{[9]} = 0$ where $\tilde{J}_9 = J_9\left(\lambda^{[9]} = 0\right)$, then $V^{[9]} = \left(0 \, , \frac{-(h_1 x_9 + h_2 h_9)}{(h_1 x_9 + h_2 h_8)} v_3^{[9]} \, , v_3^{[9]} \, , 0 \right)^T$ where $v_3^{[9]}$ represents any nonzero real number. And let $\Psi^{[9]} = \left(\psi_1^{[9]}, \psi_2^{[9]}, \psi_3^{[9]}, \psi_4^{[9]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda^{[9]} = 0$ of the matrix $\tilde{J}_9^T$, hence $\Psi^{[9]} = \left(0, \frac{-h_4 z_9}{h_3 y_9} \psi_3^{[9]} \, , \psi_3^{[9]}, 0 \right)^T$ where $\psi_3^{[9]}$ represents any nonzero real number. Now, we have

$\left(\Psi^{[9]}\right)^T\left[\mathrm{N}_{h_5}\left(E_9,h_5^{[9]}\right)\right]=0$ where $\mathrm{N}_{h_5}(X,h_5)$ represents the derivative of $\mathrm{N}(X,h_5)$ with respect to parameter h_5.

Thus, according to Sotomayor's theorem for local bifurcation the saddle-node bifurcation cannot occur. While the first condition of transcritical and pitchfork bifurcation is satisfied. Now, it is easy to observed that $\left(\Psi^{[9]}\right)^T\left[D\mathrm{N}_{h_5}\left(E_9,h_5^{[9]}\right)V^{[9]}\right]=0$

where $D\mathrm{N}_{h_5}(X,h_5)$ represents the derivative of $\mathrm{N}_{h_5}(X,h_5)$ with respect to $X=(x,y,z,P)^T$

Thus, according to Sotomayor's theorem for local bifurcation, at E_9 with the parameter $h_5=h_5^{[9]}$ both of transcritical and pitchfork bifurcation cannot occur.

6. Numerical Simulations

The system (1) is solved numerically for different sets of parameters and different sets of initial conditions, and then the time series for the trajectories of system (1) are confirm our obtained analytical results. By using $(0.75,0.75,0.75,0.75)$, $(1.75,1.75,1.75,1.75)$ and $(3.75,3.75,3.75,3.75)$ as initial points and predictor-corrector method with six order Runge-Kutta method, the numerical simulations are carried out in the following cases:

Case I For the equilibrium point E_1, we choose the following parametric values :

$$\begin{aligned}&h_1=0.8,\ h_2=50,\ h_3=0.01,\ h_4=0.01,\ h_5=0.1,\ h_6=0.5,\ h_7=0.5,\\&h_8=0.02,\ h_9=0.09,\ h_{10}=0.4,\ h_{11}=40,\ q=0.7,\ e=0.1,\ E=0.6;\end{aligned} \tag{17}$$

The conditions (7) of Theorem (2) are satisfied. Then, the equilibrium point E_1 of system (1) is globally asymptotically stable and is identical to $(50,0,0,0)$ for any time. (See Fig. 1).

Case II For the equilibrium point E_2, we choose the following parametric values :

$$\begin{aligned}&h_1=0.8,\ h_2=50,\ h_3=0.06,\ h_4=0.06,\ h_5=0.1,\ h_6=0.04,\ h_7=0.03,\\&h_8=0.02,\ h_9=0.09,\ h_{10}=0.4,\ h_{11}=40,\ q=0.7,\ e=0.1,\ E=0.1;\end{aligned} \tag{18}$$

The conditions (8) of Theorem (3) are satisfied. Then, the equilibrium point E_2 of system (1) is globally asymptotically stable and is identical to $(0,0,0,33)$ for any time. (See Fig. 2).

Case III For the equilibrium point E_3, we choose the following parametric values :

$$\begin{aligned}&h_1=0.8,\ h_2=50,\ h_3=0.06,\ h_4=0.06,\ h_5=0.1,\ h_6=0.04,\ h_7=0.03,\\&h_8=0.02,\ h_9=0.09,\ h_{10}=0.4,\ h_{11}=40,\ q=0.7,\ e=0.1,\ E=0.6;\end{aligned} \tag{19}$$

The conditions (9) are satisfied. Then, the equilibrium point E_3 of system (1) is globally asymptotically stable and is identical to $(1,21.778,0,0)$ for any time. (See Fig. 3).

Case IV For the equilibrium point E_4, we choose the following parametric values :

$$\begin{aligned}&h_1=0.8,\ h_2=50,\ h_3=0.06,\ h_4=0.2,\ h_5=0.1,\ h_6=0.04,\ h_7=0.03,\\&h_8=0.02,\ h_9=0.09,\ h_{10}=0.4,\ h_{11}=40,\ q=0.7,\ e=0.1,\ E=0.6;\end{aligned} \tag{20}$$

The conditions (10) are satisfied. Then, the equilibrium point E_4 of system (1) is globally asymptotically stable and is identical to $(0.6, 0, 4.761, 0)$ for any time. (See Fig. 4).

Case V For the equilibrium point E_5, we choose the following parametric values :

$$\begin{aligned} &h_1 = 0.8, h_2 = 50, h_3 = 0.06, h_4 = 0.2, h_5 = 0.1, h_6 = 0.04, h_7 = 0.03, \\ &h_8 = 0.02, h_9 = 0.09, h_{10} = 0.4, h_{11} = 40, q = 0.7, e = 0.1, E = 0.6; \end{aligned} \tag{21}$$

The conditions (11) are satisfied. Then, the equilibrium point E_5 of system (1) is globally asymptotically stable and is identical to $(46.9, 0, 0, 26.75)$ for any time. (See Fig. 5).

Case VI For the equilibrium point E_6, we choose the following parametric values :

$$\begin{aligned} &h_1 = 0.8, h_2 = 50, h_3 = 0.06, h_4 = 0.06, h_5 = 0.1, h_6 = 0.03, h_7 = 0.04, \\ &h_8 = 0.09, h_9 = 0.02, h_{10} = 0.4, h_{11} = 40, q = 0.7, e = 0.1, E = 0.5; \end{aligned} \tag{22}$$

The conditions (12a) and (12b) of Theorem (4) are satisfied. Then, the equilibrium point E_6 of system (1) is globally asymptotically stable and is identical to $(1, 0, 14.139, 5.5)$ for any time. (See Fig. 6).

Case VII For the equilibrium point E_7, we choose the following parametric values :

$$\begin{aligned} &h_1 = 0.8, h_2 = 50, h_3 = 0.06, h_4 = 0.06, h_5 = 0.1, h_6 = 0.04, h_7 = 0.03, \\ &h_8 = 0.02, h_9 = 0.09, h_{10} = 0.4, h_{11} = 40, q = 0.7, e = 0.1, E = 0.5; \end{aligned} \tag{23}$$

The conditions (13a) and (13b) of Theorem (5) are satisfied. Then, the equilibrium point E_7 of system (1) is globally asymptotically stable and is identical to $(1, 14.139, 0, 5.5)$ for any time. (See Fig. 7).

Case VIII For the equilibrium point E_8, we choose the following parametric values :

$$\begin{aligned} &h_1 = 0.8, h_2 = 50, h_3 = 0.8, h_4 = 0.5, h_5 = 0.1, h_6 = 0.07, h_7 = 0.18, \\ &h_8 = 0.25, h_9 = 0.02, h_{10} = 0.4, h_{11} = 40, q = 0.9, e = 0.2, E = 0.6; \end{aligned} \tag{24}$$

The conditions (14) of Theorem (6) are satisfied. Then, the equilibrium point E_8 of system (1) is locally asymptotically stable and is identical to $(0.4, 1.138, 0.973, 0)$ for any time. (See Fig. 8).

Case IX For the equilibrium point E_9, we choose the following parametric values :

$$\begin{aligned} &h_1 = 0.8, h_2 = 50, h_3 = 0.8, h_4 = 0.5, h_5 = 0.1, h_6 = 0.07, h_7 = 0.18, \\ &h_8 = 0.25, h_9 = 0.02, h_{10} = 0.4, h_{11} = 40, q = 0.7, e = 0.2, E = 0.5; \end{aligned} \tag{25}$$

The conditions (15) of Theorem (7) are satisfied. Then, the equilibrium point E_9 of system (1) is locally asymptotically stable and is identical to $(0.4, 0.55, 0.62, 5.57)$ for any time. (See Fig. 9).

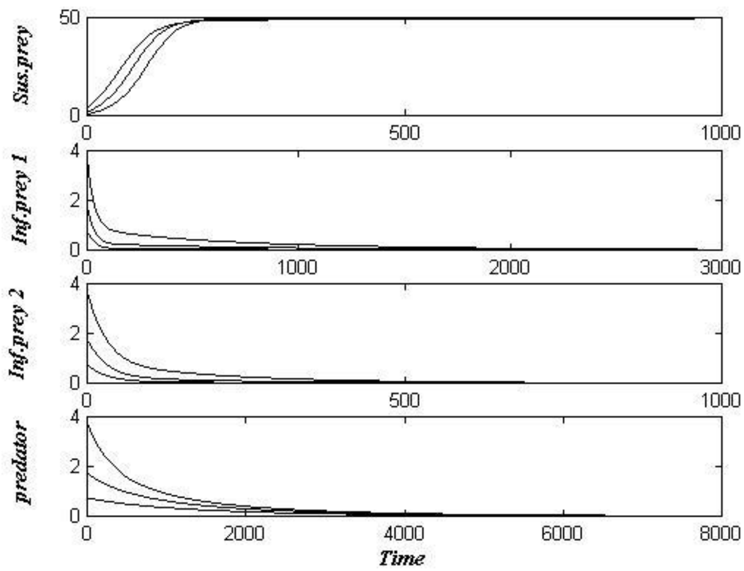

Fig.1: Time series of the trajectories of system (1) from different initial points for data given in Eq.(17) which show that E_1 is a globally asymptotically stable.

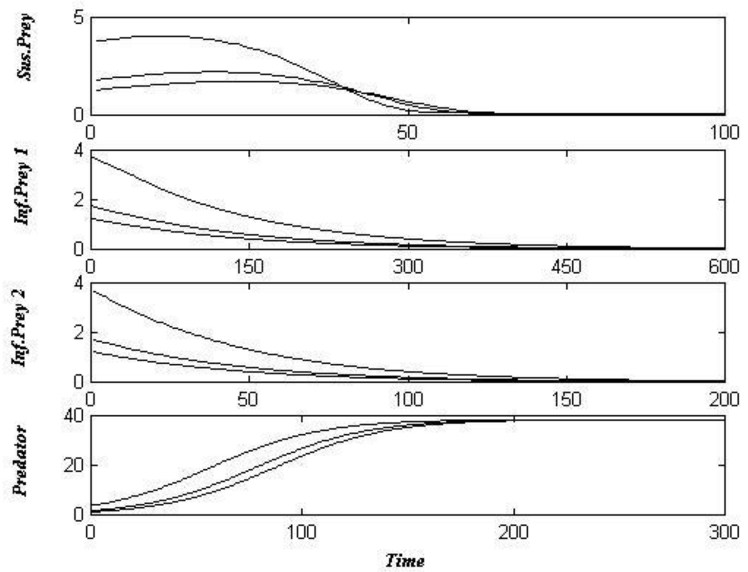

Fig.2: Time series of the trajectories of system (1) from different initial points for data given in Eq.(18) which show that E_2 is a globally asymptotically stable.

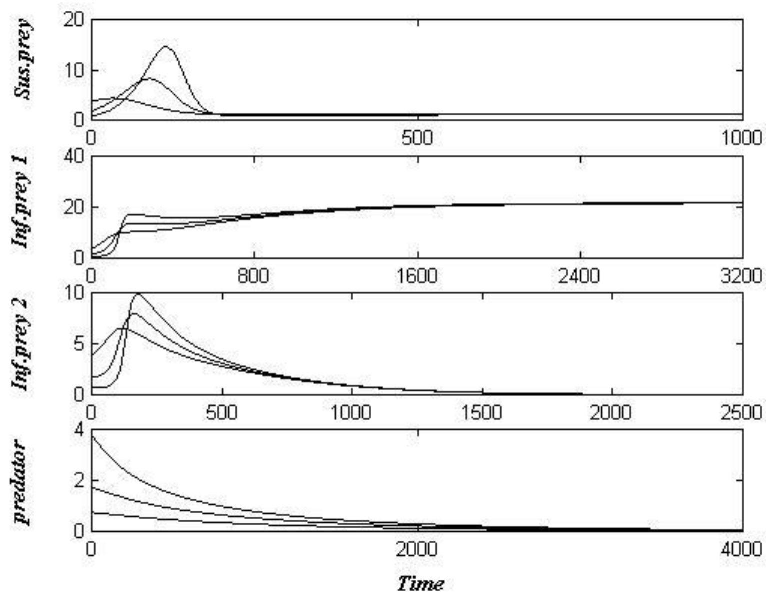

Fig.3: Time series of the trajectories of system (1) from different initial points for data given in Eq.(19) which show that E_3 is a globally asymptotically stable.

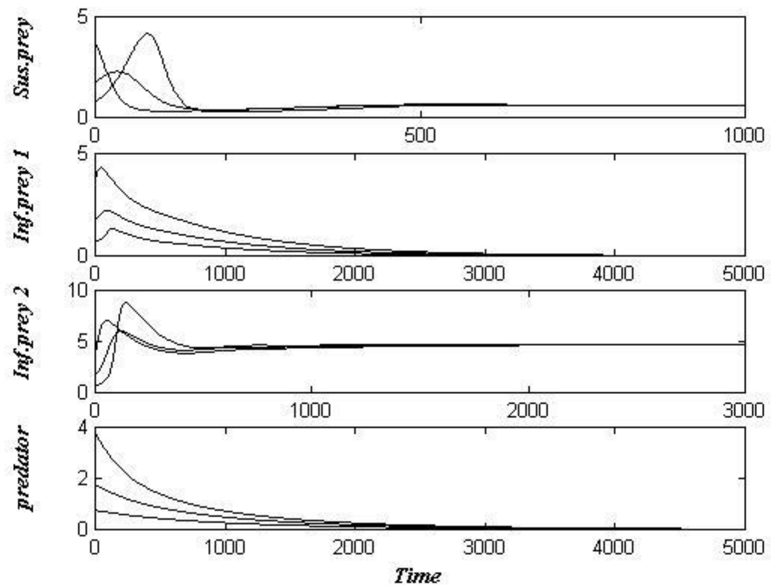

Fig.4: Time series of the trajectories of system (1) from different initial points for data given in Eq.(20) which show that E_4 is a globally asymptotically stable.

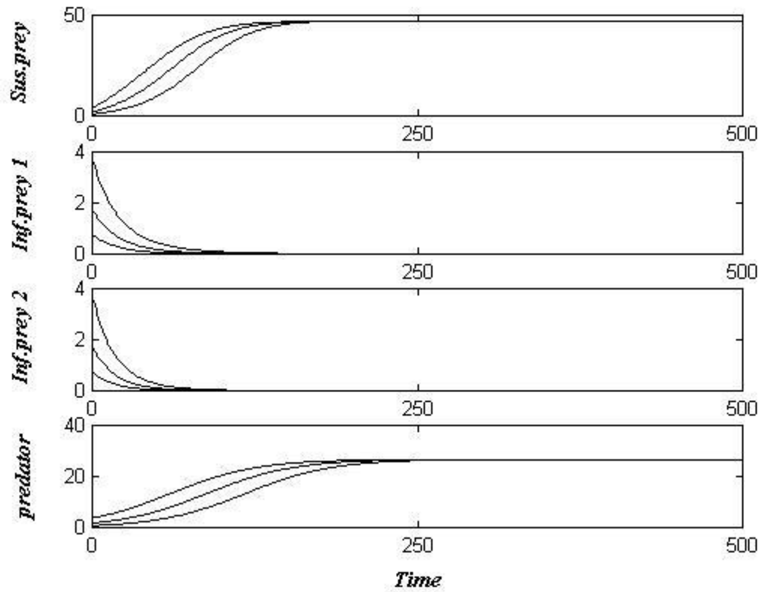

Fig.5: Time series of the trajectories of system (1) from different initial points for data given in Eq.(21) which show that E_5 is a globally asymptotically stable.

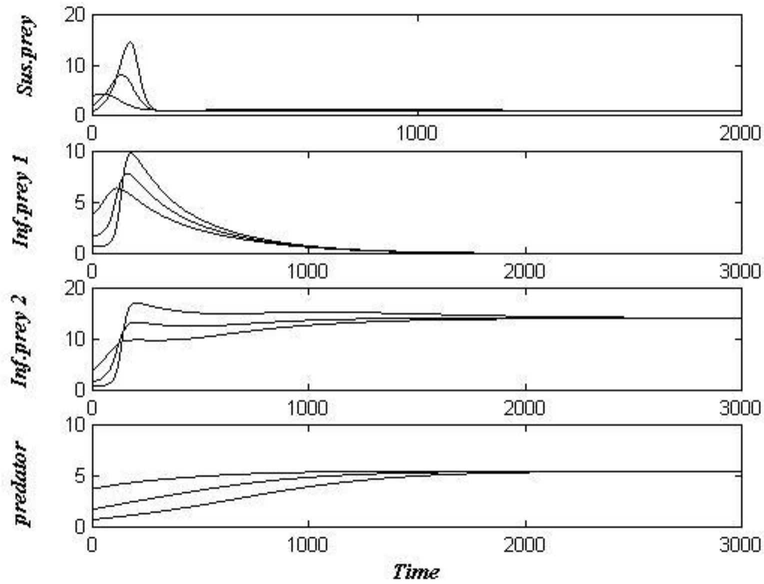

Fig.6: Time series of the trajectories of system (1) from different initial points for data given in Eq.(22) which show that E_6 is a globally asymptotically stable.

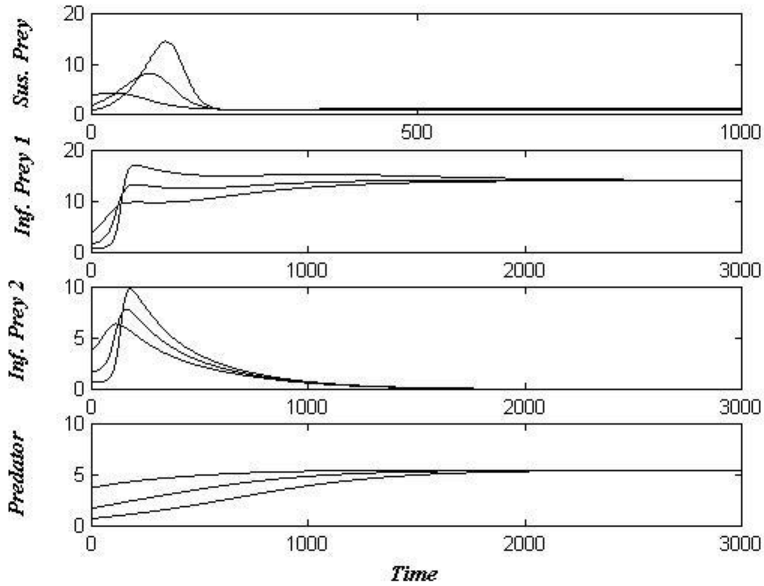

Fig.7: Time series of the trajectories of system (1) from different initial points for data given in Eq.(23) which show that E_7 is a globally asymptotically stable.

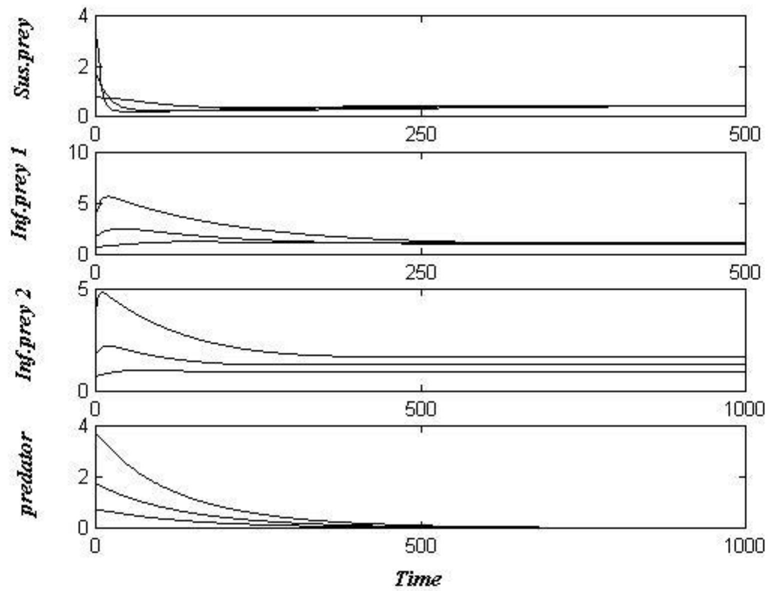

Fig.8: Time series of the trajectories of system (1) from different initial points for data given in Eq.(24) which show that E_8 is a locally asymptotically stable.

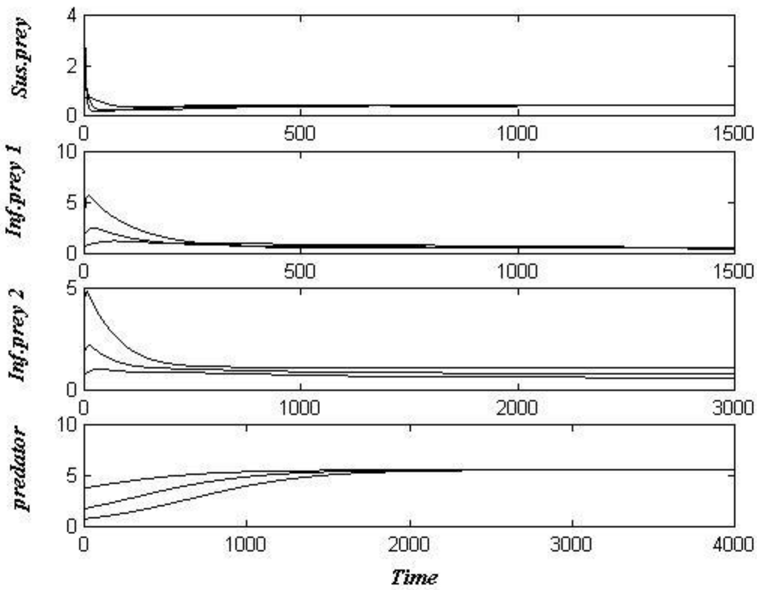

Fig.9: Time series of the trajectories of system (1) from different initial points for data given in Eq.(25) which show that E_9 is a locally asymptotically stable.

7. conclusions and Discussion

In this paper, an eco-epidemiological model has been proposed and analyzed. In order to study the effect of two infection diseases and harvesting on the dynamical behavior of the prey-predator system, the dynamical behavior of system (1) has been investigated locally as well as globally. In addition to assumed that the predator population is harvested at a constant rate the predator is harvested under optimal conditions, we used Holling type-II functional response with maximum attack rate (h_5) and linear incidence rate for the diseases in prey species. The model included four non-linear autonomous differential equations that describe the dynamics of four different populations namely susceptible prey (x), infected prey (y) by first disease, infected prey (z) by second disease and predator (p). The boundedness of the system (1) has been discussed. The conditions for existence and stability for each equilibrium points are obtained. To study the stability at each equilibrium point we used Trce-Determent theorem, Routh-Hurwitz criterion and Lyapunov function. For different sets of parameters and different sets of initial conditions show that, for all equilibrium points except E_8 and E_9 the trajectory of system (1) approaches to globally asymptotically stable point in the $Int.R_+^4$ as see Fig.(1-7), finally the trajectory of system (1) at each E_8 and E_9 approaches to locally asymptotically stable point in the $Int.R_+^4$, in Fig.(8-9). Numerically, as well as theoretically, show that for different initial points the trajectory approaches to different equilibrium point. Also we can separate the system (1) to six models as following:

- Epidemic model with first infection disease.
- Epidemic model with second infection disease.
- Simple prey-predator model with Holling type-II functional response.
- Prey-predator model involving first infection disease.
- Prey-predator model involving second infection disease.
- Epidemic model with two different SIS infection diseases.

We prove theoretically that with different candidate bifurcation parameters each equilibrium points E_1, E_2, E_4 *and* E_8 satisfy the Transcritical bifurcation and E_3, E_5 satisfy the saddle-node bifurcation but E_6, E_7 *and* E_9 did not satisfy any types of bifurcation. Numerically, notice that when change the value of the bifurcation parameter the solution of system (1) approaches to another equilibrium point.

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